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Deformed preprojective algebras of generalized Dynkin type $\mathbb{L}_n$: Classification and symmetricity

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ABSTRACT

We give a complete classification of the isomorphism classes of the deformed preprojective algebras of generalized Dynkin type $\mathbb{L}_n$ and show that all these algebras are symmetric. Moreover, we show that the deformed preprojective algebras of type $\mathbb{L}_n$ are isomorphic to the stable Auslander algebras of simple plane curve singularities of Dynkin type $\mathbb{A}_{2n}$.

Introduction and the main results

Throughout this article, K will denote a fixed algebraically closed field. By an algebra we mean an associative, finite-dimensional K-algebra with an identity, which we moreover assume to be basic

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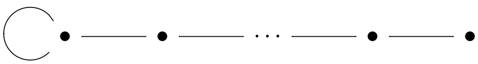
and indecomposable. Any such algebra A can be written as a bound quiver algebra, that is, $A \cong KQ/I$, where $Q = Q_A$ is the Gabriel quiver of A and I is an admissible ideal in the path algebra KQ of Q. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional right A-modules and by Ω_A the *syzygy operator* which assigns to a module M in $\operatorname{mod} A$ the kernel $\Omega_A(M)$ of a minimal projective cover $P_A(M) \to M$ of M in $\operatorname{mod} A$. Then a module M in $\operatorname{mod} A$ is called periodic if $\Omega_A^n(M) \cong M$ for some $n \geqslant 1$. Further, the category $\operatorname{mod} A$ is called *periodic* if any module M in $\operatorname{mod} A$ without non-zero projective direct summands is periodic. It is known that the periodicity of a module category $\operatorname{mod} A$ forces the algebra A to be *selfinjective*, that is, the projective and injective modules in $\operatorname{mod} A$ coincide. Many important selfinjective algebras A are even symmetric, that is there exists an associative, non-degenerate, symmetric K-bilinear form $(-,-) : A \times A \to K$. The category of finite-dimensional A-A-bimodules over an algebra A is equivalent to the category $\operatorname{mod} A^e$ over the enveloping algebra $A^e = A^{\mathrm{op}} \otimes_K A$ of A. An algebra A is called *periodic* if A is a periodic module in $\operatorname{mod} A^e$. It is well known that if A is a periodic algebra then the module category $\operatorname{mod} A$ is periodic and the period of any module M in $\operatorname{mod} A$ without non-zero projective direct summands divides the period of A in $\operatorname{mod} A^e$. The problem whether an algebra A with periodic module category $\operatorname{mod} A$ is a periodic algebra is an exciting open problem. Recently it has been proved that any selfinjective algebra A of finite representation type is a periodic algebra (see [11]). Apart from algebras of finite type, the most prominent periodic algebras are the preprojective algebras of generalized Dynkin type and their deformations.

Preprojective algebras were introduced by Gelfand and Ponomarev [20] (and implicitly in the work of Riedtmann [29]) to study the preprojective representations of finite quivers without oriented cycles, and they occur naturally in very different contexts. The finite-dimensional preprojective algebras are exactly the preprojective algebras $P(\Delta)$ associated to the Dynkin graphs $\mathbb{A}_n$ $(n \geqslant 1)$, $\mathbb{D}_n$ $(n \geqslant 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ and the graphs of the form

$\mathbb{L}_n$  $(n \geqslant 1$ vertices$)$.

Following [23] the graphs $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ and $\mathbb{L}_n$ are called *generalized Dynkin graphs*. These are precisely the graphs associated to the indecomposable finite symmetric Cartan matrices which have subadditive functions which are not additive [24]. We also mention that the preprojective algebras $P(\Delta)$ of Dynkin types $\Delta \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ are the stable Auslander algebras of the categories of maximal Cohen–Macaulay modules of the Kleinian 2-dimensional hypersurface singularities $K[[x,y,z]]/(f_\Delta)$ (see [4,5,13]). Moreover, for each $n \geqslant 1$, the preprojective algebra $P(\mathbb{L}_n)$ is the stable Auslander algebra of the category of maximal Cohen–Macaulay modules over the simple plane curve singularity $K[[x,y]]/(x^2 + y^{2n+1})$ (see [10,13]). The preprojective algebras of Dynkin types have been recently exploited by Geiss, Leclerc and Schröer to study the structure of cluster algebras related to semisimple and unipotent algebraic groups (see [19]). The Hochschild cohomology algebras of preprojective algebras of Dynkin type has been studied by Erdmann and Snashall in [14–16], and recently used by Etingof and Eu [17,18] to establish the calculus structure (Connes differential, Gerstenhaber bracket, ...) of the Hochschild homology/cohomology of preprojective algebras of Dynkin type.

In this paper we study the deformations of preprojective algebras of generalized Dynkin type which were introduced in [7]: Namely, to each generalized Dynkin graph Δ one associates a finite-dimensional (non-commutative) local selfinjective K-algebra $R(\Delta)$. Then a deformed preprojective algebra of type Δ is the deformation $P^f(\Delta)$ of $P(\Delta)$ given by an admissible element f of the radical square of $R(\Delta)$, and $P^f(\Delta) = P(\Delta)$ for $f = 0$ (see [7,13] for details). It has been proved in [7] that the deformed preprojective algebras $P^f(\Delta)$ of generalized Dynkin type are (finite-dimensional) periodic selfinjective algebras. These are precisely the indecomposable selfinjective algebras A, up to Morita equivalence, for which the third syzygy $\Omega_A^3(S)$ of any non-projective simple A-module S is isomorphic to its Nakayama shift $\mathcal{N}_A(S)$.

Therefore every indecomposable selfinjective algebra whose stable module category $\underline{\operatorname{mod}} A$ is 2-Calabi–Yau, is Morita equivalent to some deformed preprojective algebra $P^f(\Delta)$ of generalized Dynkin type Δ, and it is an interesting open problem when the converse is true. Furthermore, by a result of Amiot [1] an additively finite triangulated category $\mathcal{T}$ is 1-Calabi–Yau if and only if $\mathcal{T}$ is equivalent to the category proj $P^f(\Delta)$ of finite-dimensional projective modules over a deformed preprojective algebra $P^f(\Delta)$ of a generalized Dynkin type Δ. We refer to the survey article by Keller [26] for basic background on Calabi–Yau triangulated categories (introduced by Kontsevich in late nineties [28]). We also note that the deformed preprojective algebras of generalized Dynkin type are, with a few small exceptions, of wild representation type (see [12, Theorem 3.7]). Therefore, to classify the deformed preprojective algebras of generalized Dynkin type up to isomorphism, is an important problem.

In this paper we address these problems for the deformed preprojective algebras $P^f(\mathbb{L}_n)$ of the types $\mathbb{L}_n$, $n \geqslant 1$.

For a positive integer n, consider the quiver

$$Q_{\mathbb{L}_n}: \quad {\scriptstyle \varepsilon=\bar{\varepsilon}} \circlearrowright 0 \underset{\bar{a}_0}{\overset{a_0}{\rightleftarrows}} 1 \underset{\bar{a}_1}{\overset{a_1}{\rightleftarrows}} 2 \rightleftarrows \cdots \rightleftarrows n-2 \underset{\bar{a}_{n-2}}{\overset{a_{n-2}}{\rightleftarrows}} n-1$$

and the local K-algebra $R(\mathbb{L}_n) = K[x]/(x^{2n})$. Then, for an element $f \in \operatorname{rad}^2 R(\mathbb{L}_n)$, the *deformed preprojective algebra* $P^f(\mathbb{L}_n)$ is defined to be the bound quiver algebra $KQ_{\mathbb{L}_n}/I^f_{\mathbb{L}_n}$, where $I^f_{\mathbb{L}_n}$ is the ideal of the path algebra $KQ_{\mathbb{L}_n}$ of $Q_{\mathbb{L}_n}$ generated by the elements

$$\varepsilon^2 + a_0\bar{a}_0 + \varepsilon f(\varepsilon), \qquad \varepsilon^{2n}, \qquad \bar{a}_{n-2}a_{n-2}, \quad \text{and} \quad \bar{a}_i a_i + a_{i+1}\bar{a}_{i+1} \quad \text{for } i \in \{0, \ldots, n-3\}.$$

We distinguish also special deformed preprojective algebras of type $\mathbb{L}_n$,

$$L_n^{(r)} = P^{f_r}(\mathbb{L}_n) \quad \text{with } f_r = x^{2r} + \left(x^{2n}\right),\ r \in \{1, \ldots, n\}.$$

Then $L_n^{(n)} = P^{f_n}(\mathbb{L}_n)$ is the ordinary *preprojective algebra* $P(L_n)$ of type $\mathbb{L}_n$.

For convenience of the reader we give in this paper a detailed proof of the following fact (which is a special case of [7, Lemma 3.2]).

Theorem 1. *Let $\Lambda = P^f(\mathbb{L}_n)$ be a deformed preprojective algebra of type $\mathbb{L}_n$ over an algebraically closed field K. Then Λ is a finite-dimensional selfinjective algebra with the same Cartan matrix as the preprojective algebra $P(\mathbb{L}_n)$. In particular, we have* $\dim_K \Lambda = \dim_K P(\mathbb{L}_n)$.

The first main result of this paper is the classification of deformed preprojective algebras of type $\mathbb{L}_n$, up to isomorphism.

Theorem 2. *Let $\Lambda = P^f(\mathbb{L}_n)$ be a deformed preprojective algebra of type $\mathbb{L}_n$ over an algebraically closed field K. Then the following statements hold.*

(1) *If K is of characteristic different from 2, then Λ is isomorphic to the preprojective algebra $P(\mathbb{L}_n)$.*
(2) *If K is of characteristic 2, then Λ is isomorphic to an algebra $L_n^{(r)}$, for some $r \in \{1, \ldots, n\}$.*

It has been proved in [7, Proposition 6.1] that, for K of characteristic 2, the algebras $L_n^{(1)}, L_n^{(2)}, \ldots, L_n^{(n)} = P(\mathbb{L}_n)$ are pairwise non-isomorphic.

The second main result of the paper shows that the classification of the isomorphisms classes of deformed preprojective algebras of type $\mathbb{L}_n$ corresponds nicely (via the stable Auslander algebras) to the classification of equivalence classes of simple plane curve singularities of Dynkin type $\mathbb{A}_{2n}$ (in the sense of [2,6,21]). It has been shown in [27] that, for K of characteristic different from 2, $R = R_n^{(n)} = K[[x, y]]/(x^2 + y^{2n+1})$ is a unique such singularity, up to equivalence. For K of characteristic 2, the simple plane curve singularities

$$R_n^{(r)} = K[[x, y]]/\left(x^2 + y^{2n+1} + xy^{n+r}\right), \quad r \in \{1, \ldots, n-1\},$$

together with $R_n^{(n)}$, give representatives of the equivalence classes of all simple plane curve singularities of type $\mathbb{A}_{2n}$ (see [21, Section 1] and [27]). Moreover, it is known that, for any $r \in \{1, \dots, n\}$, the category $\mathrm{CM}(R_n^{(r)})$ of maximal Cohen–Macaulay modules over $R_n^{(r)}$ is a Frobenius (Krull–Schmidt) category having exactly $n+1$ pairwise non-isomorphic indecomposable objects, among them the unique projective indecomposable object $R_n^{(r)}$ (see [9,10,27]). Consider the direct sum $M_n^{(r)}$ of a complete set of pairwise non-isomorphic indecomposable non-projective objects in $\mathrm{CM}(R_n^{(r)})$ and the associated endomorphism algebra

$$\underline{\mathcal{A}}\big(R_n^{(r)}\big) = \mathrm{End}_{\underline{\mathrm{CM}}(R_n^{(r)})}\big(\underline{M}_n^{(r)}\big)$$

of $\underline{M}_n^{(r)} = M_n^{(r)}$ in the stable category $\underline{\mathrm{CM}}(R_n^{(r)})$ of $\mathrm{CM}(R_n^{(r)})$, called the *stable Auslander algebra* of $R_n^{(r)}$.

Theorem 3. *Let K be of characteristic 2 and n a positive integer. Then, for any $r \in \{1, \dots, n\}$, the algebras $L_n^{(r)}$ and $\underline{\mathcal{A}}(R_n^{(r)})$ are isomorphic.*

We note that an isomorphism $P(\mathbb{L}_n) = L_n^{(n)} \cong \mathcal{A}(R_n^{(n)})$, for K of arbitrary characteristic, follows from [10].

As a consequence of Theorems 2 and 3 we obtain the following fact.

Corollary 4. *Let $\Lambda = P^f(\mathbb{L}_n)$ be a deformed preprojective algebra of type $\mathbb{L}_n$. Then Λ is a symmetric algebra.*

A minimal bimodule projective resolution of a preprojective algebra $P(\mathbb{L}_n)$ of type $\mathbb{L}_n$ has been described in [7, Proposition 2.3] and one has $\Omega^3_{P(\mathbb{L}_n)^e} P(\mathbb{L}_n) \cong P(\mathbb{L}_n)$ for K of characteristic 2 and $\Omega^3_{P(\mathbb{L}_n)^e} P(\mathbb{L}_n) \ncong P(\mathbb{L}_n) \cong \Omega^6_{P(\mathbb{L}_n)^e} P(\mathbb{L}_n)$ for K of characteristic different from 2. In fact, it has been proved in [7, Proposition 2.3] that any deformed preprojective algebra $P^f(\mathbb{L}_n)$ of type $\mathbb{L}_n$ is a periodic algebra but the proof presented there does not allow us to determine the period of $P^f(\mathbb{L}_n)$. In the forthcoming paper [8], based on Theorem 2 and Corollary 4, we will determine the period of any deformed preprojective algebra of type $\mathbb{L}_n$.

We mention also the recent paper by Holm and Zimmermann [25] discussing derived and stable equivalences of deformed preprojective algebras of type $\mathbb{L}_n$.

For basic background on the representation theory applied here we refer to the book [3] and the articles [13,30], and on the singularities and Cohen–Macaulay modules to the survey article [9] and the books [6,22,31].

1. Proof of Theorem 1

For $n = 1$ we have $P(\mathbb{L}_1) = K[\varepsilon]/(\varepsilon^2)$, so this is the only deformed preprojective algebra of type $\mathbb{L}_1$. We assume from now that $n \geqslant 2$.

In $R(\mathbb{L}_n) = K[x]/(x^{2n})$, every element f of $\mathrm{rad}^2 R(\mathbb{L}_n)$ is of the form $f = (\lambda_1 x^2 + \lambda_2 x^3 + \cdots + \lambda_{2n-2} x^{2n-1}) + (x^{2n})$ for some $\lambda_1, \lambda_2, \dots, \lambda_{2n-2} \in K$. Hence, the deformed preprojective algebra $P^f(\mathbb{L}_n)$ is the bound quiver algebra given by the quiver

$$Q_{\mathbb{L}_n}: \quad \varepsilon = \bar{\varepsilon} \circlearrowright 0 \underset{\bar{a}_0}{\overset{a_0}{\rightleftarrows}} 1 \underset{\bar{a}_1}{\overset{a_1}{\rightleftarrows}} 2 \rightleftarrows \cdots \rightleftarrows n-2 \underset{\bar{a}_{n-2}}{\overset{a_{n-2}}{\rightleftarrows}} n-1$$

and the relations

$$a_0\bar{a}_0 + \varepsilon^2 + \lambda_1\varepsilon^3 + \lambda_2\varepsilon^4 + \cdots + \lambda_{2n-3}\varepsilon^{2n-1} + \lambda_{2n-2}\varepsilon^{2n} = 0,$$
$$\bar{a}_{n-2}a_{n-2} = 0, \qquad \varepsilon^{2n} = 0, \qquad \bar{a}_i a_i + a_{i+1}\bar{a}_{i+1} = 0 \quad \text{for } i \in \{0, \dots, n-3\}.$$

Observe that we may omit the parameter λ_{2n-2} in the above relations, because $\varepsilon^{2n}=0$. Note that the relation $\varepsilon^{2n}=0$ is also satisfied in $P(\mathbb{L}_n)$, because we have there

$$\varepsilon^{2n}=(-1)^n(a_0\bar{a}_0)^n=(-1)^{\frac{n(n+1)}{2}}a_0\cdots a_{n-2}\bar{a}_{n-2}a_{n-2}\bar{a}_{n-2}\cdots\bar{a}_0=0.$$

Therefore, a deformed preprojective algebra of type $\mathbb{L}_n$ is an algebra $L_n(\lambda_1,\lambda_2,\ldots,\lambda_{2n-3})$, for $\lambda_1,\lambda_2,\ldots,\lambda_{2n-3}\in K$, given by the quiver $Q_{\mathbb{L}_n}$ and the relations

$$a_0\bar{a}_0+\varepsilon^2+\lambda_1\varepsilon^3+\lambda_2\varepsilon^4+\cdots+\lambda_{2n-3}\varepsilon^{2n-1}=0,$$
$$\bar{a}_{n-2}a_{n-2}=0,\qquad \varepsilon^{2n}=0,\qquad \bar{a}_ia_i+a_{i+1}\bar{a}_{i+1}=0\quad\text{for } i\in\{0,\ldots,n-3\}.$$

With this, we have $L_n(0,\ldots,0)=P(\mathbb{L}_n)$.

We assume now that $\Lambda=L_n(\lambda_1,\lambda_2,\ldots,\lambda_{2n-3})$ for fixed elements $\lambda_1,\lambda_2,\ldots,\lambda_{2n-3}$ of K.

For the proof of Theorem 1 we will use the following lemma. For a path w in the quiver of Λ, we denote by $r(w)$ the number of arrows a_i in w with even indices i, and similarly we denote by $\bar{r}(w)$ the number of arrows $\bar{a}_i$ with even indices i.

Lemma 1.1. *For $k=0,\ldots,n-1$, the following hold in Λ:*

(A_k) *All paths from 0 to k of length greater than $2n-k-1$ are zero paths.*
(B_k) *All paths from 0 to k of length $2n-k-1$ are equal to $(-1)^{\bar{r}(w)}\varepsilon^{2n-2k-1}a_0\cdots a_{k-1}$.*
(A'_k) *All paths from k to 0 of length greater than $2n-k-1$ are zero paths.*
(B'_k) *All paths from k to 0 of length $2n-k-1$ are equal to $(-1)^{r(w)}\bar{a}_{k-1}\cdots\bar{a}_0\varepsilon^{2n-2k-1}$.*

Proof. We will prove this lemma by induction on k.

We prove first statements (A_0) and (B_0) for all paths which only have arrows ε, a_0, $\bar{a}_0$. We proceed by induction on the number of arrows different from ε. If w is a path of length greater than $2n-1$ with source and target equal to 0 and has only arrows ε, then the claim $w=0$ in (A_0) follows since we have the relation $\varepsilon^{2n}=0$. Moreover, if $w=\varepsilon^{2n-1}$, the claim for w in (B_0) is trivial. Assume the claims from (A_0) and (B_0) are satisfied for all paths containing at most s arrows a_0. Let w be a path of length $l\geqslant 2n-1$ with source and target equal 0, containing exactly $s+1$ arrows a_0. Then noting that ε and $a_0\bar{a}_0$ commute, we can write $w=a_0\bar{a}_0\varepsilon^iw'$ for some path w' of length $l-i-2$ with source and target equal 0, containing exactly s arrows a_0.

By the inductive assumption we have the equality

$$\varepsilon^{i+2}w'=(-1)^s\varepsilon^l. \tag{1}$$

Indeed, if $l<2n$, then from (B_0) follows that $\varepsilon^{i+2}w'=\varepsilon^l$ if l is even and $\varepsilon^{i+2}w'=-\varepsilon^l$ if l is odd. On the other hand, if $l\geqslant 2n$, then from (A_0) we have $\varepsilon^{i+2}w'=0$ and from the relation $\varepsilon^{2n}=0$ we have $\varepsilon^l=0=-\varepsilon^l$. Further, using again the relation $\varepsilon^{2n}=0$ and (1) we obtain

$$\varepsilon^{i+3}w'=(-1)^s\varepsilon^{l-(2n-1)}\varepsilon^{2n}=0. \tag{2}$$

Finally, using the relation

$$a_0\bar{a}_0+\varepsilon^2+\lambda_1\varepsilon^3+\lambda_2\varepsilon^4+\cdots+\lambda_{2n-3}\varepsilon^{2n-1}=0$$

for w, and equalities (1) and (2), we obtain the required claim

$$w = a_0\bar{a}_0\varepsilon^i w' = -\left(\varepsilon^2 + \lambda_1\varepsilon^3 + \lambda_2\varepsilon^4 + \cdots + \lambda_{2n-3}\varepsilon^{2n-1}\right)\varepsilon^i w'$$
$$= -\left(1 + \sum_{i=1}^{2n-3}\lambda_i\varepsilon^i\right)\varepsilon^{i+2}w' = -\varepsilon^{i+2}w' - \sum_{i=1}^{2n-3}\lambda_i\varepsilon^{i-1}\varepsilon^{i+3}w' = (-1)^{s+1}\varepsilon^l.$$

Hence the statements (A_0) and (B_0) hold for all paths consisting only of the arrows ε, a_0, $\bar{a}_0$ with at most $s+1$ arrows a_0. This proves (by induction) that the statements (A_0) and (B_0) are satisfied for all paths only with arrows ε, a_0, $\bar{a}_0$.

In order to show the statements (A_0) and (B_0) for arbitrary paths, we may inductively prove these statements for paths consisting only of the arrows ε, a_i, $\bar{a}_i$, $i \in \{0,\dots,s\}$ (induction on s). Indeed, assume that the statements are satisfied for some s and let w be a path consisting only of the arrows ε, a_i, $\bar{a}_i$, $i \in \{0,\dots,s+1\}$, having exactly t_i arrows a_i, for $i \in \{0,\dots,s+1\}$. Then, applying t_{s+1} times the equality $a_{s+1}\bar{a}_{s+1} = -\bar{a}_s a_s$ to w, we obtain that $w = (-1)^{t_{s+1}}w'$ for some path w' consisting only of the arrows ε, a_i, $\bar{a}_i$, $i \in \{0,\dots,s\}$, and having exactly t_i arrows a_i, for $i \in \{0,\dots,s-1\}$, and $t_s + t_{s+1}$ arrows a_s.

This ends the proof of the statements (A_0) and (B_0).

Assume now that the statements (A_k) and (B_k) are satisfied for some $k \in \{0,\dots,n-2\}$. We will prove the statement (A_{k+1}).

Let w be a path from 0 to $k+1$ of length $l > 2n-k-1$. Applying to w some relations $a_{i+1}\bar{a}_{i+1} = -\bar{a}_i a_i$ with $i \geqslant k$, if necessary, we obtain that w is equal up to sign to $w'a_k$ for some path w' of length $l-1 \geqslant 2n-k-1$. Then, applying (B_k), we conclude that w' is up to sign equal to the path $\varepsilon^{l-k-1}a_0\cdots a_{k-1}$. Hence, w is equal up to sign to the path $\varepsilon^{l-k-1}a_0\cdots a_{k-1}a_k$. Further, from ($\mathrm{A}_k$) we know that, for $l-1 > 2n-k-1$, $\varepsilon^{l-k-1}a_0\cdots a_{k-1} = 0$ holds, and hence $w = 0$. So assume that $l = 2n-k$. Applying again (B_k) and the relations $\bar{a}_i a_i + a_{i+1}\bar{a}_{i+1} = 0$, $i \in \{0,\dots,n-3\}$, and $\bar{a}_{n-2}a_{n-2} = 0$, we obtain that

$$\begin{aligned}\varepsilon^{l-k-1}a_0\cdots a_{k-1}a_k &= \varepsilon^{2(n-k)-1}a_0\cdots a_k = (-1)^{n-k-1}\varepsilon(a_0\bar{a}_0)^{n-k-1}a_0\cdots a_k\\ &= (-1)^{(n-k-1)(k+1)}\varepsilon a_0\cdots a_k(\bar{a}_k a_k)^{n-k-1}\\ &= (-1)^{(n-k-1)(k+1)+(n-k-1)(n-k)/2-1}\varepsilon a_0\cdots a_{n-2}\bar{a}_{n-2}a_{n-2}\bar{a}_{n-2}\cdots\bar{a}_{k+1} = 0.\end{aligned}$$

Therefore $w = 0$.

Assume now that w is a path from 0 to $k+1$ of length $2n-k-1$. Applying to w the relations $\bar{a}_i a_i + a_{i+1}\bar{a}_{i+1} = 0$, $i \in \{0,\dots,n-3\}$, we conclude that w is equal up to a sign to $w'a_0\cdots a_k$, where w' is a path from 0 to 0 consisting of s arrows a_0, s arrows $\bar{a}_0$ and $2(n-k-s-1)$ arrows ε, for some $s \in \{0,\dots,n-k\}$. We note that, by the above arguments, all paths from 0 to $k+1$ of length greater than $2n-k-1$ are zero paths. Hence, applying s times the relation

$$a_0\bar{a}_0 + \varepsilon^2 + \lambda_1\varepsilon^3 + \lambda_2\varepsilon^4 + \cdots + \lambda_{2n-3}\varepsilon^{2n-1} = 0$$

to the path $w'a_0\cdots a_k$, we obtain that

$$w'a_0\cdots a_k = (-1)^s\varepsilon^{2(n-k-1)}a_0\cdots a_k = (-1)^{s+n-k-1}(a_0\bar{a}_0)^{n-k-1}a_0\cdots a_k.$$

Applying again the relations $\bar{a}_i a_i + a_{i+1}\bar{a}_{i+1} = 0$, $i \in \{0,\dots,n-3\}$, and $\bar{a}_{n-2}a_{n-2} = 0$, we conclude that

$$\begin{aligned}(a_0\bar{a}_0)^{n-k-1}a_0\cdots a_k &= (-1)^{(n-k-1)k}a_0\cdots a_k(\bar{a}_k a_k)^{n-k-1}\\ &= (-1)^{(n-k-1)(k+1)+(n-k-1)(n-k)/2-1}a_0\cdots a_{n-2}\bar{a}_{n-2}a_{n-2}\bar{a}_{n-2}\cdots\bar{a}_{k+1} = 0.\end{aligned}$$

Hence w is the zero path, and this shows the statement (A_{k+1}).

The statement (B_{k+1}) will be proved similarly. Let w be a path from 0 to $k+1$ of length $2n-k-2$. As before, by applying to w the relations $a_{i+1}\bar{a}_{i+1} = -\bar{a}_i a_i$, $i \in \{0, \dots, n-3\}$, we obtain the path $w' a_0 \cdots a_k$, where w' is a path from 0 to 0 consisting from s arrows a_0, s arrows $\bar{a}_0$ and $2(n-k-s)-3$ arrows ε, for some $s \in \{0, \dots, n-k-1\}$. Notice that each use of the relation changes the sign, decreases by one the number of arrows a_{i+1} and increases by one the number of arrows a_i. Then it follows from (A_{k+1}) that applying s times the relation

$$a_0\bar{a}_0 + \varepsilon^2 + \lambda_1\varepsilon^3 + \lambda_2\varepsilon^4 + \cdots + \lambda_{2n-3}\varepsilon^{2n-1} = 0$$

to $w' a_0 \cdots a_k$ we obtain the equality

$$w' a_0 \cdots a_k = (-1)^s \varepsilon^{2(n-k)-3} a_0 \cdots a_k.$$

Therefore (B_{k+1}) holds.

The proofs of the statements (A'_{k+1}) and (B'_{k+1}) are dual. □

Proposition 1.2. *In the algebra Λ the following hold*:

(i) *For $s, t \in \{0, \dots, n-1\}$, all paths from s to t of length greater than $2n - |s-t| - 1$ are zero.*
(ii) *For $s, t \in \{0, \dots, n-1\}$, any w from s to t of length $2n - |s-t| - 1$ is equal to*

$$w = (-1)^{r(w)+r(a_0 \cdots a_{t-1})} \bar{a}_{s-1} \cdots \bar{a}_0 \varepsilon^{2n-2\max(s,t)-1} a_0 \cdots a_{t-1}.$$

(iii) *For $k \in \{0, \dots, n-1\}$, all paths from k to k of length $2n-1$ are maximal non-zero paths* (*and they are equal up to sign*).

Proof. For the proof of (i), we assume first that $t \geqslant s$. Let w be a path from s to t of length greater than $2n+s-t-1$ with $t \geqslant s$. Then applying to w the relations $a_{i+1}\bar{a}_{i+1} = -\bar{a}_i a_i$, $i \in \{0, \dots, n-3\}$, we obtain the path $\bar{a}_{s-1} \cdots \bar{a}_0 w'$, where w' is a path from 0 to t of length greater than $2n-t-1$. Hence w is up to sign equal to $\bar{a}_{s-1} \cdots \bar{a}_0 w'$. By Lemma 1.1($A_t$) we conclude that $w' = 0$, and so $w = 0$. Dually, in the case $t < s$, by applying to a path w from s to t of length greater than $2n+t-s-1$ the relations $a_{i+1}\bar{a}_{i+1} = -\bar{a}_i a_i$, $i \in \{0, \dots, n-3\}$, we obtain a path $w'' a_0 \cdots a_{t-1}$, with the subpath w'' from s to 0 of length $2n-s-1$. It follows from Lemma 1.1(A'_s) that $w'' = 0$, and so $w = 0$.

Similarly, one may prove that (ii) follows from Lemma 1.1(B'_s).

To prove (iii) observe first that each path of length $2n-1$ with the same source and target k is non-zero. Indeed, such a path has to pass through the vertex 0, because it is of odd length and hence contains an arrow ε, so it has to either pass through the vertex $n-1$ at most once, if $k \neq n-1$, or to have the source and target as the unique vertex k on the path in the case $k = n-1$. In both cases no such path has a subpath $\bar{a}_{n-2}a_{n-2}$, hence is non-zero. Uniqueness (up to sign) of the path in (iii) follows from (ii), while its maximality follows from (i) since all paths of length $2n$ are zero in Λ. Its existence is obvious. This proves (iii). □

Proposition 1.3. *Let $l \in \{0, \dots, 2n-1\}$ and k, t be fixed vertices of the Gabriel quiver $Q_{\mathbb{L}_n}$ of Λ with $|k-t| \leqslant l$. Consider the quotient algebra $\bar{\Lambda}_l = \Lambda / I_l$ of Λ by the ideal I_l generated by all paths of length $l+1$. Then in $\bar{\Lambda}_l$ the following hold*:

(i) *if $k+t+1 \leqslant l \leqslant 2n-1-|k-t|$ and $k+t+l$ is odd, then all paths of length l from k to t are non-zero and are equal up to sign to the path*

$$\bar{a}_{k-1} \cdots \bar{a}_0 \varepsilon^{l-(k+t)} a_0 \cdots a_{t-1};$$

(ii) *if* $|k-t| \leqslant l \leqslant 2(n-1)-(k+t)$ *and* $k+t+l$ *is even, then all paths of length* l *from* k *to* t *are non-zero and are equal up to sign to the path*

$$a_k \cdots a_{\frac{k+t+l}{2}-1} \bar{a}_{\frac{k+t+l}{2}-1} \cdots \bar{a}_t;$$

(iii) *all paths of length* l *from* k *to* t *with* $l > 2(n-1)-(k+t)$ *and* $k+t+l$ *even and all paths of length* l *from* k *to* t *with* $l > 2n-1-|k-t|$ *and* $k+t+l$ *odd (if exist) are zero paths.*

Proof. The proof of (i) is similar to the proof of Proposition 1.2(ii). Let w be a path of length l from k to t with $k+t+1 \leqslant l \leqslant 2n-1-|k-t|$ and $k+t+l$ odd. Applying to w the relations $a_{i+1}\bar{a}_{i+1} = -\bar{a}_i a_i$, $i \in \{0,\dots,n-3\}$, we obtain the path $\bar{a}_{k-1}\cdots\bar{a}_0 w' a_0 \cdots a_{t-1}$, where w' is a path from 0 to 0 of length $l-k-t > 0$ consisting of s arrows a_0, s arrows $\bar{a}_0$, and $l-k-t-2s$ arrows ε, for some integer s. Because in Λ/I_l all paths of length greater than l are zero paths, then it follows from the relation at the vertex 0 that

$$\bar{a}_{k-1}\cdots\bar{a}_0 w' a_0\cdots a_{t-1} = (-1)^s \bar{a}_{k-1}\cdots\bar{a}_0 \varepsilon^{l-k-t} a_0 \cdots a_{t-1}.$$

Finally, the path $\bar{a}_{k-1}\cdots\bar{a}_0\varepsilon^{l-(k+t)}a_0\cdots a_{t-1}$ is non-zero, because by Proposition 1.2(iii) it is a subpath of a maximal non-zero path.

Now we will prove (ii). Let w be a path of length l from k to t with $|k-t| \leqslant l \leqslant 2(n-1)-(k+t)$ and $k+t+l$ even. If w does not contain the arrow ε then we may obtain from w the path $a_k\cdots a_{\frac{k+t+l}{2}-1}\bar{a}_{\frac{k+t+l}{2}-1}\cdots\bar{a}_t$ by applying the relations $a_{i+1}\bar{a}_{i+1} = -\bar{a}_i a_i$, $i \in \{0,\dots,n-3\}$. If w contains the arrow ε then, in general case, we may obtain from w (as in the proof of (i)) the path $\bar{a}_{k-1}\cdots\bar{a}_0\varepsilon^{l-k-t}a_0\cdots a_{t-1}$. Note that in Λ/I_l we have

$$\bar{a}_{k-1}\cdots\bar{a}_0\varepsilon^{l-k-t}a_0\cdots a_{t-1} = (-1)^{\frac{l-k-t}{2}}\bar{a}_{k-1}\cdots\bar{a}_0(a_0\bar{a}_0)^{\frac{l-k-t}{2}}a_0\cdots a_{t-1}.$$

Then, applying again the relations $a_{i+1}\bar{a}_{i+1} = -\bar{a}_i a_i$, $i \in \{0,\dots,n-3\}$, to the path $\bar{a}_{k-1}\cdots(\bar{a}_0 a_0)^{\frac{l-k-t}{2}+1}\cdots a_{t-1}$, we obtain the path $a_k\cdots a_{\frac{k+t+l}{2}-1}\bar{a}_{\frac{k+t+l}{2}-1}\cdots\bar{a}_t$. Moreover, following Proposition 1.2(iii), the path $a_k\cdots a_{\frac{k+t+l}{2}-1}\bar{a}_{\frac{k+t+l}{2}-1}\cdots\bar{a}_t$ is a subpath of a maximal path, and hence it is non-zero.

We know from Proposition 1.2(i) that all paths of length l from k to t with $l > 2n-1-|k-t|$ and $k+t+l$ odd (if they exist) are zero paths. Moreover, all paths of length l from k to t with $l > 2(n-1)-(k+t)$ and $k+t+l$ even (if they exist) are (up to sign) equal to the path

$$a_k\cdots a_{n-2}(\bar{a}_{n-2}a_{n-2})^{n-1-\frac{k+t+l}{2}}\bar{a}_{n-2}\cdots\bar{a}_t = 0,$$

because $\bar{a}_{n-2}a_{n-2} = 0$. This ends the proof of (iii). □

We complete now our proof of Theorem 1.

Applying Proposition 1.3 and Proposition 1.2(i), we conclude that, for each pair $s,t \in \{0,\dots,n-1\}$ of vertices of $Q_{\mathbb{L}_n}$, we have the equalities

$$\begin{aligned}\dim e_t\Lambda e_s &= \#\bigl\{l\in\mathbb{N} \bigm| s+t+1 \leqslant l \leqslant 2n-1-|s-t| \wedge s+t+l \text{ odd}\bigr\}\\ &\quad + \#\bigl\{l\in\mathbb{N} \bigm| |s-t| \leqslant l \leqslant 2(n-1)-(s+t) \wedge s+t+l \text{ even}\bigr\}\\ &= 2\bigl(n-\max(s,t)\bigr).\end{aligned}$$

Hence, the Cartan matrix of the algebra Λ is of the form

$$\begin{bmatrix} 2n & 2n-2 & \cdots & 6 & 4 & 2 \\ 2n-2 & 2n-2 & \cdots & 6 & 4 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 6 & 6 & \cdots & 6 & 4 & 2 \\ 4 & 4 & \cdots & 4 & 4 & 2 \\ 2 & 2 & \cdots & 2 & 2 & 2 \end{bmatrix}$$

and is equal to the Cartan matrix of the algebra $P(\mathbb{L}_n)$. In particular, Λ is finite-dimensional.

This completes the proof of Theorem 1.

2. Proof of Theorem 2

We divide the proof of Theorem 2 into several steps. The first lemma will help us to identify isomorphisms.

Lemma 2.1. *Let $n \geqslant 2$ and $\lambda_1, \ldots, \lambda_{2n-3}, \lambda'_1, \ldots, \lambda'_{2n-3} \in K$. Assume that there exists a K-algebra homomorphism $\varphi : L_n(\lambda_1, \ldots, \lambda_{2n-3}) \to L_n(\lambda'_1, \ldots, \lambda'_{2n-3})$ given by*

$$\varphi(\varepsilon) = \sum_{i=0}^{2n-2} \gamma_i \varepsilon^{i+1}, \qquad \varphi(a_l) = a_l, \qquad \varphi(\bar{a}_l) = \bar{a}_l, \quad \textit{for } l = 0, \ldots, n-2,$$

with $\gamma_0, \ldots, \gamma_{2n-2} \in K$, $\gamma_0 \neq 0$. Then φ is an isomorphism of K-algebras.

Proof. We will construct a K-algebra homomorphism $\psi : L_n(\lambda'_1, \ldots, \lambda'_{2n-3}) \to L_n(\lambda_1, \ldots, \lambda_{2n-3})$ given by

$$\psi(\varepsilon) = \sum_{i=0}^{2n-2} \delta_i \varepsilon^{i+1}, \qquad \psi(a_l) = a_l, \qquad \psi(\bar{a}_l) = \bar{a}_l, \quad \text{for } l = 0, \ldots, n-2,$$

with $\delta_0, \ldots, \delta_{2n-2} \in K$, $\delta_0 \neq 0$, such that $\psi\varphi = \mathrm{id}_{L_n(\lambda_1, \ldots, \lambda_{2n-3})}$.

Let $r_0 = 0$, $\delta_0 = \gamma_0^{-1}$ and

$$r_l = \sum_{i=1}^{l} \gamma_i \left(\sum_{\substack{0 \leqslant a_1, a_2, \ldots, a_{i+1} \\ a_1 + a_2 + \cdots + a_{i+1} = l - i}} \prod_{j=1}^{i+1} \delta_{a_j} \right) \quad \text{and} \quad \delta_l = -\gamma_0^{-1} r_l,$$

for $l \in \{1, \ldots, 2n-2\}$.

Note that we have

$$\psi\varphi(\varepsilon) = \psi\left(\sum_{i=0}^{2n-2} \gamma_i \varepsilon^{i+1} \right) = \sum_{i=0}^{2n-2} \gamma_i \psi\left(\varepsilon^{i+1}\right) = \sum_{i=0}^{2n-2} \gamma_i \psi(\varepsilon)^{i+1} = \sum_{i=0}^{2n-2} \gamma_i \left(\sum_{j=0}^{2n-2} \delta_j \varepsilon^{j+1} \right)^{i+1}$$

$$= \sum_{i=0}^{2n-2} \gamma_i \left(\sum_{l=0}^{2n-i-2} \left(\sum_{\substack{0 \leqslant a_1, a_2, \ldots, a_{i+1} \\ a_1 + a_2 + \cdots + a_{i+1} = l}} \prod_{t=1}^{i+1} \delta_{a_t} \right) \varepsilon^{i+1+l} \right)$$

$$= \sum_{j=0}^{2n-2} \left(\sum_{i=0}^{j} \gamma_i \left(\sum_{\substack{0 \leqslant a_1, a_2, \dots, a_{i+1} \\ a_1 + a_2 + \dots + a_{i+1} = j-i}} \prod_{t=1}^{i+1} \delta_{a_t} \right) \right) \varepsilon^{j+1}$$

$$= \sum_{j=0}^{2n-2} \left(\gamma_0 \delta_j + \sum_{i=1}^{j} \gamma_i \left(\sum_{\substack{0 \leqslant a_1, a_2, \dots, a_{i+1} \\ a_1 + a_2 + \dots + a_{i+1} = j-i}} \prod_{t=1}^{i+1} \delta_{a_t} \right) \right) \varepsilon^{j+1}$$

$$= \sum_{j=0}^{2n-2} (\gamma_0 \delta_j + r_j) \varepsilon^{j+1} = \gamma_0 \delta_0 \varepsilon + \sum_{j=1}^{2n-2} \left(\gamma_0 \left(-\gamma_0^{-1} r_j \right) + r_j \right) \varepsilon^{j+1} = \varepsilon.$$

From the definition of φ and ψ we also have $\psi\varphi(a_l) = a_l$ and $\psi\varphi(\bar{a}_l) = \bar{a}_l$ for all $l = 0, \dots, n-2$. This shows that $\psi\varphi = \mathrm{id}_{L_n(\lambda_1, \dots, \lambda_{2n-3})}$. Since Λ is finite-dimensional it follows that ψ is the 2-sided inverse of φ, and it also follows that ψ is an algebra homomorphism. Hence $\varphi = \varphi'$ is a K-algebra isomorphism. □

The following proposition proves part (1) of Theorem 2.

Proposition 2.2. *Let K be of characteristic different from 2, and $\Lambda = L_n(\lambda_1, \dots, \lambda_{2n-3})$ for $n \geqslant 2$ and $\lambda_1, \dots, \lambda_{2n-3} \in K$. Then Λ is isomorphic to $P(\mathbb{L}_n)$.*

Proof. We will choose elements $\gamma_0, \gamma_1, \dots, \gamma_{2n-3} \in K$ such that, for each $k \in \{0, \dots, 2n-3\}$, the equality

$$\left(\sum_{i=0}^{k} \gamma_i \varepsilon^{i+1} \right)^2 + \left(\varepsilon^{k+3} \right) = \left(\varepsilon^2 + \sum_{i=1}^{k} \lambda_i \varepsilon^{i+2} \right) + \left(\varepsilon^{k+3} \right)$$

holds, in the quotient algebra $L_n(\lambda_1, \dots, \lambda_{2n-3})/(\varepsilon^{k+3})$.

Observe that

$$\left(\varepsilon^2 + \sum_{i=1}^{2n-3} \lambda_i \varepsilon^{i+2} \right) + \left(\varepsilon^{k+3} \right) = \left(\varepsilon^2 + \sum_{i=1}^{k} \lambda_i \varepsilon^{i+2} \right) + \left(\varepsilon^{k+3} \right).$$

For $k = 0$, the required equality is of the form

$$(\gamma_0 \varepsilon)^2 + \left(\varepsilon^3 \right) = \varepsilon^2 + \left(\varepsilon^3 \right),$$

and hence $\gamma_0^2 \varepsilon^2 = \varepsilon^2$, $\gamma_0^2 = 1$. Hence, we may choose either $\gamma_0 = 1$ or $\gamma_0 = -1$. Let $\gamma_0 = 1$.

Assume now that, for some $k \geqslant 1$, elements $\gamma_0, \gamma_1, \dots, \gamma_{k-1} \in K$ satisfying the equalities

$$\left(\sum_{i=0}^{j} \gamma_i \varepsilon^{i+1} \right)^2 + \left(\varepsilon^{j+3} \right) = \left(\varepsilon^2 + \sum_{i=1}^{j} \lambda_i \varepsilon^{i+2} \right) + \left(\varepsilon^{j+3} \right),$$

for $j \in \{0, \dots, k-1\}$, are defined. Observe that we have the equalities

$$\left(\sum_{i=0}^{k}\gamma_i\varepsilon^{i+1}\right)^2+\left(\varepsilon^{k+3}\right)=\left(\gamma_k\varepsilon^{k+1}+\sum_{i=0}^{k-1}\gamma_i\varepsilon^{i+1}\right)^2+\left(\varepsilon^{k+3}\right)$$
$$=\left(\gamma_k^2\varepsilon^{2k+2}+2\gamma_k\sum_{i=0}^{k-1}\gamma_i\varepsilon^{k+i+2}+\left(\sum_{i=0}^{k-1}\gamma_i\varepsilon^{i+1}\right)^2\right)+\left(\varepsilon^{k+3}\right)$$
$$=\left(2\gamma_k\gamma_0\varepsilon^{k+2}+\left(\sum_{i=0}^{k-1}\gamma_i\varepsilon^{i+1}\right)^2\right)+\left(\varepsilon^{k+3}\right)$$

and

$$\left(\varepsilon^2+\sum_{i=1}^{k}\lambda_i\varepsilon^{i+2}\right)+\left(\varepsilon^{k+3}\right)=\left(\lambda_k\varepsilon^{k+2}+\varepsilon^2+\sum_{i=1}^{k-1}\lambda_i\varepsilon^{i+2}\right)+\left(\varepsilon^{k+3}\right),$$

because $2k+2\geqslant k+3$ for $k\geqslant 1$. Moreover, from the choice of $\gamma_0,\dots,\gamma_{k-1}$, we have

$$\left(\sum_{i=0}^{k-1}\gamma_i\varepsilon^{i+1}\right)^2+\left(\varepsilon^{k+2}\right)=\left(\varepsilon^2+\sum_{i=1}^{k-1}\lambda_i\varepsilon^{i+2}\right)+\left(\varepsilon^{k+2}\right).$$

Hence, the required equality

$$\left(\sum_{i=0}^{k}\gamma_i\varepsilon^{i+1}\right)^2+\left(\varepsilon^{k+3}\right)=\left(\varepsilon^2+\sum_{i=1}^{k}\lambda_i\varepsilon^{i+2}\right)+\left(\varepsilon^{k+3}\right)$$

forces γ_k to satisfy the equality

$$2\gamma_k\gamma_0\varepsilon^{k+2}+\sum_{\substack{0\leqslant i,j\leqslant k-1\\ i+j=k}}\gamma_i\gamma_j\varepsilon^{(i+1)+(j+1)}=\lambda_k\varepsilon^{k+2},$$

or equivalently

$$2\gamma_k\gamma_0=\lambda_k-\sum_{i=1}^{k-1}\gamma_i\gamma_{k-i}.$$

Therefore, we define

$$\gamma_k=\frac{\gamma_0^{-1}}{2}\left(\lambda_k-\sum_{i=1}^{k-1}\gamma_i\gamma_{k-i}\right)=\frac{1}{2}\left(\lambda_k-\sum_{i=1}^{k-1}\gamma_i\gamma_{k-i}\right).$$

Finally, for $k=2n-3$, we have $\varepsilon^{k+3}=\varepsilon^{2n}=0$, and hence in $L_n(\lambda_1,\dots,\lambda_{2n-3})$ the equality

$$\left(\sum_{i=0}^{2n-3}\gamma_i\varepsilon^{i+1}\right)^2=\varepsilon^2+\sum_{i=1}^{2n-3}\lambda_i\varepsilon^{i+2}$$

holds. Therefore, the homomorphism $\varphi : L_n = L_n(0, \dots, 0) \to L_n(\lambda_1, \dots, \lambda_{2n-3})$ given by

$$\varphi(\varepsilon) = \sum_{i=0}^{2n-3} \gamma_i \varepsilon^{i+1}, \qquad \varphi(a_l) = a_l, \qquad \varphi(\bar{a}_l) = \bar{a}_l, \quad \text{for } l \in \{0, \dots, n-2\},$$

is well defined. By Lemma 2.1, we conclude that φ is an isomorphism of K-algebras. □

Proposition 2.3. *Let K have characteristic 2, $n \geqslant 3$, and $\lambda_1, \dots, \lambda_{2n-3} \in K$ with $\lambda_1 = \dots = \lambda_{2k-1} = 0$ and $\lambda_{2k} \neq 0$ for some $k \in \{1, \dots, n-2\}$. Then there exist elements $\lambda'_1, \dots, \lambda'_{2n-3} \in K$ with $\lambda'_1 = \dots = \lambda'_{2k-1} = \lambda'_{2k} = 0$ and $\lambda'_{2k+1} = \lambda_{2k+1}$ such that $L_n(\lambda_1, \dots, \lambda_{2n-3})$ and $L_n(\lambda'_1, \dots, \lambda'_{2n-3})$ are isomorphic.*

Proof. We will define elements $\lambda'_{2k+1}, \dots, \lambda'_{2n-3} \in K$ so that there is an isomorphism of K-algebras $\varphi : L_n(\lambda_1, \dots, \lambda_{2n-3}) \to L_n(0, 0, \dots, \lambda'_{2k+1}, \dots, \lambda'_{2n-3})$ given by

$$\varphi(\varepsilon) = \varepsilon + \gamma \varepsilon^{k+1}, \qquad \varphi(a_l) = a_l, \qquad \varphi(\bar{a}_l) = \bar{a}_l, \quad \text{for } l \in \{0, \dots, n-2\}, \tag{$*$}$$

where $\gamma^2 = \lambda_{2k}$.

For integers $k \geqslant 1$, $i \geqslant 2k$, denote

$$m(k,i) = \min\left(\left\lfloor \frac{i+2}{k+1} \right\rfloor, \left\lfloor \frac{i}{k} - 2 \right\rfloor\right).$$

We note that $m(k,i)$ is a nonnegative integer and $m(k,i) \leqslant \lfloor \frac{i-1}{k} \rfloor$, because $\lfloor \frac{i}{k} - 2 \rfloor \leqslant \lfloor \frac{i-1}{k} \rfloor$.

For $i \in \{2k+1, \dots, 2n-3\}$ we define

$$\lambda'_i = \sum_{j=0}^{m(k,i)} \lambda_{i-jk} \binom{i-jk+2}{j} \gamma^j,$$

for $i \in \{2k+1, \dots, 2n-3\}$.

In order to prove that the map φ in $(*)$ is a well-defined homomorphism of K-algebras, it is enough to show that $\varphi(a_0\bar{a}_0 + \varepsilon^2 + \lambda_{2k}\varepsilon^{2k+2} + \dots + \lambda_{2n-3}\varepsilon^{2n-1}) = 0$ in $L_n(0, \dots, 0, \lambda'_{2k+1}, \lambda'_{2k+2}, \dots, \lambda'_{2n-3})$. Indeed, we have in $L_n(0, \dots, 0, \lambda'_{2k+1}, \lambda'_{2k+2}, \dots, \lambda'_{2n-3})$ the equalities

$$\begin{aligned}
\varphi\left(a_0\bar{a}_0 + \varepsilon^2 + \sum_{i=2k}^{2n-3} \lambda_i \varepsilon^{i+2}\right) &= \varphi(a_0)\varphi(\bar{a}_0) + \varphi(\varepsilon)^2 + \sum_{i=2k}^{2n-3} \lambda_i \varphi(\varepsilon)^{i+2} \\
&= a_0\bar{a}_0 + \left(\varepsilon + \gamma\varepsilon^{k+1}\right)^2 + \sum_{i=2k}^{2n-3} \lambda_i \varphi\left(\varepsilon + \gamma\varepsilon^{k+1}\right)^{i+2} \\
&= a_0\bar{a}_0 + \varepsilon^2 + \gamma^2\varepsilon^{2k+2} + \sum_{i=2k}^{2n-3} \lambda_i \sum_{j=0}^{i+2} \binom{i+2}{j} \gamma^j \varepsilon^{(k+1)j+((i+2)-j)} \\
&= a_0\bar{a}_0 + \varepsilon^2 + \lambda_{2k}\varepsilon^{2k+2} + \sum_{i=2k}^{2n-3} \sum_{j=0}^{i+2} \lambda_i \binom{i+2}{j} \gamma^j \varepsilon^{kj+i+2} \\
&= a_0\bar{a}_0 + \varepsilon^2 + \lambda_{2k}\varepsilon^{2k+2} + \sum_{l=2k}^{2n-3} \left(\sum_{j=0}^{m(k,l)} \lambda_{l-jk} \binom{l-jk+2}{j} \gamma^j \right) \varepsilon^{l+2}
\end{aligned}$$

$$= a_0\bar{a}_0 + \varepsilon^2 + 2\lambda_{2k}\varepsilon^{2k+2} + \sum_{l=2k+1}^{2n-3}\left(\sum_{j=0}^{m(k,l)} \lambda_{l-jk}\binom{l-jk+2}{j}\gamma^j\right)\varepsilon^{l+2}$$

$$= a_0\bar{a}_0 + \varepsilon^2 + \sum_{l=2k+1}^{2n-3} \lambda'_l\varepsilon^{l+2} = 0,$$

because for $l = 2k$ we have $\lfloor \frac{l}{k} - 2\rfloor = 0$, and hence

$$\sum_{j=0}^{m(k,l)} \lambda_{l-jk}\binom{l-jk+2}{j}\gamma^j\varepsilon^{l+2} = \lambda_{2k}\binom{2k+2}{0}\gamma^0\varepsilon^{2k+2} = \lambda_{2k}\varepsilon^{2k+2}.$$

Hence φ is a homomorphism of K-algebras, and consequently, by Lemma 2.1, an isomorphism.

We note also that $\lambda'_{2k+1} = \lambda_{2k+1}$. Indeed, we have

$$\lambda'_{2k+1} = \sum_{j=0}^{m(k,2k+1)} \lambda_{2k+1-jk}\binom{2k+1-jk+2}{j}\gamma^j$$

and

$$m(k, 2k+1) = \min\left(\left\lfloor \frac{(2k+1)+2}{k+1}\right\rfloor, \left\lfloor \frac{2k+1}{k} - 2\right\rfloor\right) = \left\lfloor \frac{1}{k}\right\rfloor.$$

Hence for $k > 1$ we have

$$\lambda'_{2k+1} = \lambda_{2k+1}\binom{2k+3}{0}\gamma^0 = \lambda_{2k+1},$$

and for $k = 1$ we obtain

$$\lambda'_3 = \lambda_{2+1}\binom{2+1+2}{0}\gamma^0 + \lambda_2\binom{2+1-1+2}{1}\gamma^1 = \lambda_3 + 4\lambda_2\gamma = \lambda_3.$$

This completes our proof. □

We will now prove the crucial step for part (2) of Theorem 2.

Proposition 2.4. *Let K be of characteristic 2, $n \geqslant 2$, $k \in \{0, \ldots, n-2\}$ and $\lambda_{2k+1}, \ldots, \lambda_{2n-3} \in K$, $\lambda_{2k+1} \neq 0$. Then $L_n(0, \ldots, 0, \lambda_{2k+1}, \ldots, \lambda_{2n-3})$ is isomorphic to $L_n^{(k+1)}$.*

Proof. Recall that $L_n^{(k+1)} = L_n(\underbrace{0, \ldots, 0}_{2k}, 1, 0, \ldots, 0)$, for $k \in \{0, \ldots, n-2\}$.

We will construct a homomorphism of K-algebras

$$\varphi : L_n(\underbrace{0, \ldots, 0}_{2k}, 1, \ldots, 0) \to L_n(0, \ldots, 0, \lambda_{2k+1}, \ldots, \lambda_{2n-3}),$$

given by

$$\varphi(\varepsilon)=\sum_{i=0}^{2n-2}\gamma_i\varepsilon^{i+1},\qquad \varphi(a_l)=a_l,\qquad \varphi(\bar{a}_l)=\bar{a}_l,\quad \text{for } l\in\{0,\dots,n-2\}.$$

Such a map is an algebra homomorphism provided $\varphi(a_0\bar{a}_0+\varepsilon^2+\varepsilon^{2k+3})=0$ in $L_n(0,\dots,\lambda_{2k+1},\dots,\lambda_{2n-3})$. So we will choose elements $\gamma_0,\gamma_1,\dots,\gamma_{2n-3}\in K$ which will satisfy the equalities

$$\left(\left(\sum_{i=0}^{j}\gamma_i\varepsilon^{i+1}\right)^2+\left(\sum_{i=0}^{j}\gamma_i\varepsilon^{i+1}\right)^{2k+3}\right)+\left(\varepsilon^{m(j)}\right)=\left(\varepsilon^2+\sum_{i=1}^{j}\lambda_i\varepsilon^{i+2}\right)+\left(\varepsilon^{m(j)}\right),$$

for $j=0,\dots,2n-2$, $m(j)=\min(2j+3,j+2k+4)$, in $L_n(\lambda_1,\dots,\lambda_{2n-3})/(\varepsilon^{m(j)})$.

Let $\gamma_0=1$ and $\gamma_j=0$ for $0<j\leqslant k$. Then, for $0\leqslant j\leqslant k$, we have

$$\begin{aligned}\left(\left(\sum_{i=0}^{j}\gamma_i\varepsilon^{i+1}\right)^2+\left(\sum_{i=0}^{j}\gamma_i\varepsilon^{i+1}\right)^{2k+3}\right)+\left(\varepsilon^{2j+3}\right)&=\left(\gamma_0^2\varepsilon^2+\gamma_0^{2k+3}\varepsilon^{2k+3}\right)+\left(\varepsilon^{2j+3}\right)\\&=\gamma_0^2\varepsilon^2+\left(\varepsilon^{2j+3}\right)=\varepsilon^2+\left(\varepsilon^{2j+3}\right)\\&=\left(\varepsilon^2+\sum_{i=1}^{j}\lambda_i\varepsilon^{i+2}\right)+\left(\varepsilon^{2j+3}\right).\end{aligned}$$

From now on assume that we have chosen $\gamma_0,\dots,\gamma_{j-1}$, for some $j>0$, satisfying the above equalities. For $l=0,\dots,j$, we denote

$$r_l=\sum_{\substack{0\leqslant a_1,a_2,\dots,a_{2k+3}<l\\ a_1+a_2+\cdots+a_{2k+3}=l}}\prod_{i=1}^{2k+3}\gamma_{a_i}.$$

Then we have, for each $l\in\{0,\dots,j-1\}$, the equalities

$$\begin{aligned}\left(\sum_{i=0}^{l}\gamma_i\varepsilon^{i+1}\right)^{2k+3}+\left(\varepsilon^{l+2k+4}\right)&=\sum_{i=0}^{l}\left(\sum_{\substack{0\leqslant a_1,a_2,\dots,a_{2k+3}\leqslant i\\ a_1+a_2+\cdots+a_{2k+3}=i}}\prod_{t=1}^{2k+3}\gamma_{a_t}\right)\varepsilon^{i+2k+3}+\left(\varepsilon^{l+2k+4}\right)\\&=\sum_{i=0}^{l}\left((2k+3)\gamma_i\gamma_0^{2k+2}+r_i\right)\varepsilon^{i+2k+3}+\left(\varepsilon^{l+2k+4}\right)\\&=\sum_{i=0}^{l}(\gamma_i+r_i)\varepsilon^{i+2k+3}+\left(\varepsilon^{l+2k+4}\right)\end{aligned}$$

and

$$\left(\sum_{i=0}^{l}\gamma_i\varepsilon^{i+1}\right)^2=\sum_{i=0}^{l}\gamma_i^2\varepsilon^{2(i+1)}.$$

We will now consider four cases.

If $k < j \leqslant 2k$, then the required γ_j should satisfy the equality

$$\left(\gamma_j^2\varepsilon^{2(j+1)} + (\gamma_{2(j-k)-1} + r_{2(j-k)-1})\varepsilon^{2(j+1)}\right) + \left(\varepsilon^{2j+3}\right) = \lambda_{2j}\varepsilon^{2(j+1)} + \left(\varepsilon^{2j+3}\right),$$

which is equivalent to the equality

$$\gamma_j^2 + \gamma_{2(j-k)-1} + r_{2(j-k)-1} = \lambda_{2j}.$$

Hence, we define γ_j as the square root of $\gamma_{2(j-k)-1} + r_{2(j-k)-1} + \lambda_{2j}$.

Assume $j = 2k+1$. Note that in this case $j = 2(j-k)-1$. Then the required γ_j should satisfy

$$\left(\gamma_j^2\varepsilon^{2(j+1)} + (\gamma_j + r_j)\varepsilon^{2(j+1)}\right) + \left(\varepsilon^{2j+3}\right) = \lambda_{2j}\varepsilon^{2(j+1)} + \left(\varepsilon^{2j+3}\right),$$

and this is equivalent to

$$\gamma_j^2 + \gamma_j + r_j = \lambda_{2j}.$$

Hence, we define γ_j as a root of the polynomial $x^2 + x + r_j + \lambda_{2j} \in K[x]$.

Let $j > 2k+1$ and assume j is odd. Observe that in this case $2(k + \frac{j+1}{2} + 1) = j + 2k + 3$. Then the required γ_j should satisfy the equality

$$\left(\gamma_{k+\frac{j+1}{2}}^2\varepsilon^{2(k+\frac{j+1}{2}+1)} + (\gamma_j + r_j)\varepsilon^{j+2k+3}\right) + \left(\varepsilon^{j+2k+4}\right) = \lambda_{j+2k+1}\varepsilon^{j+2k+3} + \left(\varepsilon^{j+2k+4}\right),$$

which is equivalent to the equality

$$\gamma_{k+\frac{j+1}{2}}^2 + \gamma_j + r_j = \lambda_{j+2k+1}.$$

Therefore, we define $\gamma_j = \gamma_{k+\frac{j+1}{2}}^2 + r_j + \lambda_{j+2k+1}$.

Finally, assume that $j > 2k+1$ and j is even. Then the required γ_j should satisfy

$$(\gamma_j + r_j)\varepsilon^{j+2k+3} + \left(\varepsilon^{j+2k+4}\right) = \lambda_{j+2k+1}\varepsilon^{j+2k+3} + \left(\varepsilon^{j+2k+4}\right),$$

which is clearly equivalent to the equality

$$\gamma_j + r_j = \lambda_{j+2k+1}.$$

Hence, we define $\gamma_j = r_j + \lambda_{j+2k+1}$.

It follows from the above construction of $\gamma_0, \dots, \gamma_{2n-2}$ that in $L_n(\lambda_1, \dots, \lambda_{2n-3})/(\varepsilon^{m(2n-2)})$ the following equality holds

$$\left(\left(\sum_{i=0}^{2n-2}\gamma_i\varepsilon^{i+1}\right)^2 + \left(\sum_{i=0}^{j}\gamma_i\varepsilon^{i+1}\right)^{2k+3}\right) + \left(\varepsilon^{m(2n-2)}\right) = \left(\varepsilon^2 + \sum_{i=1}^{2n-2}\lambda_i\varepsilon^{i+2}\right) + \left(\varepsilon^{m(2n-2)}\right).$$

We note that in $L_n(0, \dots, 0, \lambda_{2k+1}, \dots, \lambda_{2n-3})$ we have $\varepsilon^{m(2n-2)} = 0$, because $m(2n-2) \geqslant 2n$ and $\varepsilon^{2n} = 0$. So the equalities

$$\left(\sum_{i=0}^{2n-2} \gamma_i \varepsilon^{i+1}\right)^2 + \left(\sum_{i=0}^{j} \gamma_i \varepsilon^{i+1}\right)^{2k+3} = \varepsilon^2 + \sum_{i=1}^{2n-2} \lambda_i \varepsilon^{i+2} = a_0 \bar{a}_0$$

hold in $L_n(0, \ldots, 0, \lambda_{2k+1}, \ldots, \lambda_{2n-3})$. Hence φ is a homomorphism of K-algebras, and consequently, by Lemma 2.1, an isomorphism. This completes our proof. □

Proof of part (2) of Theorem 2. Assume K has characteristic 2. Observe first that

$$L_n^{(r)} = P^{f_r}(\mathbb{L}_n) = L_n(\underbrace{0, \ldots, 0}_{2(r-1)}, 1, \ldots, 0), \quad \text{for } r \in \{1, \ldots, n-1\},$$

and

$$L_n^{(n)} = P^{f_n}(\mathbb{L}_n) = L_n(0, \ldots, 0).$$

Let $\lambda_1, \ldots, \lambda_{2n-3} \in K$ and $\Lambda = L_n(\lambda_1, \ldots, \lambda_{2n-3})$. We claim that $\Lambda \cong L_n^{(r)}$ for some $r \in \{1, \ldots, n\}$. Clearly, if $\lambda_1 = \cdots = \lambda_{2n-3} = 0$, then $\Lambda \cong L_n^{(n)}$. Assume $\lambda_i \neq 0$ for some $i \in \{1, \ldots, 2n-3\}$. Take the minimal index $m \in \{1, \ldots, 2n-3\}$ with $\lambda_m \neq 0$. If m is odd, say $m = 2r-1$ for some $r \in \{1, \ldots, n-1\}$, then it follows from Proposition 2.4 that $\Lambda \cong L_n^{(r)}$. On the other hand, if m is even, then, by Proposition 2.3, there exist elements $\lambda'_1, \ldots, \lambda'_{2n-3} \in K$ such that $\lambda'_1 = \cdots = \lambda'_m = 0$ and $\Lambda \cong L_n(\lambda'_1, \ldots, \lambda'_{2n-3})$. Applying Propositions 2.3 and 2.4, we conclude, by induction on m, that Λ is isomorphic to an algebra $L_n^{(r)}$ for some $r \in \{1, \ldots, n\}$. □

We end this section with the following complementary result.

Proposition 2.5. *Let K be of characteristic 2, $n \geqslant 2$, and $\lambda_1, \ldots, \lambda_{2n-3} \in K$ with $\lambda_{2i+1} = 0$ for all $i \in \{0, \ldots, n-2\}$. Then $L_n(\lambda_1, \ldots, \lambda_{2n-3})$ is isomorphic to $P(\mathbb{L}_n)$.*

Proof. We show that there exists a homomorphism of K-algebras

$$\varphi : P(\mathbb{L}_n) = L_n(0, \ldots, 0) \to L_n(0, \lambda_2, 0, \lambda_4, 0, \ldots, 0, \lambda_{2n-4}, 0) = L_n(\lambda_1, \ldots, \lambda_{2n-3})$$

such that

$$\varphi(\varepsilon) = \sum_{i=0}^{n-2} \gamma_i \varepsilon^{i+1}, \qquad \varphi(a_l) = a_l, \qquad \varphi(\bar{a}_l) = \bar{a}_l, \quad \text{for } l \in \{0, \ldots, n-2\},$$

where $\gamma_0, \ldots, \gamma_{n-2} \in K$ satisfy the conditions $\gamma_0 = 1$ and $\gamma_i^2 = \lambda_{2i}$, for $i \in \{0, \ldots, n-2\}$. Then φ will be an isomorphism, by Lemma 2.1.

Indeed, we have

$$\begin{aligned}
\varphi\big(a_0\bar{a}_0 + \varepsilon^2\big) &= \varphi(a_0\bar{a}_0) + \varphi\big(\varepsilon^2\big) = \varphi(a_0)\varphi(\bar{a}_0) + \varphi(\varepsilon)^2 \\
&= a_0\bar{a}_0 + \left(\sum_{i=0}^{n-2} \gamma_i \varepsilon^{i+1}\right)^2 = a_0\bar{a}_0 + \sum_{i=0}^{n-2} \gamma_i^2 \varepsilon^{2i+2} \\
&= a_0\bar{a}_0 + \gamma_0^2 \varepsilon^2 + \sum_{i=1}^{n-2} \gamma_i^2 \varepsilon^{2i+2} = a_0\bar{a}_0 + \varepsilon^2 + \sum_{i=1}^{n-2} \lambda_{2i} \varepsilon^{2i+2} = 0,
\end{aligned}$$

$$\varphi(\bar{a}_{n-2}a_{n-2}) = \bar{a}_{n-2}a_{n-2} = 0,$$

and

$$\varphi(\bar{a}_i a_i + a_{i+1}\bar{a}_{i+1}) = \bar{a}_i a_i + a_{i+1}\bar{a}_{i+1} = 0$$

for $i \in \{0, \dots, n-3\}$, and hence φ is a well-defined homomorphism of K-algebras. $\Box$

3. Proofs of Theorem 3 and Corollary 4

For an integer $n \geqslant 2$ and $r \in \{1, \dots, n-1\}$, we denote by $\Lambda_n^{(r)}$ the bound quiver algebra $KQ_{\mathbb{L}_n}/I_n^{(r)}$, where $I_n^{(r)}$ is the ideal in the path algebra $KQ_{\mathbb{L}_n}$ generated by the elements

$$\varepsilon^2 + a_0\bar{a}_0 + \varepsilon(a_0\bar{a}_0)^r, \qquad \bar{a}_{n-2}a_{n-2}, \quad \text{and} \quad \bar{a}_i a_i + a_{i+1}\bar{a}_{i+1} \quad \text{for } i \in \{0, \dots, n-3\}.$$

Proposition 3.1. *Let K be of characteristic 2, $n \geqslant 2$ an integer, and $r \in \{1, \dots, n-1\}$. Then the algebras $L_n^{(r)}$ and $\Lambda_n^{(r)}$ are isomorphic.*

Proof. Fix $r \in \{1, \dots, n-1\}$. First, we prove by induction on i that in $\Lambda_n^{(r)}$ the equalities $\varepsilon^{2i}(a_0\bar{a}_0)^{n-i} = 0$, for $i \in \{0, \dots, n\}$, hold. Indeed, we have in $\Lambda_n^{(r)}$ the equalities

$$(a_0\bar{a}_0)^n = a_0(a_1\bar{a}_1)^{n-1}\bar{a}_0 = \cdots = a_0 a_1 \cdots a_{n-2}\bar{a}_{n-2}a_{n-2}\bar{a}_{n-2}\cdots\bar{a}_1\bar{a}_0 = 0.$$

Assume now that $\varepsilon^{2i}(a_0\bar{a}_0)^{n-i} = 0$ for some $i \in \{0, \dots, n-1\}$. Then, from the equality $\varepsilon^2 + a_0\bar{a}_0 + \varepsilon(a_0\bar{a}_0)^r = 0$, we conclude that

$$\begin{aligned}
\varepsilon^{2(i+1)}(a_0\bar{a}_0)^{n-(i+1)} &= \varepsilon^{2i}\varepsilon^2(a_0\bar{a}_0)^{n-(i+1)} = \varepsilon^{2i}\big(a_0\bar{a}_0 + \varepsilon(a_0\bar{a}_0)^r\big)(a_0\bar{a}_0)^{n-(i+1)} \\
&= \varepsilon^{2i}(a_0\bar{a}_0)^{1+n-(i+1)} + \varepsilon^{2i}\varepsilon(a_0\bar{a}_0)^r(a_0\bar{a}_0)^{n-i-1} \\
&= \varepsilon^{2i}(a_0\bar{a}_0)^{n-i} + \varepsilon\big(\varepsilon^{2i}(a_0\bar{a}_0)^{n-i}\big)(a_0\bar{a}_0)^{r-1} = 0.
\end{aligned}$$

In particular, for $i = n$, we obtain $\varepsilon^{2n} = 0$.

We claim now that there exist elements $\lambda_{2r}, \dots, \lambda_{2n-3} \in K$ such that the identity endomorphism of $KQ_{\mathbb{L}_n}$ induces an epimorphism of K-algebras $L_n(0, \dots, 0, 1, \lambda_{2r}, \dots, \lambda_{2n-3}) \to \Lambda_n^{(r)}$. Observe that it is sufficient to find elements $\lambda_{2r}, \dots, \lambda_{2n-3}$ in K such that the equality $\varepsilon^2 + a_0\bar{a}_0 + \varepsilon^{2r+1} + \sum_{i=2r}^{2n-3}\lambda_i\varepsilon^{i+2} = 0$ holds in $\Lambda_n^{(r)}$. Since K is of characteristic 2, we have in $\Lambda_n^{(r)}$ the equality $a_0\bar{a}_0 = \varepsilon^2 + \varepsilon(a_0\bar{a}_0)^r$. Then we obtain the sequence of equalities in $\Lambda_n^{(r)}$

$$\begin{aligned}
\varepsilon(a_0\bar{a}_0)^r &= \varepsilon\big(\varepsilon^2 + \varepsilon(a_0\bar{a}_0)^r\big)(a_0\bar{a}_0)^{r-1} = \varepsilon^3(a_0\bar{a}_0)^{r-1} + \varepsilon^2(a_0\bar{a}_0)^{2r-1} \\
&= \varepsilon^5(a_0\bar{a}_0)^{r-2} + \varepsilon^4(a_0\bar{a}_0)^{2r-2} + \varepsilon^4(a_0\bar{a}_0)^{2r-2} + \varepsilon^3(a_0\bar{a}_0)^{3r-2} \\
&= \varepsilon^5(a_0\bar{a}_0)^{r-2} + \varepsilon^3(a_0\bar{a}_0)^{3r-2} = \cdots \\
&= \varepsilon^{2r+1} + \sum_{i=2r}^{2n-3}\lambda_i\varepsilon^{i+2}
\end{aligned}$$

for some elements $\lambda_{2r}, \dots, \lambda_{2n-3} \in \{0, 1\} \subseteq K$, because $\varepsilon^{2n} = 0$. Obviously then

$$a_0\bar{a}_0 + \varepsilon^2 + \varepsilon^{2r+1} + \sum_{i=2r}^{2n-3} \lambda_i \varepsilon^{i+2} = a_0\bar{a}_0 + \varepsilon^2 + \varepsilon(a_0\bar{a}_0)^r = 0$$

in $\Lambda_n^{(r)}$.

Conversely, we show that there exist elements $\lambda'_1, \dots, \lambda'_{2n-3} \in K$ such that the identity endomorphism of $KQ_{\mathbb{L}_n}$ induces an epimorphism of K-algebras $\Lambda_n^{(r)} \to L_n(\lambda'_1, \dots, \lambda'_{2n-3})$. Therefore, we have to find elements $\lambda'_1, \dots, \lambda'_{2n-3}$ in K such that the equality $\varepsilon^2 + a_0\bar{a}_0 + \varepsilon(a_0\bar{a}_0)^r = 0$ holds in $L_n(\lambda'_1, \dots, \lambda'_{2n-3})$.

Now we will construct elements $\lambda'_1, \dots, \lambda'_{2n-3} \in K$ such that the equality

$$\sum_{i=1}^{2n-3} \lambda'_i \varepsilon^{i+2} = \varepsilon^{2r+1} \left(1 + \sum_{i=1}^{2n-3} \lambda'_i \varepsilon^i\right)^r \tag{$*$}$$

holds in the quotient algebra $KQ_{\mathbb{L}_n}/(\varepsilon^{2n})$.

We note that, if we calculate the right side of equality $(*)$, we will obtain a sum of elements of the form $(\prod_j \lambda'_{i_j})\varepsilon^i$ with all indices i_j less than $i-2$. Hence, we may inductively calculate λ'_k for $k = 1, \dots, 2n-3$ from the following equalities

$$\lambda'_k \varepsilon^{k+2} + \left(\varepsilon^{k+3}\right) = \left(\varepsilon^{2r+1}\left(1 + \sum_{i=1}^{2n-3} \lambda'_i \varepsilon^i\right)^r + \sum_{i=1}^{k-1} \lambda'_i \varepsilon^i\right) + \left(\varepsilon^{k+3}\right)$$

(obtained from $(*)$) in the quotient algebras $KQ_{\mathbb{L}_n}/(\varepsilon^{k+3})$. Observe that this procedure uniquely determines the elements $\lambda'_1, \dots, \lambda'_{2n-3} \in K$ satisfying the equality $(*)$ in the quotient algebra $KQ_{\mathbb{L}_n}/(\varepsilon^{2n})$. We note also that such the chosen elements $\lambda'_1, \dots, \lambda'_{2n-3} \in K$ satisfy the conditions $\lambda'_1 = \dots = \lambda'_{2r-2} = 0$, $\lambda'_{2r-1} = 1$ and $\lambda'_{2r}, \dots, \lambda'_{2n-3} \in \{0, 1\}$, and hence $L_n(\lambda'_1, \dots, \lambda'_{2n-3}) = L_n(0, \dots, 0, 1, \lambda'_{2r}, \dots, \lambda'_{2n-3})$.

Consider now the algebra $L_n(0, \dots, 0, 1, \lambda'_{2r}, \dots, \lambda'_{2n-3})$. Observe that it is a quotient algebra of $KQ_{\mathbb{L}_n}/(\varepsilon^{2n})$. Hence, the equality $(*)$ holds also in the algebra $L_n(0, \dots, 0, 1, \lambda'_{2r}, \dots, \lambda'_{2n-3})$. Moreover, we have in $L_n(0, \dots, 0, 1, \lambda'_{2r}, \dots, \lambda'_{2n-3})$ the equality $a_0\bar{a}_0 = \varepsilon^2 + \sum_{i=1}^{2n-3} \lambda'_i \varepsilon^{i+2}$. Then we obtain the equalities

$$\begin{aligned} \varepsilon^2 + a_0\bar{a}_0 + \varepsilon(a_0\bar{a}_0)^r &= \varepsilon^2 + \left(\varepsilon^2 + \sum_{i=1}^{2n-3} \lambda'_i \varepsilon^{i+2}\right) + \varepsilon\left(\varepsilon^2 + \sum_{i=1}^{2n-3} \lambda'_i \varepsilon^{i+2}\right)^r \\ &= \sum_{i=1}^{2n-3} \lambda'_i \varepsilon^{i+2} + \varepsilon^{2r+1}\left(1 + \sum_{i=1}^{2n-3} \lambda'_i \varepsilon^i\right)^r = 0 \end{aligned}$$

in $L_n(0, \dots, 0, 1, \lambda'_{2r}, \dots, \lambda'_{2n-3})$.

Summing up, we have epimorphisms of K-algebras

$$L_n(0, \dots, 0, 1, \lambda_{2r}, \dots, \lambda_{2n-3}) \to \Lambda_n^{(r)}$$
$$\Lambda_n^{(r)} \to L_n\left(0, \dots, 0, 1, \lambda'_{2r}, \dots, \lambda'_{2n-3}\right).$$

Moreover, by Proposition 2.4, the algebras $L_n(0, \dots, 0, 1, \lambda_{2r}, \dots, \lambda_{2n-3})$ and $L_n(0, \dots, 0, 1, \lambda'_{2r}, \dots, \lambda'_{2n-3})$ are isomorphic to $L_n^{(r)}$. Therefore, the algebras $L_n^{(r)}$ and $\Lambda_n^{(r)}$ are isomorphic. □

The following proposition and its proof has been indicated by the referee.

Proposition 3.2. *Let K be of characteristic 2, $n \geqslant 2$ an integer, and $r \in \{1, \dots, n-1\}$. Then the algebras $\Lambda_n^{(r)}$ and $\underline{\mathcal{A}}(R_n^{(r)})$ are isomorphic.*

Proof. Fix $r \in \{1, \dots, n-1\}$. It follows from [27, (3.1)] that the fractional ideals

$$M_i = R_n^{(r)} + R_n^{(r)} \frac{x}{y^{n-i}}, \quad i \in \{0, 1, \dots, n-1\},$$

form a complete set of pairwise non-isomorphic indecomposable non-projective objects in $\mathrm{CM}(R_n^{(r)})$. Then there is an isomorphism of algebras $\varphi : \Lambda_n^{(r)} \xrightarrow{\sim} \underline{\mathcal{A}}(R_n^{(r)})$ which assigns to the trivial paths at the vertices i of $Q_{\mathbb{L}_n}$ the identity maps on M_i, to the arrows a_i the multiplication maps $\cdot y : M_i \to M_{i+1}$, to the arrows $\bar{a}_i$ the inclusion maps $M_{i+1} \hookrightarrow M_i$, and to the loop ε the multiplication map $\cdot \frac{x}{y^n} : M_0 \to M_0$. We note first that the stable Auslander–Reiten quiver of $\mathrm{CM}(R_n^{(r)})$ is isomorphic to $Q_{\mathbb{L}_n}$ and that the representative irreducible morphisms are given by the inclusion maps $M_{i+1} \hookrightarrow M_i$, the multiplication maps $\cdot y : M_i \to M_{i+1}$ and the multiplication map $\cdot \frac{x}{y^n} : M_0 \to M_0$. This shows that the described above homomorphism $\varphi : \Lambda_n^{(r)} \to \underline{\mathcal{A}}(R_n^{(r)})$ is an epimorphism. In order to prove that φ is a monomorphism, it is enough to show that the non-zero elements of the socle of $\Lambda_n^{(r)}$ are sent by φ to non-zero elements of $\underline{\mathcal{A}}(R_n^{(r)})$. It follows from Propositions 1.3(iii) and 3.1 that the socle of $\Lambda_n^{(r)}$ is the K-linear subspace of $\Lambda_n^{(r)}$ generated by the maximal non-zero paths $\bar{a}_{k-1} \cdots \bar{a}_0 \varepsilon^{2(n-k)-1} a_0 \cdots a_{k-1}$ from k to k, for $k \in \{0, 1, \dots, n-1\}$. Further, for $k \in \{0, 1, \dots, n-1\}$, $\varphi(\bar{a}_{k-1} \cdots \bar{a}_0 \varepsilon^{2(n-k)-1} a_0 \cdots a_{k-1})$ is the stable class of the multiplication map $\cdot y^k t^{2(n-k)-1} : M_k \to M_k$, with $t = \frac{x}{y^n}$. Moreover, if $\cdot y^k t^{2(n-k)-1}$ factors through a projective module in $\mathrm{CM}(R_n^{(r)})$, then $y^k t^{2(n-k)-1} \in R_n^{(r)}$. On the other hand, using the identity $t^2 = y + y^r t$ (which is obtained by dividing $x^2 = y^{2n+1} + xy^{n+r}$ by y^{2n}), we deduce by induction on $j \in \{1, \dots, n\}$, that simultaneously we have

(a) $y^{n-j} t^{2j-1} \notin R_n^{(r)}$;
(b) $y^{n-j} t^{2j} \in R_n^{(r)}$;
(c) $y^{n-j+1} t^{2j-1} \in R_n^{(r)}$.

In particular, for $j = n-k$, we conclude from (a) that $y^k t^{2(n-k)-1} \notin R_n^{(r)}$. Therefore, $\varphi : \Lambda_n^{(r)} \to \underline{\mathcal{A}}(R_n^{(r)})$ is a monomorphism, and hence an isomorphism. □

Theorem 3 is a direct consequence of Propositions 3.1 and 3.2.

We note (as pointed out by the referee) that in the stable category $\underline{\mathrm{CM}}(R)$ of the category $\mathrm{CM}(R)$ of maximal Cohen–Macaulay modules over a simple plane curve singularity R there are bifunctorial isomorphisms

$$\mathrm{Hom}_{\underline{\mathrm{CM}}(R)}(\underline{M}, \underline{N}) \cong D\,\mathrm{Hom}_{\underline{\mathrm{CM}}(R)}(\underline{N}, \underline{M})$$

for any modules M, N in $\mathrm{CM}(R)$ (see [9, (9.7)]). In particular, for the direct sum U_R of a complete set of pairwise non-isomorphic indecomposable non-projective objects in $\mathrm{CM}(R)$ we obtain that

$$\underline{\mathcal{A}}(R) = \mathrm{End}_{\underline{\mathrm{CM}}(R)}(\underline{U_R}) \quad \text{and} \quad D\,\mathrm{End}_{\underline{\mathrm{CM}}(R)}(\underline{U_R}) = D\underline{\mathcal{A}}(R)$$

are isomorphic as $\underline{\mathcal{A}}(R)$-bimodules, and consequently the stable Auslander algebra $\underline{\mathcal{A}}(R)$ of R is a symmetric algebra. This, together with Theorems 2 and 3, provides the proof of Corollary 4.

In the forthcoming paper [8] we will provide a direct proof of Corollary 4.

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