JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **123**, 461–479 (1987)

Boundedness Criteria for Solutions of Certain Second Order Nonlinear Differential Equations

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Submitted by Edward Angel

Received October 24, 1985

INTRODUCTION

We shall study the boundedness of all solutions and their first derivatives over the half interval $t \geqslant 0$ for the following second order nonlinear differential equations:

$$(a(t)\,x')' + b(t) f(x, x') + c(t) g(x)\, h(x') = p(t, x, x') \qquad \left(' = \frac{d}{dt}\right), \tag{1}$$

and

$$a(t)\,x'' + b(t) f(x, x') + c(t) g(x)\, h(x') = p(t, x, x'), \tag{2}$$

where a, b, c, f, g, h, and p are real valued functions which depend at most on the arguments displayed explicitly.

Both Eqs. (1) and (2) include the so-called generalized Liénard equation. Boundedness and stability problems of generalized Liénard equations have been extensively (perhaps even exhaustively) investigated. Several surveys of the literature dealing with these problems have been made, and in fact, the report of Bushaw [10] contains an excellent summary of the results obtained up to 1957, and the book by Sansone and Conti [29] contains a list of results obtained up to 1960. Reissig, Sansone, and Conti [29] updated the previous volume to 1962, and the papers by Burton and Townsend [9], Graef [13], Müller [25], and Wong [38] continued the efforts up to about 1970. Supplementary bibliographies and results prior to 1979 can be found in Kartsatos [20] and Knowles [21].

Equation (1) includes as special cases equations of the type

$$(a(t)\,x')' + c(t)\, g(x)\, h(x') = q(t)$$

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which have been the centre of a considerable amount of research, and there are a number of papers about these equations; see, in particular, Bihari [3], Borodin and Mamii [4], Burton and Grimmer [8], Chang [11], Graef and Spikes [14], Klokov [21], Lalli [23], Olehnik [26, 27], Willett and Wong [35], Wong [36, 37], Wong and Burton [39], and the references therein. Equations (1) and (2) include also as special cases equations of the type

$$x'' + f_1(x) f_2(x')\, x' + g(x)\, h(x') = q(t)$$

which have been studied by Antosiewicz [1], Burton [5, 6, 7], Heidel [16, 17], Willett and Wong [34], and others. The greatest part of these papers is concerned with the boundedness of solutions of the equations being considered and the majority of the obtained results require intermediate or direct use of energy (Liapunov) functions.

In the present paper, sufficient conditions are given for the boundedness of all solutions and their derivatives of Eqs. (1) and (2). In contrast to the results in the above cited papers, we do not have to find Liapunov functions. We will present a quite different approach to the problem of boundedness. The heart of this approach is the method of "integral inequalities" [2]. A standard technique used in this method is the integration by parts. We replace this technique by using two forms of the mean value theorem for integrals. Such a device seems to be new. Some of our results are concerned with the relationship between the boundedness behavior of the solutions of (1) and (2) and the monotonic behavior of the quotient $a(t)/c(t)$ or $c(t)/a(t)$. We will use Stieltjes integral inequalities in the case that $a(t)/c(t)$ or $c(t)/a(t)$ is continuous, positive and locally of bounded variation. To our knowledge this approach to the study of boundedness behavior has not been tried before. Moreover, we will be able to replace many of the monotonicity conditions placed on $a(t)$ and $c(t)$ by integral conditions involving their derivatives. Finally, in the process of our discussion, we not only achieve a certain degree of generalizations, but also discover improved versions of earlier results even in the simple cases of the Eqs. (1) and (2).

Notation and Preliminaries

The following notation is used. We denote by R the real line, by R^+ and I the intervals $(0, \infty)$ and $[0, \infty)$, respectively, and by $|\cdot|$ an absolute value. $C(A, R)$ and $C^1(A, R)$ denote the sets of R-valued functions defined on the set A that are, respectively, continuous and continuously differentiable with respect to each variable. $L_1(A)$ denotes the set of Lebesgue integrable functions on A. The solution $x(t)$ (of the equation being causidered) through the initial point (t_0, x_0) is bounded, by definition, if there exists a positive number P such that $|x(t)| < P$ for all $t \geqslant t_0$. This P may be determined for each solution. When all solutions of (1) are bounded, we say that the solutions of (1) are bounded.

It is assumed throughout that all solutions of (1) and (2) are continuously extendable throughout the entire nonnegative real axis I. In this regard see Hastings [15], Coffman and Urlich [12], and Willett and Wong [35]. Without further mention, we note that the results in this paper pertain only to continuable solutions of Eqs. (1) and (2).

The next lemmas will be useful in the proofs of the main results.

LEMMA 1. *If u and v are real valued functions, defined and nonnegative for $t \geqslant t_0$, $u, v \in L_1[t_0, \infty)$ and if*

$$u(t) \leqslant c + \int_{t_0}^{t} u(s)\, v(s)\, ds,$$

for some positive constant c, then

$$u(t) \leqslant c \exp\left(\int_{t_0}^{t} v(s)\, ds\right).$$

This very useful lemma is due to Bellman [2] (also known as Gronwall's inequality).

We now state the following two forms of the second mean value theorem for integrals. For example, one can refer to Hildebrandt [18].

LEMMA 2. *If $u \in L_1[\alpha, \beta]$ and v is a positive, bounded and nonincreasing function on $[\alpha, \beta]$, then there is a number $\delta \in [\alpha, \beta]$ such that*

$$\int_{\alpha}^{\beta} u(s)\, v(s)\, ds = v(\alpha + 0) \int_{\alpha}^{\delta} u(s)\, ds.$$

LEMMA 3. *If $u \in L_1[\alpha, \beta]$ and v is a positive, bounded and nondecreasing function on $[\alpha, \beta]$, then there is a number $\delta \in [\alpha, \beta]$ such that*

$$\int_{\alpha}^{\beta} u(s)\, v(s)\, ds = v(\beta - 0) \int_{\delta}^{\beta} u(s)\, ds.$$

The following generalization of Lemma 1 for Riemann–Stieltjes integrals is a modification of a result given by Jones [19].

LEMMA 4. *Let u, v, w be real valued functions, defined and continuous on $[\alpha, \beta]$. Let u, v be nonnegative and let w be nondecreasing on $[\alpha, \beta]$. If*

$$u(t) \leqslant c + \int_{\alpha}^{t} u(s)\, v(s)\, dw(s)$$

for some positive constant c, then

$$u(t) \leqslant c \exp\left(\int_\alpha^t v(s)\, dw(s)\right)$$

for all $t \in [\alpha, \beta]$.

Finally, we state the following lemma.

LEMMA 5. *If, for $t \in [\alpha, \beta]$, $u(t)$ is real valued, continuous, of bounded variation and if $u(t) > 0$, then*

$$\int_\alpha^\beta (1/u(s))\, du(s) = \log u(\beta) - \log u(\alpha).$$

The proof is a matter of straightforward application of elementary properties of Riemann–Stieltjes integrals, and thus omitted. For an account of the theory of Riemann–Stieltjes integrals the reader is referred to [18].

MAIN RESULTS

We assume that the functions defining the differential equations (1) and (2) satisfy the following conditions:

(c_1) $a, c \in C(I, R^+)$, $b \in C(I, I)$;

(c_2) $f \in C(R^2, R)$, $g \in C(R, R)$, $h \in C(R, R^+)$, $p \in C(I \times R^2, R)$;

(c_3) $f(x, y)\, y > 0$ for all $(x, y) \in R^2$, $y \neq 0$;

(c_4) $G(x) \to \infty$ as $|x| \to \infty$, where $G(x) = \int_0^x g(\tau)\, d\tau \geqslant 0$;

(c_5) $H(y) \to \infty$ as $|y| \to \infty$, where $H(y) = \int_0^y (\tau/h(t))\, d\tau$;

(c_6) there is a nonnegative function $e(t) \in L_1(I)$ such that

$$|p(t, x, y)\, y| \leqslant e(t)\, h(y) \qquad \text{for all } (t, x, y) \in I \times R^2;$$

(c_7) there are positive constants M and k such that

$$y^2/h(y) \leqslant MH(y) \qquad \text{for all } |y| \geqslant k$$

and a nonnegative function $e_1(t) \in L_1(I)$ such that

$$|p(t, x, y)| \leqslant e_1(t) \qquad \text{for all } (t, x, y) \in I \times R^2.$$

Before giving the main results of this paper, we make some observations and remarks concerning the above conditions. Conditions (c_1) and (c_2) are sufficient to guarantee the local existence of solutions of (1) and (2). Conditions (c_3) is standard in the case when $a(t) = b(t) = c(t) \equiv 1$ and $h(x') \equiv 1$;

see Antosiewicz [1], Burton [5], and Willett and Wong [34]. In (c_4) we do not require that $xg(x)>0$ if $x \neq 0$, as do most authors (see, e.g., Burton and Grimmer [8] or Willet and Wong [34]), but only ask that $G(x) \geqslant 0$. Since $G(x) \to \infty$ as $|x| \to \infty$, we have $G(x) \geqslant 0$; generally speaking this is the case only for sufficiently large $|x|$. Conditions (c_4) and (c_5) are used in a number of papers to establish boundedness and continuability theorems; see, Burton and Grimmer [8], Graef and Spikes [14], Lalli [23], and Wong and Burton [39]. Condition (c_6) is a generalization of a condition of Legatos [24] and it has been used by Wong [36]. The first part of (c_7) is less restrictive than bounding h from above and below or asking $y^2/h(y) \leqslant MH(y)$ for all y (see Burton and Grimmer [8], Olehnik [27], and Opial [28]), and it does not violate the condition (c_5). The second part of (c_7) generalizes a condition of Tejumola [31]. Since $h(x') \equiv 1$ satisfies (c_7) it is clear that any theorem proved for (1) and (2) using (c_7) will hold for the equations obtained from (1) and (2) by setting $h(x') \equiv 1$.

First, we state and prove theorems on the boundedness of solutions for the equation

$$x'' + b(t)f(x, x') + c(t)g(x)\,h(x') = p(t, x, x') \tag{3}$$

which is a special case of (2) when $a(t) \equiv 1$. The obtained results are then specialized to (2) as corollaries.

THEOREM 1. *Suppose that conditions* (c_1)–(c_6) *hold and* $c(t)$ *is nondecreasing on* I. *Then any solution* $x(t)$ *of* (3) *is bounded. If, in addition,* $c(t)$ *is bounded above on* I, *then* $x'(t)$ *is also bounded.*

Proof. Let $x(t)$ be a solution of (3) defined on $[0, t]$. Multiplying (3) by $x'(t)/c(t)\,h(x'(t))$ and integrating both sides of the resulting equation from 0 to t, we obtain

$$\begin{aligned}
&\int_0^t [x'(\tau)\,x''(\tau)/c(\tau)\,h(x'(\tau))]\,d\tau \\
&\quad + \int_0^t [b(\tau)\,f(x(\tau), x'(\tau))\,x'(\tau)/c(\tau)\,h(x'(\tau))]\,d\tau \\
&\quad + \int_0^t g(x(\tau))\,x'(\tau)\,d\tau \leqslant \int_0^t [|p(t, x(\tau), x'(\tau))\,x'(\tau)|/c(\tau)\,h(x'(\tau))]\,d\tau.
\end{aligned}$$

The integral in the second term on the left is nonnegative because of (c_1)–(c_3) and, hence, using (c_6) we get

$$\begin{aligned}
&\int_0^t [x'(\tau)\,x''(\tau)/c(\tau)\,h(x'(\tau))]\,d\tau + G(x(t)) - G(x(0)) \\
&\quad \leqslant (1/c(0)) \int_0^t e(\tau)\,d\tau.
\end{aligned}$$

From Lemma 2 it follows that there is $\delta \in [0, t]$ such that

$$(1/c(0)) \int_0^{\delta} [x'(\tau)\, x''(\tau)/h(x'(\tau))]\, d\tau + G(x(t)) - G(x(0))$$

$$\leqslant (1/c(0)) \int_0^t e(\tau)\, d\tau$$

which because of (c_1) and (c_5) leads to the estimate

$$G(x(t)) \leqslant G(x(t)) + (1/c(0))\, H(x'(\delta))$$

$$\leqslant G(x(0)) + (1/c(0)) \left[H(x'(0)) + \int_0^{\infty} e(\tau)\, d\tau \right].$$

The right side of the last inequality is a constant indepenent of t, say K, and therefore (c_4) implies that $x(t)$ is bounded on I.

We now suppose that $c(t) \leqslant c_0$ on I. Substitute $x(t)$ into (3), multiply on both sides by $x'(t)/h(x'(t))$ and integrate from 0 to t. By (c_1)–(c_3) and (c_6) the result may be written

$$\int_0^t [x'(\tau)\, x''(\tau)/h(x'(\tau))]\, d\tau + \int_0^t c(\tau)\, g(x(\tau))\, x'(\tau)\, d\tau \leqslant \int_0^t e(\tau)\, d\tau.$$

Using Lemma 3, there exists $\delta \in [0, t]$ such that

$$H(x'(t)) - H(x'(0)) + c(t) \int_{\delta}^t g(x(\tau))\, x'(\tau)\, d\tau \leqslant \int_0^t e(\tau)\, d\tau$$

which, since $c(t)\, H(x(t))$ is nonnegative on I, yields

$$H(x'(t)) \leqslant H(x'(t)) + C(t)\, G(x(t))$$

$$\leqslant H(x'(0)) + c(t)\, G(x(\delta)) + \int_0^{\infty} e(\tau)\, d\tau$$

$$\leqslant H(x'(0)) + c_0 K + \int_0^{\infty} e(\tau)\, d\tau = L,$$

a constant independent of t. Hence (c_5) implies that $x'(t)$ is bounded on I.

Remark. In the case when $b(t) \equiv 0$, $h(x') \equiv 1$ and $p(t, x, x') \equiv 0$, Theorem 1 generalizes Theorem 1 of Klokov [21] and Theorem 1 of Waltman [33]. If $b(t) = c(t) \equiv 1$ and $h(x') \equiv 1$ it reduces to Theorem 1 of Antosiewicz [1] and includes Theorems 1 and 2 of Utz [32]. When $b(t) \equiv 0$ and $p(t, x, x') \equiv 0$ it generalizes Theorem 8 of Bihari [3] and, when only for $b(t) \equiv 0$, Theorem 6 of Wong [37].

As a consequence of Theorem 1 we have the following result.

COROLLARY 2. *Suppose that conditions* (c_1)–(c_6) *hold an* $a(t)$ *is bounded away from zero on* I. *If the quotient* $c(t)/a(t)$ *is nondecreasing on* I, *then any solution* $x(t)$ *of* (2) *is bounded. If, in addition,* $c(t)/a(t)$ *is bounded above on* I, *then* $x'(t)$ *is also bounded.*

Proof. Multiplying both sides of (2) by $1/a(t)$, we see that the conclusion follows from Theorem 1.

Remark. If $p(t, x, x') \equiv 0$, then the requirement that $a(t)$ is bounded away from zero on I can be disregarded. In this case, Corollary 2 generalizes Theorem 3 of Lalli [23].

THEOREM 3. *Suppose that conditions* (c_1)–(c_5) *and* (c_7) *hold and let* $c(t)$ *be nonincreasing on* I. *Then, for any solution* $x(t)$ *of* (3), $x'(t)$ *is bounded. If, in addition,* $c(t)$ *is bounded away from zero on* I, *then* $x(t)$ *is also bounded.*

Proof. Multiplying (3) by $x'(t)/h(x'(t))$, integrating both sides of the obtaining equation between 0 and t, and using (c_1)–(c_3), we obtain

$$\int_0^t [x'(\tau)\, x''(\tau)/h(x'(\tau))]\, d\tau + \int_0^t c(\tau)\, g(x(\tau))\, x'(\tau)\, d\tau$$

$$\leqslant \int_0^t [|p(\tau, x(\tau), x'(\tau))\, x'(\tau)|/h(x'(\tau))]\, d\tau.$$

If $|y| \leqslant \max\{a, k\}$, $y^2/h(y) \leqslant d_1$ for some $d_1 > 0$, so $y^2/h(y) \leqslant d_1 + MH(y)$ for all y. Also, for $|y| \leqslant \max\{1, k\}$, $|y|/h(y) \leqslant d_2$, $d_2 > 0$, and for $|y| \geqslant \max\{1, k\}$, $|y|/h(y) \leqslant y^2/h(y)$, so $|y|/h(y) \leqslant d_2 \pm y^2/h(y) \leqslant d_1 + d_2 + MH(y) = D + MH(y)$ for all y. Thus, using (c_7) and Lemma 2, we see that there is $\delta \in [0, t]$ such that

$$H(x'(t)) + c(0) \int_0^\delta g(x(\tau))\, x'(\tau)\, d\tau$$

$$\leqslant H(x'(0)) + M \int_0^t e_1(\tau)\, H(x'(\tau))\, d\tau + D \int_0^t e_1(\tau)\, d\tau$$

which, since $c(0)\, G(x(\delta))$ is nonnegative, yields

$$H(x'(t)) \leqslant K_1 + M \int_0^t e_1(\tau)\, H(x'(\tau))\, d\tau,$$

where

$$K_1 = H(x'(0)) + c(0)\, G(x(0)) + D \int_0^\infty e_1(\tau)\, d\tau$$

is a nonnegative constant. By Lemma 1 it follows that

$$H(x'(t)) \leqslant K_1 \exp\left(M \int_0^{\infty} e_1(\tau)\, d\tau\right) = L_1,$$

a constant independent of t. Thus condition (c_5) implies that $x'(t)$ is bounded on I.

We now suppose that $c(t) \geqslant c_0 > 0$ on I. Multiplying (3) by $x'(t)/c(t)\, h(x'(t))$, integrating between 0 and t, and using (c_1)–(c_3) and (c_7) we get

$$\int_0^t [x'(\tau)\, x''(\tau)/c(\tau)\, h(x'(\tau))]\, d\tau + \int_0^t g(x(\tau))\, x'(\tau)\, d\tau$$

$$\leqslant (M/c_0) \int_0^t e_1(\tau)\, H(x'(\tau))\, d\tau + (D/c_0) \int_0^t e_1(\tau)\, d\tau.$$

By Lemma 3 we see that there is $\delta \in [0, t]$ such that

$$(1/c(t))[H(x'(t)) - H(x'(\delta))] + G(x(t)) - G(x(0))$$

$$\leqslant (ML_1 + D)(1/c_0) \int_0^t e_1(\tau)\, d\tau$$

which, since $(1/c(t))\, H(x'(t))$ is nonnegative on I, yields

$$G(x(t)) \leqslant G(x(0)) + L_1/c_0 + (ML_1 + D)(1/c_0) \int_0^{\infty} e_1(\tau)\, d\tau = L_2,$$

a constant independent of t. Thus (c_4) implies that $x(t)$ is bounded on I and the proof is complete.

Remark. Theorem 3 generalizes Theorem 1 of Wong and Burton [39] and those of Antosiewicz and Utz mentioned before.

COROLLARY 4. *Suppose that conditions* (c_1)–(c_5) and (c_7) *hold and let* $a(t)$ *be bounded away from zero on* I. *If the quotient* $c(t)/a(t)$ *is nonincreasing on* I, *then for any solution* $x(t)$ *of* (2), $x'(t)$ *is bounded. If in addition,* $c(t)/a(t)$ *is bounded below on* I, *then* $x(t)$ *is bounded.*

Proof. Multiplying both sides of (2) by $1/a(t)$, we see that the conclusion follows from Theorem 3.

Remark. If $p(t, x, x')$ is identically zero, then the condition that $a(t)$ is bounded away from zero can be dropped. In this case, Corollary 4 generalizes Theorem 5 of Lalli [23].

We begin to discuss the boundedness of solutions of Eq. (1). The proof of the following theorem resembles that of Theorem 3.

THEOREM 5. *Suppose that conditions* (c_1)–(c_5) *and* (c_7) *hold and let* $c(t)$ *be nonincreasing on* I. *If* $a(t) \in C^1(I, R^+)$, $a'(t) \leqslant 0$, *and if* $a(t)$ *is bounded away from zero on* I, *then for any solutions* $x(t)$ *of* (1), $x'(t)$ *is bounded. If, in addition,* $c(t)$ *is bounded away from zero on* I, *then* $x(t)$ *is bounded.*

Proof. Multiplying (1) by $x'(t)/h(x'(t))$, integrating from 0 to t, and using (c_1)–(c_3) we get

$$\int_0^t [a'(\tau)(x'(\tau))^2/h(x'(\tau))]\,d\tau + \int_0^t [a(\tau)\,x'(\tau)\,x''(\tau)/h(x'(\tau))]\,d\tau$$
$$+\int_0^t c(\tau)\,g(x(\tau))\,x'(\tau)\,d\tau$$
$$\leqslant \int_0^t [|p(\tau, x(\tau), x'(\tau))\,x'(\tau)|/h(x'(\tau))]\,d\tau.$$

As in the proof of Theorem 3 there are $d_1 > 0$ and $d_2 > 0$ such that $y^2/h(y) \leqslant d_1 + MH(y)$ and $|y|/h(y) \leqslant d_1 + d_2 + MH(y)$ for all y. Thus using the supposition that $a'(t) \leqslant 0$ and (c_7) we obtain from the above inequality

$$d_1 \int_0^t a'(\tau)\,d\tau + M \int_0^t a'(\tau)\,H(x'(\tau))\,d\tau + \int_0^t a(\tau)(H(x'(\tau)))'d\tau$$
$$+\int_0^t c(\tau)\,g(x(\tau))\,x'(\tau)\,d\tau \leqslant D \int_0^t e_1(\tau)\,d\tau + M \int_0^t e_1(\tau)\,H(x'(\tau))\,d\tau,$$

where $D = d_1 + d_2$. Integrating the third integral and applying Lemma 2 to the fourth integral we obtain, after some computations, that

$$a(t)\,H(x'(t)) \leqslant N + (1-M) \int_0^t a'(\tau)\,H(x'(\tau))\,d\tau$$
$$+ M \int_0^t e_1(\tau)\,H(x'(\tau))\,d\tau,$$

where

$$N = d_1 a(0) + a(0)\,H(x'(0)) + c(0)\,G(x(0)) + D \int_0^\infty e_1(\tau)\,d\tau.$$

If $M \leqslant 1$ then Lemma 1 gives

$$H(x'(t)) \leqslant (N/a_0) \exp((M/a_0) \int_0^\infty e_1(\tau)\,d\tau) = R_1$$

and if $M > 1$,

$$H(x'(t)) \leqslant (N/a_0) \exp((M-1)\, a(0)/a_0) \exp\left((M/a_0) \int_0^\infty e_1(\tau)\, d\tau\right) = R_2,$$

where a_0 is the lower bound (positive) of $a(t)$ on I. Thus (c_5) implies that $x'(t)$ is bounded on I.

By hypothesis there is $c_0 > 0$ such that $c(t) \geqslant c_0$ on I. Multiplying (1) by $x'(t)/c(t)\, h(x'(t))$, integrating from 0 to t, using the conditions (c_1)–(c_3) and (c_7) and applying Lemma 3, one can show that

$$\begin{aligned} G(x(t)) \leqslant{} & G(x(0)) - a(t)\, H(x'(t))/c(t) + a(\delta_3)\, H(x'(\delta_3))/c(t) \\ & - (d_1/c(t)) \int_{\delta_1}^t a'(\tau)\, d\tau - (M/c(t)) \int_{\delta_2}^t a'(\tau)\, H(x'(\tau))\, d\tau \\ & + (1/c(t)) \int_{\delta_3}^t a'(\tau)\, H(x'(\tau))\, d + (D/c_0) \int_0^t e_1(\tau)\, d\tau \\ & + (M/c_0) \int_0^t e_1(\tau)\, H(x'(\tau))\, d\tau, \end{aligned}$$

where $\delta_1, \delta_2, \delta_3 \in [0, t]$. Put $R = \max(R_1, R_2)$ and then, since $H(x'(t)) \leqslant R$ on I, we obtain

$$\begin{aligned} G(x(t)) \leqslant{} & G(x(0)) + (Ra(0)/c_0)(1 + M + d_1/R) \\ & + ((D + MR)/c_0) \int_0^\infty e_1(\tau)\, d\tau. \end{aligned}$$

Therefore $G(x(t))$ is bounded on I and (c_4) implies that $x(t)$ is bounded. This completes the proof.

Remark. We observe that in the above theorem as in Theorem 3, the condition that $c(t)$ is bounded away from zero is essential in order to prove the boundedness of $x(t)$.

In the following two theorems, by appealing to Riemann–Stieltjes integrals and examining the quotients $a(t)/c(t)$ and $c(t)/a(t)$, we are able to obtain boundedness results for the solutions of (1).

THEOREM 6. *Suppose that conditions* (c_1)–(c_6) *hold. If* $a(t) \in C^1(I, R^+)$, $a'(t) \geqslant 0$, *and the quotient* $a(t)/c(t)$ *is nondecreasing and bounded above on* I, *then any solution* $x(t)$ *of* (1), *along with its derivative* $x'(t)$, *is bounded for all* $t \in I$.

Proof. Multiplying (1) by $x'(t)/c(t)\, h(x'(t))$, integrating both sides of the obtained equation from zero to some $t \geqslant 0$, using (c_1)–(c_3), (c_6) and the fact that $a'(t) \geqslant 0$ for all $t \in I$, we obtain

$$\int_0^t [(a(\tau)/c(\tau))(H(x'(\tau))'] \, d\tau + \int_0^t g(x(\tau)) \, x' \, d\tau$$
$$\leqslant \int_0^t [e(\tau)/c(\tau)] \, d\tau.$$

By the assumed monotonicity of the quotient $a(t)/c(t)$ we conclude that $a(\tau)/c(\tau)$ is of bounded variation on $[0, t]$. From this it follows that $H(x'(\tau))$ is Riemann–Stieltjes integrable with respect to $a(\tau)/c(\tau)$ on $[0, t]$. Using the integration by parts formula for Riemann–Stieltjes integrals we find that $a(\tau)/c(\tau)$ is Riemann–Stieltjes integrable with respect to $H(x'(\tau))$ in $[0, t]$. Thus, by the theorem of reduction of a Riemann–Stieltjes integral to a Riemann integral, we obtain that

$$\int_0^t [(a(\tau)/c(\tau))(H(x'(\tau)))'] \, d\tau = \int_0^t (a(\tau)/c(\tau)) \, dH(x'(\tau)),$$

where the second integral is a Riemann–Stieltjes integral. Substituting this in the inequality above, we get

$$\int_0^t (a(\tau)/c(\tau)) \, dH(x'(\tau)) + G(x(t)) - G(x(0)) \leqslant \int_0^t [e(\tau)/c(\tau)] \, d\tau.$$

By assumption, there exists $r_0 > 0$ such that $a(t)/c(t) \leqslant r_0$ for all $t \in I$. Then $1/c(t) \leqslant r$ on I, where $r = r_0/a(0)$. We now use the integration by parts formula for Riemann–Stieltjes integrals and obtain that

$$\int_0^t (a(\tau)/c(\tau)) \, dH(x'(\tau)) = (a(t)/c(t)) \, H(x'(t))$$
$$- (a(0)/c(0)) \, H(x'(0)).$$

Then the last inequality above becomes

$$(a(t)/c(t)) \, H(x'(t)) + G(x(t)) \leqslant N + \int_0^t H(x'(\tau)) \, d(a(\tau)/c(\tau)),$$

where

$$N = G(x(0)) + (a(0)/c(0)) \, H(x'(0)) + \int_0^\infty e(\tau) \, d\tau.$$

Since $G(x(t)) \geqslant 0$ on I, we have

$$(a(t)/c(t))\, H(x'(t)) \leqslant N + \int_0^t ((a(\tau)/c(\tau))/a(\tau)\, c(\tau))\, H(x'(\tau))\, d(a(\tau)/c(\tau)).$$

By Lemma 5, we obtain

$$(a(t)/c(t))\, H(x'(t)) \leqslant N \exp\left(\int_0^t (1/(a(\tau)/c(\tau))\, d(a(\tau)/c(\tau)))\right)$$

which, using Lemma 4, becomes

$$(a(t)/c(t))\, H(x'(t)) \leqslant Nc(0)\, a(t)/c(t)\, a(0).$$

Hence $H(x'(t)) \leqslant Nc(0)/a(0)$ and (c_5) implies that $x'(t)$ is bounded on I.

Since $H(x'(t)) \geqslant 0$ and $a(t)/c(t)$ is bounded above on I, we have from the above that

$$\begin{aligned} G(x(t)) &\leqslant N + \int_0^t H(x'(\tau))\, d(a(\tau)/c(\tau)) \\ &\leqslant N + N(c(0)/a(0)) \int_0^t d(a(\tau)/c(\tau)) \\ &\leqslant Nrc(0)/a(0). \end{aligned}$$

Therefore (c_4) implies that $x(t)$ is bounded on I and this completes the proof.

THEOREM 7. *Suppose that conditions* (c_1)–(c_6) *hold. If* $a(t) \in C^1(I, R^+)$, $a'(t) \geqslant 0$ *and the quotient* $c(t)/a(t)$ *is nondecreasing and bounded above on* I, *then any solution* $x(t)$ *of* (1), *along with its derivative* $x'(t)$, *is bounded for all* $t \in I$.

Proof. Multiplying (1) by $x'(t)/a(t)\, h(x'(t))$ and arguing as in the proof of Theorem 6, we obtain

$$\begin{aligned} H(x'(t)) - H(x'(0)) &+ \int_0^t [(c(\tau)/a(\tau))\, g(x(\tau))\, x'(\tau)]\, d\tau \\ &\leqslant \int_0^t [e(\tau)/a(\tau)]\, d\tau. \end{aligned}$$

By following an argument similar to that used in the proof of Theorem 6 we see that $c(\tau)/a(\tau)$ is Riemann–Stieltjes integrable with respect to $G(x(\tau))$ on $[0, t]$. Then

$$\int_0^t [(c(\tau)/a(\tau))\, g(x(\tau))\, x'(\tau)]\, d\tau = \int_0^t [c(\tau)/a(\tau)]\, dG(x(\tau)),$$

where the second integral is a Riemann–Stieltjes integral. With this and by the integration by parts formula for Riemann–Stieltjes integrals the above inequality becomes

$$H(x'(t)) + (c(t)/a(t))\, G(x(t)) \leqslant N + \int_0^t G(x(\tau))\, d(c(\tau)/a(\tau)),$$

where

$$N = H(x'(0)) + (c(0)/a(0))\, G(x(0)) + (1/a(0)) \int_0^\infty e(\tau)\, d\tau.$$

It then follows that

$$\begin{aligned}(c(t)/a(t))\, G(x(t)) &\leqslant N \\ &\quad + \int_0^t (a(\tau)/c(\tau))(c(\tau)/a(\tau))\, G(x(\tau))\, d(c(\tau)/a(\tau)).\end{aligned}$$

From Lemma 4 we see that

$$(c(t)/a(t))\, G(x(t)) \leqslant N \exp\left(\int_0^t (a(\tau)/c(\tau))\, d(c(\tau)/a(\tau))\right)$$

and Lemma 5 gives

$$(c(t)/a(t))\, G(x(t)) \leqslant Na(0)\, c(t)/c(0)\, a(t)$$

which implies that $G(x(t)) \leqslant Na(0)/c(0)$ on I. The boundedness of $x(t)$ follows by (c_4).

By hypothesis $c(t)/a(t) \leqslant \rho$ for some $\rho > 0$ and for all $t \in I$. Then

$$\begin{aligned} H(x'(t)) &\leqslant N + \int_0^t G(x(\tau))\, d(c(\tau)/a(\tau)) \\ &\leqslant N + N(a(0)/c(0)) \int_0^t d(c(\tau)/a(\tau)) \leqslant N\rho a(0)/c(0)\end{aligned}$$

and (c_5) implies that $x'(t)$ is bounded on I. The proof is now completed.

Remark. A number of papers have dealt with boundedness of solutions of (1) for the case $b(t) \equiv 0$. Theorems 6 and 7 generalize the corresponding results in [11, 23, 40].

Throughout the remainder of this paper we replace the monoticity conditions on $a(t)$ and $c(t)$ by integral conditions on the derivatives $a'(t)$, $c'(t)$ and $(c(t)/a(t))'$.

THEOREM 8. *Let conditions* (c_1)–(c_6) *hold and let* $c(t) \in C^1(I, R^+)$ *be bounded away from zero and* $|c'(t)| \in L_1(I)$. *Then both the solution* $x(t)$ *of* (3) *and its derivative* $x'(t)$ *are bounded.*

Proof. Multiplying (3) by $x'(t)/h(x'(t))$, integrating from 0 to t, and using (c_1)–(c_3) and (c_6), we obtain the inequality

$$H(x'(t)) - H(x'(0)) + \int_0^t c(\tau)\, g(x(\tau))\, x'(\tau)\, d\tau \leqslant \int_0^t e(\tau)\, d\tau,$$

which, after integration by parts, becomes

$$c(t)\, G(x(t)) \leqslant N + \int_0^t |c'(\tau)|\, G(x(\tau))\, d\tau,$$

where

$$N = H(x'(0)) + c(0)\, G(x(0)) + \int_0^\infty e(\tau)\, d\tau.$$

By hypothesis $c(t) \geqslant c_0$ for some $c_0 > 0$ and all $t \in I$ and Lemma 1 then gives

$$G(x(t)) \leqslant (N/c_0) \exp\left(\int_0^\infty |c'(\tau)\, d\tau/c_0\right) = M.$$

Again, the condition (c_4) implies that $x(t)$ is bounded.

Furthermore, it is easy to show that

$$H(x'(t)) \leqslant N + M \int_0^\infty |c'(\tau)|\, d\tau$$

and (c_5) implies that $x'(t)$ is bounded.

From Theorem 8 we get the following result.

COROLLARY 9. *Suppose that conditions* (c_1)–(c_6) *hold. If* $a(t)$ *and* $c(t)/a(t) \in C^1(I, R^+)$ *are bounded away from zero and* $|(c(t)/a(t))'| \in L_1(I)$, *then any solution* $x(t)$ *of* (2), *along with its derivate* $x'(t)$, *is bounded.*

Remark. Theorem 8 and its corollary generalize Theorem 7 of Bihari [3] and Theorem 4 of Wong [36].

THEOREM 10. *Let conditions* (c_1)–(c_6) *hold and let* $a(t)$ *be nonincreasing on* I. *If* $c(t) \in C^1(I, R^+)$ *is bounded away from zero and* $|c'(t)| \in L_1(I)$, *then any solution* $x(t)$ *of* (2) *is bounded. If, in addition,* $a(t)$ *is bounded away from zero, then* $x'(t)$ *is also bounded.*

Proof. Multiplying (2) by $x'(t)/h(x'(t))$, integrating between 0 and t, and using (c_1)–(c_3) and (c_6), we obtain

$$\int_0^t [a(\tau)\, x'(\tau)\, x''(\tau)/h(x'(\tau))\, d + \int_0^t c(\tau)\, g(x(\tau))\, x'(\tau)\, d\tau$$

$$\leqslant \int_0^t e(\tau)\, d\tau.$$

Applying Lemma 2 on the first integral and integrating the second integral by parts, we obtain

$$a(0)\, H(x'(\delta)) - a(0)\, H(x'(0) + c(t)\, G(x(t)) - c(0)\, G(x(0))$$

$$\leqslant \int_0^t |c'(\tau)|\, G(x(\tau))\, d\tau + \int_0^t e(\tau)\, d\tau,$$

where $\delta \in [0, t]$. Since $c(t) \geqslant c_0$ for some $c_0 > 0$ and all $t \in I$, we have

$$G(x(t)) \leqslant N(c_0 + (1/c_0) \int_0^t |c'(\tau)|\, G(x(\tau))\, d\tau,$$

where

$$N = a(0)\, H(x'(0)) + c(0)\, G(x(0)) + \int_0^\infty e(\tau)\, d\tau.$$

From Lemma 1 and (c_4) it follows that $x(t)$ is bounded on I.

Suppose now that $a(t) \geqslant a_0$ for some $a_0 > 0$ and all $t \in I$. Multiplying (2) by $x'(t)/a(t)\, h(x'(t))$ and proceeding as before, we have

$$\int_0^t [x'(\tau)\, x''(\tau)/h(x'(\tau))]\, d\tau + \int_0^t [(c(\tau)/a(\tau))\, g(x(\tau))\, x'(\tau)]\, d\tau$$

$$\leqslant \int_0^t e(\tau)\, d\tau.$$

Integrating the first integral by parts and applying Lemma 3 on the second integral, we obtain

$$H(x'(t)) - H(x'(0)) + (1/a(t)) \int_\delta^t c(\tau)\, g(x(\tau))\, x'(\tau)\, d\tau$$

$$\leqslant (1/a_0) \int_0^t e(\tau)\, d\tau,$$

where $\delta \in [0, t]$. Integrating the integral on the left by parts, we get

$$H(x'(t)) - H(x'(0)) + (c(t)/a(t))\, G(x(t)) - (c(\delta)/a(t))\, G(x(\delta)) \leqslant (1/a(t)) \int_\delta^t |c'(\tau)|\, G(x(\tau))\, d\tau + (1/a_0) \int_0^t e(\tau)\, d\tau.$$

From the conditions on $c(t)$ we see that $c(t)$ tends to a positive limit as $t \to \infty$ and then, $c(t)$ is bounded above, say by c_1. Thus, from the above inequality, we get the following estimate

$$H(x'(t)) \leqslant H(x'(0)) + c_1 L/a_0 + (L/a_0) \int_0^\infty |c'(\tau)|\, d\tau + (1/a_0) \int_0^\infty e(\tau)\, d\tau,$$

where

$$L = (N/c_0) \exp\left(\int_0^\infty |c'(\tau)| d\tau / c_0 \right).$$

Therefore (c_5) implies the boundedness of $x'(t)$ and this completes the proof.

Remark. Changing the roles of $a(t)$ and $c(t)$ in Theorem 10 we may obtain a similar result. Since the procedure is clear the statement and proof will be omitted.

THEOREM 11. *Let conditions* (c_1)–(c_5) *and* (c_7) *hold, and let* $a(t)$ *and* $c(t)$ *be bounded away from zero. If,* $a(t), c(t) \in C^1(I, R^+)$ *and* $|a'(t)|$, $|c'(t)| \in L_1(I)$, *then any solution* $x(t)$ *of* (2), *along with its derivative* $x'(t)$, *is bounded on* I.

Proof. Multiplying (2) by $x'(t)/h(x'(t))$, integrating from 0 to t and using (c_1)–(c_3), we obtain

$$\int_0^t [a(\tau)\, x'(\tau)\, x''(\tau)/h(x'(\tau))]\, d\tau + \int_0^t c(\tau)\, g(x(\tau))\, x'(\tau)\, d\tau \leqslant \int_0^t [\,|p(\tau, x(\tau), x'(\tau))| |x'(\tau)/h(x'(\tau))]\, d\tau.$$

Arguing as in the proof of Theorem 3, we see that there is $D > 0$ such that $|y|/h(y) \leqslant D + MH(y)$ for all y. Thus, we have

$$\int_0^t [a(\tau)(H(x'(\tau))'] \, d\tau + \int_0^t [c(\tau)(G(x(\tau)))']\, d\tau \leqslant D \int_0^t e(\tau)\, d\tau + M \int_0^t e(\tau)\, H(x'(\tau))\, d\tau.$$

Integrating the integrals on the left, we obtain

$$r[H(x'(t))+G(x(t))]$$
$$\leqslant R+\int_0^t [|a'(\tau)|+|c^*(\tau)|+Me(\tau)][H(x'(\tau))+G(x(\tau))]\,d\tau,$$

where $r=\min(a_0, c_0)$, a_0 and c_0 being the lower bounds of $a(t)$ and $c(t)$, respectively, and

$$R=a(0)\,H(x'(0))+c(0)\,G(x(0))+D\int_0^\infty e(\tau)\,d\tau.$$

By Lemma 1 it follows that $H(x'(t))+G(x(t))$ is bounded and then the conditions (c_4) and (c_5) imply that $x(t)$ and $x'(t)$ are bounded on I. The proof is now completed.

Remark. Theorem 11 generalizes Theorem 7 of Bihari [3] and Theorem 4 of Wang [36]. An analogous result has been proved by Burton and Grimmer [8] for (1) where $b(t)\equiv 0$. Their proof is different from the proof given here.

Acknowledgments

Aside from the generosity of my family, this research has not been financially supported by any public or private institution.

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