Computers and Mathematics with Applications 62 (2011) 367–375

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On a new class of analytic functions associated with conic domain

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A R T I C L E I N F O

Article history:
Received 16 February 2011
Received in revised form 5 May 2011
Accepted 6 May 2011

Keywords:
Analytic functions
Conic domains
k-uniformly convex functions
k-starlike functions

A B S T R A C T

The aim of this paper is to generalize the conic domain defined by Kanas and Wisniowska, and define the class of functions which map the open unit disk E onto this generalized conic domain. A brief comparison between these conic domains is the main motivation of this paper. A correction is made in selecting the range interval of order of conic domain.

1. Introduction and preliminaries

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$. Also let S be the class of functions from A which are univalent in E. The classes S^* and C are the well known classes of starlike and convex univalent functions respectively, for details see [1].

Kanas and Wisniowska [2,3] introduced and studied the classes of k-uniformly convex denoted by k-UCV and the corresponding class of k-starlike functions denoted by k-ST related by the Alexandar type relation. They defined these classes as follows:

A function $f(z) \in A$ is said to be in the class k-UCV, if and only if,

$$\mathrm{Re}\left(\frac{(zf'(z))'}{f'(z)}\right) > k\left|\frac{(zf'(z))'}{f'(z)} - 1\right|, \quad k \geq 0.$$

A function $f(z) \in A$ is said to be in the class k-ST, if and only if,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad k \geq 0.$$

Geometrically, a function $f(z) \in A$ is said to be in the class k-UCV(or k-ST), if and only if, the function $\frac{(zf'(z))'}{f'(z)}$ (or $\frac{zf'(z)}{f(z)}$) takes all values in the conic domain Ω_k which is defined as:

$$\Omega_k = \{u + iv : u > k\sqrt{(u-1)^2 + v^2}\}.$$

This domain represents the right half plane when $k = 0$, a hyperbola when $0 < k < 1$, a parabola when $k = 1$ and an ellipse when $k > 1$ as shown in Fig. 1.

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2011.05.018

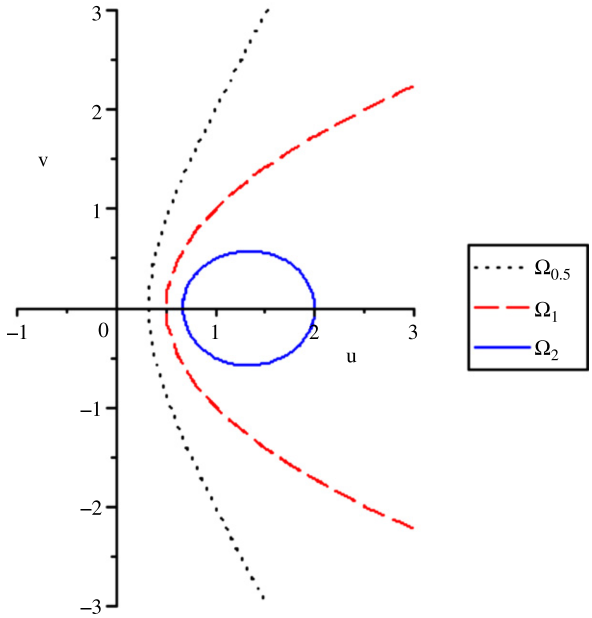

Fig. 1. The curve $u = k\sqrt{(u-1)^2 + v^2}$.

The functions which play the role of extremal functions for these conic regions are given as:

$$p_k(z) = \begin{cases} \dfrac{1+z}{1-z}, & k = 0, \\ 1 + \dfrac{2}{\pi^2}\left(\log \dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1, \\ 1 + \dfrac{2}{1-k^2}\sinh^2\left[\left(\dfrac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right], & 0 < k < 1, \\ 1 + \dfrac{1}{k^2-1}\sin\left(\dfrac{\pi}{2R(t)}\displaystyle\int_0^{\frac{u(z)}{\sqrt{t}}} \dfrac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\mathrm{d}x\right) + \dfrac{1}{k^2-1}, & k > 1, \end{cases} \tag{1.2}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0,1)$, $z \in E$ and z is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$, see [2,3]. These conic regions are extensively studied with regard to real and complex orders by Noor [4,5]. We generalize this conic domain and define the following.

Definition 1.1. A function $p(z)$ is said to be in the class $k - P(a, b)$, if and only if,

$$p(z) \prec p_k(a, b; z), \tag{1.3}$$

where $k \in [0, \infty)$,

$$\begin{aligned} p_k(a, b; z) &= 1 + a + (1-b)\{p_k(z) - 1\} \\ &= a + b + (1-b)p_k(z), \end{aligned} \tag{1.4}$$

and $p_k(z)$ is defined by (1.2). Also a and b must be chosen accordingly as:

$$\left.\begin{aligned} &\text{(i) For } k = 0, \text{ we take } b = 0, \\ &\text{(ii) For } k \in \left(0, \frac{1}{\sqrt{2}}\right), \text{ we take } b \in \left[\frac{1}{2k^2-1}, 1\right), \\ &\text{(iii) For } k \in \left[\frac{1}{\sqrt{2}}, 1\right], \text{ we take } b \in (-\infty, 1), \\ &\text{(iv) For } k \in (1, \infty), \text{ we take } b \in \left(-\infty, \frac{1}{2k^2-1}\right]. \end{aligned}\right\} \tag{1.5}$$

and

$$\left.\begin{aligned} &\frac{k^2(1-b)}{1-k^2}-\eta \le a < 1-\frac{k^2(1-b)}{k^2-1}+\eta, && 0\le k<1,\\ &-\frac{1+b}{2}\le a<\frac{1-b}{2}, && k=1,\\ &\max\left(\frac{k^2(1-b)}{1-k^2}-\eta,\, 1-\frac{k^2(1-b)}{k^2-1}-\eta\right)\le a<1-\frac{k^2(1-b)}{k^2-1}+\eta, && k>1, \end{aligned}\right\} \tag{1.6}$$

where $\eta = \frac{k\sqrt{k^2(1-b)^2+(1-k^2)(1-b^2)}}{k^2-1}$.

Geometrically, the function $p(z) \in k - P(a, b)$ takes all values from the conic domain $\Omega_k(a, b)$ which is defined as:

$$\Omega_k(a, b) = \{u + iv : (u - a)^2 > k^2[(u - a + b - 1)^2 + v^2 + 2b(1 - b)]\}. \tag{1.7}$$

The conic domain $\Omega_k(a, b)$ represents the right half plane when $k = 0$, a hyperbola when $0 < k < 1$, a parabola when $k = 1$ and an ellipse when $k > 1$.

It can be seen that $\Omega_k(0, 0) = \Omega_k$, the conic domain defined by Kanas and Wisniowska [2], consequently, $k - P(0, 0) = P(p_k)$, the well-known class introduced by Kanas and Wisniowska [2]. The function $p_1(a, b; z) = Q_{a,b}(z)$ is defined by Kanas in [6]. Here are some basic facts about the class $k - P(a, b)$.

Remark 1.2. (1) $k - P(a, b) \subset P(\alpha)$, where

$$\alpha = \begin{cases} a + \dfrac{1+b}{2}, & k = 1,\\[2mm] a + \dfrac{k^2(1-b) - k\sqrt{k^2(1-b)^2 + (1-k^2)(1-b^2)}}{k^2-1}, & k \ne 1. \end{cases} \tag{1.8}$$

(2) $k - P(a_1, b) \subset k - P(a_2, b)$, $a_1 > a_2$, $k \in [0, 1]$.
(3) $k - P(a, b_1) \subset k - P(a, b_2)$, $b_1 > b_2$, $k \in (0, \infty)$.

The domain $\Omega_k(a, b)$ always ensures that the point $(1, 0)$ is contained inside it whereas the domain $\Omega_{k,\xi}$, studied by several authors, defined by

$$\Omega_{k,\xi} = (1 - \xi)\Omega_k + \xi, \quad 0 \le \xi < 1,\ k \ge 0, \tag{1.9}$$

is not always well defined because in general $(1, 0) \notin \Omega_{k,\xi}$ (for example, in particular $(1, 0) \notin \Omega_{2,0.5}$). We see that the conic domain $\Omega_k(0, b)$ coincides with $\Omega_{k,b}$ only when b is chosen according to (1.5). This means that for $\Omega_{k,\xi}$ to contain the point $(1, 0)$, ξ must be chosen according as:

$$\xi \in \begin{cases} [0, 1), & \text{if } 0 \le k \le 1,\\[2mm] \left[0, 1 - \dfrac{\sqrt{k^2-1}}{k}\right), & \text{if } k > 1. \end{cases} \tag{1.10}$$

The domain $\Omega_{k,\xi}$ gives only the contraction of Ω_k whereas the domain $\Omega_k(a, b)$ gives contraction as well as magnification of Ω_k depending upon b. For $b > 0$, the domain $\Omega_k(a, b)$ gives the contraction and for $b < 0$, the domain gives the magnification of Ω_k as can be seen from the Figs. 2 and 3.

Definition 1.3. A function $f(z) \in A$ is said to be in the class k-UCV(a, b), $k \ge 0$, a, b satisfying (1.5) and (1.6), if and only if,

$$\left[\operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)} - a\right\}\right]^2 > k^2\left[\left|\frac{(zf'(z))'}{f'(z)} - a + b - 1\right|^2 + 2b(1 - b)\right] \tag{1.11}$$

or equivalently

$$\frac{(zf'(z))'}{f'(z)} \prec p_k(a, b; z), \tag{1.12}$$

where $p_k(a, b; z)$ is defined by (1.4).

Definition 1.4. A function $f(z) \in A$ is said to be in the class k-ST(a, b), $k \ge 0$, a, b satisfying (1.5) and (1.6), if and only if,

$$\left[\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - a\right\}\right]^2 > k^2\left[\left|\frac{zf'(z)}{f(z)} - a + b - 1\right|^2 + 2b(1 - b)\right] \tag{1.13}$$

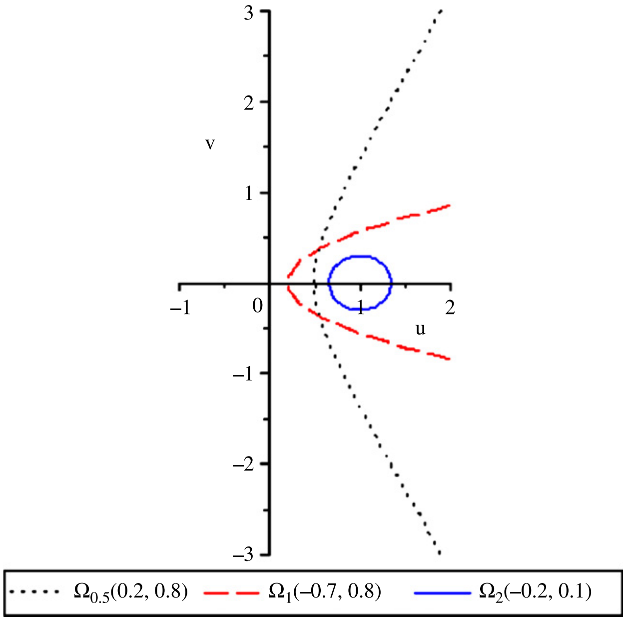

Fig. 2. The curve $(u-a)^2 = k^2[(u-a+b-1)^2 + v^2 + 2b(1-b)]$.

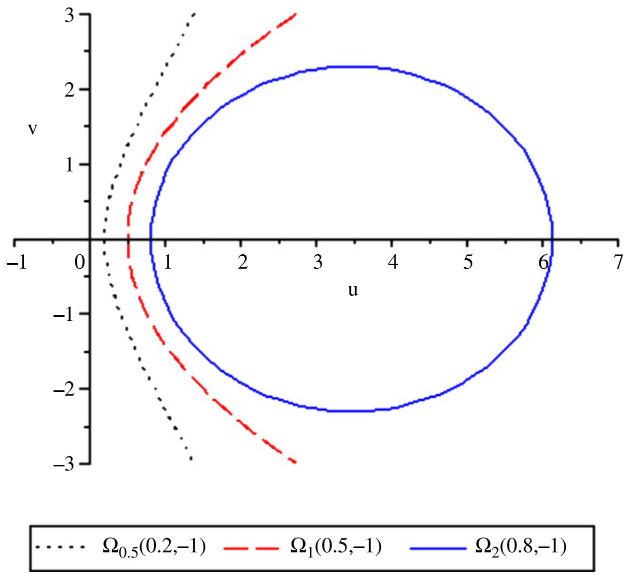

Fig. 3. The curve $(u-a)^2 = k^2[(u-a+b-1)^2 + v^2 + 2b(1-b)]$.

or equivalently

$$\frac{zf'(z)}{f(z)} \prec p_k(a, b; z), \tag{1.14}$$

where $p_k(a, b; z)$ is defined by (1.4).

It can be easily seen that

$$f(z) \in k\text{-UCV}(a, b) \Longleftrightarrow zf'(z) \in k\text{-ST}(a, b).$$

Special cases.

i. k-UCV$(0, 0) = k$-UCV, the well-known class of k-uniformly convex functions, introduced by Kanas and Wisniowska [2].
ii. k-ST$(0, 0) = k$-ST, the well-known class of k-starlike functions, introduced by Kanas and Wisniowska [3].

Lemma 1.5 ([7]). *Let the function $w(z)$ be non-constant analytic in E with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then*

$$z_0 w'(z_0) = c w(z_0),$$

c is real and $c \geq 1$.

2. Main results

Theorem 2.1. *If $f(z) \in A$ satisfies the inequality*

$$\operatorname{Re}\left\{\frac{\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)} - 1}\right\} < \frac{3-\alpha}{2-\alpha},$$

where α is defined by (1.8), then $f(z) \in k$-ST(a, b), $k \in [0, 1]$, $b \leq 0$ with a and b satisfying (1.5) and (1.6).

Proof. We consider the function $w(z)$ as

$$\frac{zf'(z)}{f(z)} - 1 = (1-\alpha)w(z), \tag{2.1}$$

where α is defined by (1.8). We see that $w(z)$ is analytic in E and $w(0) = 0$. Logarithmic differentiation of (2.1) gives us

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} = \frac{(1-\alpha)zw'(z)}{(1-\alpha)w(z)+1}.$$

This implies that

$$\frac{zf''(z)}{f'(z)} = (1-\alpha)w(z) + \frac{(1-\alpha)zw'(z)}{(1-\alpha)w(z)+1}. \tag{2.2}$$

Now from (2.1) and (2.2), we have

$$\frac{\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)} - 1} = 1 + \frac{zw'(z)}{w(z)\left\{(1-\alpha)\, w(z)+1\right\}}.$$

Suppose that there exists a point $z_0 \in E$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1, \; |w(z_0)| \neq 1$$

and also $w(z_0) = e^{i\theta}$. Then applying Lemma 1.5, we have

$$z_0 w'(z_0) = c w(z_0), \quad c \geq 1.$$

Using this, we can have

$$\begin{aligned}
\operatorname{Re}\left\{\frac{\frac{zf''(z_0)}{f'(z_0)}}{\frac{zf'(z_0)}{f(z_0)} - 1}\right\} &= \operatorname{Re}\left\{1 + \frac{z_0 w'(z_0)}{w(z_0)\left\{(1-\alpha)w(z_0)+1\right\}}\right\} \\
&= \operatorname{Re}\left\{1 + \frac{c w(z_0)}{w(z_0)\left\{(1-\alpha)w(z_0)+1\right\}}\right\} \\
&= 1 + c\operatorname{Re}\left\{\frac{1}{(1-\alpha)w(z_0)+1}\right\} \\
&= 1 + c\operatorname{Re}\left\{\frac{1}{(1-\alpha)e^{i\theta}+1}\right\} \\
&= 1 + c\frac{1+(1-\alpha)\cos\theta}{(1-\alpha)^2 + 2(1-\alpha)\cos\theta + 1} = F(\theta), \quad \text{say.}
\end{aligned}$$

Now as we know that $F(\theta) \geq \min F(\theta)$ and it can easily be seen that

$$\begin{aligned}
\min F(\theta) &= F(\pi) \\
&= 1 + c\frac{1+(1-\alpha)\cos\pi}{(1-\alpha)^2 + 2(1-\alpha)\cos\pi + 1}
\end{aligned}$$

$$
\begin{aligned}
&= 1+\frac{c}{\alpha}\\
&\geq 1+\frac{1}{\alpha}\\
&> 1+\frac{1}{2-\alpha}, \quad \alpha<1\\
&= \frac{3-\alpha}{2-\alpha}.
\end{aligned}
$$

Therefore, we have

$$\mathrm{Re}\left\{\frac{\frac{zf''(z_0)}{f'(z_0)}}{\frac{zf'(z_0)}{f(z_0)}-1}\right\} \geq \frac{3-\alpha}{2-\alpha},$$

which is a contradiction to our hypothesis. Thus, we must have $|w(z)| < 1$ for all $z \in E$ and therefore we have from (2.1),

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\alpha,$$

which shows that $\frac{zf'(z)}{f(z)}$ lies inside a circle centered at $(1, 0)$ and having radius $1-\alpha$ and we know from (1.7) that this circle lies inside the conic domain $\Omega_k(a, b)$, $k \in [0, 1]$, $b \leq 0$ with a and b satisfying (1.5) and (1.6). This implies that $f(z) \in k\text{-ST}(a, b)$, $k \in [0, 1]$, $b \leq 0$ with a and b satisfying (1.5) and (1.6). □

From the Theorem 2.1, we see that when $a = 0$, $b = 0$ and $k = 1$, we have the following result which is the special case (when $p = 1$) of the result proved by Al-Kharsani et al. [8].

Corollary 2.2. *If $f(z) \in A$ satisfies the inequality*

$$\mathrm{Re}\left\{\frac{\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}-1}\right\} < \frac{5}{3},$$

then $f(z)$ is uniformly starlike in E (that is $f(z) \in 1\text{-ST}$).

Theorem 2.3. *For $b_1 > b_2$,*

i. $k\text{-UCV}(a, b_1) \subset k\text{-UCV}(a, b_2)$.
ii. $k\text{-ST}(a, b_1) \subset k\text{-ST}(a, b_2)$.

Proof follows directly from Remark 1.2(3), (1.3), (1.12) and (1.14).

Theorem 2.4. *Let $f(z) \in S$. Then $f(z) \in k\text{-UCV}(a, b)$ for $|z| < r_0 < 1$ with*

$$r_0 = \frac{2-\sqrt{3+\alpha^2}}{1+\alpha},$$

where α is defined by (1.8).

Proof. Let $f(z) \in S$. Then, for $|z| = r < 1$, we have

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2},$$

for detail, see [1]. This implies that

$$\left|\frac{(zf'(z))'}{f'(z)} - \frac{1+r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2}. \tag{2.3}$$

This disk intersects the real axis at the points $\left(\frac{1-4r+r^2}{1-r^2}, 0\right)$ and $\left(\frac{1+4r+r^2}{1-r^2}, 0\right)$. Now we have to find the largest value of r such that the disk (2.3) lies completely inside the conic domain $\Omega_k(a, b)$, that is $\left(\frac{1-4r+r^2}{1-r^2}, 0\right) \in \Omega_k(a, b)$. For this, we must have

$$\frac{1-4r+r^2}{1-r^2} \geq \alpha,$$

where α is defined by (1.8). This gives us

$$(1+\alpha)r^2 - 4r + 1 - \alpha \geq 0, \quad 0 < r < 1.$$

This holds only if

$$r \le r_0 = \frac{2-\sqrt{3+\alpha^2}}{1+\alpha}.$$

Now it can also be seen that the curves

$$(u-a)^2 = k^2(u-a+b-1)^2 + k^2v^2 + 2k^2b(1-b)$$

and

$$\left(u - \frac{1+r^2}{1-r^2}\right)^2 + v^2 = \frac{16r^2}{(1-r^2)^2}$$

do not intersect any where except the possibility that the points $(\alpha, 0)$ and $\left(\frac{1-4r+r^2}{1-r^2}, 0\right)$ coincides. Therefore, the disk (2.3) lies completely inside the conic domain $\Omega_k(a, b)$. Hence the proof. □

When $a = 0$ and $b = 0$, then we have the following result, proved by Kanas and Wisniowska [2].

Corollary 2.5. *Let $f(z) \in S$. Then $f(z) \in k$-UCV for $|z| < r_0 < 1$ with*

$$r_0 = \frac{2(k+1) - \sqrt{4k^2+6k+3}}{2k+1}.$$

When $a = 0,\ b = 0$ and $k = 1$, then we have the following result, proved in [9].

Corollary 2.6. *Let $f(z) \in S$. Then $f(z) \in$ UCV for $|z| < r_0 < 1$ with*

$$r_0 = \frac{4-\sqrt{13}}{3}.$$

When $a = 0,\ b = 0$ and $k = 0$, then we have the following result, proved in [1].

Corollary 2.7. *Let $f(z) \in S$. Then $f(z) \in C$ for $|z| < r_0 < 1$ with $r_0 = 2 - \sqrt{3}$.*

Now we have an extension of the result proved in [6].

Lemma 2.8. *Let $0 \le k < \infty$. Also, let $\beta, \gamma \in \mathbb{C}$ be such that $\beta \ne 0$ and $\mathrm{Re}(\alpha\beta + \gamma) > 0$, where α is defined by (1.8). If $p(z)$ is analytic in E, $p(0) = 1$, $p(z)$ satisfying*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec p_k(a, b; z), \tag{2.4}$$

and $q(z)$ is an analytic solution of

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = p_k(a, b; z),$$

then $q(z)$ is univalent, $p(z) \prec q(z) \prec p_k(a, b; z)$ and $q(z)$ is the best dominant of (2.4).

The proof follows similarly as given in [6].

As a special case, when $\beta = 1$ and $\gamma = 0$, we have the function $q(z)$ as

$$q(z) = \left[\int_0^1 \left(\exp \int_z^{tz} \frac{p_k(a, b; u) - 1}{u} \mathrm{d}u\right) \mathrm{d}t\right]^{-1}.$$

Now we see a few applications of the Lemma 2.8.

When $k > 1$, the conic domain $\Omega_k(a, b)$ may be characterized by the circular domain having its diameter end points as the vertices of the ellipse. As we see that the vertices of ellipse are $(\alpha, 0)$ and $(\alpha_1, 0)$, where α is defined by (1.8) and

$$\alpha_1 = a + \frac{k^2(1-b) + k\sqrt{k^2(1-b)^2 + (1-k^2)(1-b^2)}}{k^2-1}.$$

The circle $K(X, R)$ having diameter end points $(\alpha, 0)$, $(\alpha_1, 0)$ has its center at $X\left(\frac{a(k^2-1)+(1-b)k^2}{k^2-1}, 0\right)$ and radius R as

$$R = \frac{k\sqrt{k^2(1-b)^2 + (1-k^2)(1-b^2)}}{k^2-1}.$$

The point $z = 1$ is contained inside the circle $K(X, R)$ and then the function $\phi_{a,b} : E \longrightarrow K(X, R)$ has the form

$$\phi_{a,b}(z) = a + \frac{k^2(1-b) + kz\sqrt{(1-b)(1-b-2b(k^2-z^2))}}{k^2 - z^2}.$$

For $b = 0$, we have

$$\phi_{a,0}(z) = a + \frac{k}{k-z}.$$

Theorem 2.9. *Let $k \in (1, \infty)$ and $b = 0$. Also, let $p(z)$ be analytic in E with $p(0) = 1$ and $p(z)$ satisfies (2.4). Then*

$$p(z) \prec \frac{1}{(k-z)\int_0^1 \frac{t^a}{k-tz}\mathrm{d}t}$$

and

$$\mathrm{Re}\,p(z) > \frac{1}{(k+1)\int_0^1 \frac{t^a}{k+t}\mathrm{d}t},$$

where $-\frac{k}{k+1} \le a < \frac{1}{k+1}$.

The proof follows directly by taking $q(z) = \phi_{a,0}(z)$ in Lemma 2.8.

When $a = 0$, we have the following result, proved by Kanas [6].

Corollary 2.10. *Let $k \in (1, \infty)$ and let $p(z)$ be analytic in E with $p(0) = 1$ and $p(z)$ satisfies (2.4). Then*

$$p(z) \prec \frac{z}{(z-k)\log\left(1 - \frac{z}{k}\right)}$$

and

$$\mathrm{Re}\,p(z) > \frac{1}{(k+1)\log\left(1 + \frac{1}{k}\right)}.$$

Theorem 2.11. *Let $f(z), g(z) \in k$-ST(a, b) and let μ, c and δ be positive reals. Then the function $F(z)$, defined by*

$$F(z) = \left[cz^{\mu-c}\int_0^z t^{c-\mu-1}\,(f(t))^{\delta}\,(g(t))^{\mu-\delta}\,\mathrm{d}t\right]^{\frac{1}{\mu}} \tag{2.5}$$

belongs to k-ST(a, b).

Proof. From (2.5), we have

$$z^{c-\mu}(F(z))^{\mu} = c\int_0^z t^{c-\mu-1}(f(t))^{\delta}(g(t))^{\mu-\delta}\mathrm{d}t.$$

This implies that

$$(c-\mu)z^{c-\mu-1}(F(z))^{\mu} + \mu z^{c-\mu}(F(z))^{\mu-1}F'(z) = cz^{c-\mu-1}(f(z))^{\delta}(g(z))^{\mu-\delta}.$$

Let $h(z) = \frac{zF'(z)}{F(z)}$. Then, we have

$$(F(z))^{\mu}\{(c-\mu) + \mu h(z)\} = c(f(z))^{\delta}(g(z))^{\mu-\delta}.$$

Differentiating logarithmically, we have

$$\mu\frac{zF'(z)}{F(z)} + \frac{\mu zh'(z)}{\mu h(z) + (c-\mu)} = \delta\frac{zf'(z)}{f(z)} + (\mu-\delta)\frac{zg'(z)}{g(z)}.$$

Now let $h_1(z) = \frac{zf'(z)}{f(z)}$ and $h_2(z) = \frac{zg'(z)}{g(z)}$. Then we have

$$h(z) + \frac{\frac{1}{\mu}zh'(z)}{h(z) + \frac{c-\mu}{\mu}} = \frac{\delta}{\mu}h_1(z) + \left(1 - \frac{\delta}{\mu}\right)h_2(z).$$

Since $f(z), g(z) \in k$-ST(a, b), so $h_1(z), h_2(z) \in k-P(a, b)$. And we know by subordination technique that the class $k-P(a, b)$ is convex. Therefore,

$$h(z) + \frac{\frac{1}{\mu}zh'(z)}{h(z) + \frac{c-\mu}{\mu}} \prec p_k(a, b; z),$$

which implies by using Lemma 2.8,

$$h(z) \prec p_k(a, b; z).$$

This shows that $F(z) \in k\text{-ST}(a, b)$. □

Acknowledgments

The authors are thankful to Dr. S.M. Junaid Zaidi (Rector CIIT) for providing excellent research facilities and also to the Higher Education Commission of Pakistan for financial assistance.

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