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Classification of Two-Dimensional F-Regular and F-Pure Singularities

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1. INTRODUCTION

The notions of F-regularity and F-purity for rings of characteristic $p > 0$,

but also via singularity theory. In [MS], Mehta and Srinivas studied two-dimensional normal F-pure singularities and proved that such a singularity is either (a) a simple elliptic singularity with an ordinary elliptic exceptional curve, (b) a cusp singularity, or (c) a rational singularity. Moreover, they showed that a singularity of type (a) or (b) is F-pure (see also [W1] for the Gorenstein case). But being a rational singularity is not sufficient to be F-pure.

The aim of this paper is to study two-dimensional F-regular and rational F-pure singularities and to complete the classification of these singularities in terms of the dual graph of the minimal resolution and characteristic p. To do this we heavily use the result of Kei-ichi Watanabe [W3], which tells us that F-regular (resp. F-pure) normal surface singularities are log terminal (resp. log canonical). Then we can apply the classification of the dual graphs of two-dimensional log terminal and log canonical singularities [Wk], and a refinement of the method in [MS] enables us to reduce our problem to the case of graded rings [W2].

Let $(A, \mathfrak{m})$ be a two-dimensional Noetherian normal local ring containing an algebraically closed field k of characteristic $p > 0$ such that $A/\mathfrak{m} = k$. Let $\pi: X \to Y = \mathrm{Spec}(A)$ be the minimal resolution of the singularity. (Note that a surface singularity has a resolution even in positive characteristic [Li2].) We call the dual graph of the exceptional divisor $E = \pi^{-1}(\mathfrak{m})$ simply the graph of the singularity of Y. If the graph is star-shaped with r branches, then we can associate to a branch of length l a natural number d such that $(-1)^l d$ is the determinant of the intersection matrix of the

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branch. The r-ple $(d_1, d_2, \ldots, d_r)$, $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_r$, of these numbers is called the *type* of the star-shaped graph.

Under this assumption and notation, our results are described as follows:

THEOREM (1.1). *A is F-regular if and only if* $Y = \mathrm{Spec}(A)$ *has only a rational singularity and one of the following holds*:

(1) *The graph of the singularity is a chain.*

(2) *The graph is star-shaped and either*

(i) *of type* $(2, 2, d)$, $d \geqslant 2$ *and* $p \neq 2$,

(ii) *of type* $(2, 3, 3)$ *or* $(2, 3, 4)$ *and* $p > 3$, *or*

(iii) *of type* $(2, 3, 5)$ *and* $p > 5$.

THEOREM (1.2). *Assume that* $Y = \mathrm{Spec}(A)$ *has only a rational singularity. If A is F-pure, then one of the following holds*:

(1) *A is F-regular.*

(2) *A is a rational double point, and either*

(i′) *the graph is* D_{n+2}, $n \geqslant 2$, *and* $p = 2$,

(ii′) *the graph is* E_6 *or* E_7, *and* $p = 2$ *or* 3, *or*

(iii′) *the graph is* E_8, *and* $p = 2$, 3, *or* 5.

(3) *The graph is star-shaped and either*

(iv) *of type* $(3, 3, 3)$ *or* $(2, 3, 6)$, *and* $p \equiv 1 \pmod 3$,

(v) *of type* $(2, 4, 4)$, *and* $p \equiv 1 \pmod 4$, *or*

(vi) *of type* $(2, 2, 2, 2)$, $p = 2n + 1$ *for some integer n, and if the branches intersect the central curve* E_0 *at points* $0, \infty, -1, \lambda$ $(0, -1 \neq \lambda \in k)$ *under some coordinate change of* $E_0 \cong \mathbf{P}^1$, *then the coefficient of* x^n *in the expansion of* $(x+1)^n (x-\lambda)^n$ *is not zero.*

(4) *The graph is* ${}_*\tilde{D}_{n+3}$, $n \geqslant 2$ *(see Fig. 1 in (4.6)), and* $p \neq 2$.

Conversely if (1), (3), *or* (4) *holds, then A is F-pure.*

Remark (1.3). (i) There are both of F-pure and non F-pure cases for (i′), (ii′), and (iii′) of Theorem (1.2) (2). But Artin has completely classified the defining equations of rational double points in characteristic $p = 2, 3, 5$ [A3], and we can determine whether a rational double point with a given defining equation is F-pure or not using Fedder's criterion for F-purity [F1] (see also (2.2)). In this sense Theorem (1.2), together with the results in [MS], gives a complete classification of F-pure normal surface singularities in characteristic $p > 0$.

(ii) The graphs which appear in Theorem (1.1) (resp. Theorem (1.2)) exactly correspond to those of log terminal (resp. rational log canonical) singularities. We list up these graphs in the Appendix following Kimio Watanabe [Wk].

(iii) Srinivas gave a classification of normal surface singularities of F-pure type in characteristic zero [Sr]. On the other hand, results in characteristic $p>0$ are found in [MS, W1, W2].

(iv) There is another concept defined for rings of characteristic $p>0$, namely, the notion of "F-rational" rings [FW]. F-rationality is the right one which corresponds to rational singularity [F2, Sm, Ha2].

2. PRELIMINARIES

Let A be a Noetherian ring of characteristic $p>0$ and $F: A \to A$ be the Frobenius map defined by $F(a)=a^p$. We always use the letter q for a power p^e of p. The ring A viewed as an A-module via the map $F^e: A \to A$ is denoted by ${}^eA$. For simplicity, we always assume that A is reduced and F-finite, i.e., $F: A \to {}^1A$ is a finite ring extension. We can identify the map $F^e: A \to {}^eA$ with the inclusion $A \hookrightarrow A^{1/q}$.

Definition (2.1). (i) [HH2]. A is said to be *strongly F-regular* if for any element $c \in A$ which is not contained in any minimal prime ideal of A, there exists $q=p^e$ such that the A-linear map $A \to A^{1/q}$ defined by $a \mapsto c^{1/q}a$ is a split injective map.

(ii) [HR]. A is said to be *F-pure* if for every A-module M, the map $1_M \otimes F: M = M \otimes_A A \to M \otimes_A {}^1A$ is injective. In our assumption this is equivalent to saying that $F: A \to {}^1A$ splits as an A-linear map.

The notion of strong F-regularity is easier to treat geometrically than that of "weak F-regularity," which is the original form defined by using the notion of "tight closure" [HH1]. We know the implications "regular $\Rightarrow$ strongly F-regular $\Rightarrow$ weakly F-regular $\Rightarrow$ F-pure" and that strong and weak F-regularities are exactly the same if the ring A is Gorenstein (we hope that this is true in the absence of Gorensteinness). Recently Williams [Wi] proved that strong F-regularity and weak F-regularity coincide for two- and three-dimensional cases under some weak condition: The ring A is an F-finite local domain which is a homomorphic image of a Gorenstein ring. It turns out that for our Theorems (1.1) and (1.2) the two notions are equivalent. In these cases we will say simply that the ring is *F-regular* if it has the above equivalent properties.

For hypersurface singularities, the following criteria are known (cf. [Ha1]).

PROPOSITION (2.2) [F1, G]. *Let S be a regular local ring of characteristic $p>0$ with maximal ideal $\mathfrak{m}$. We denote by $\mathfrak{m}^{[q]}$ the ideal of S generated by the qth powers of the elements of $\mathfrak{m}$. Let f be a non-zero element of S, and put $A=S/(f)$.*

(i) *The ring A is F-pure if and only if $f^{p-1}\notin\mathfrak{m}^{[p]}$.*

(ii) *A is F-regular if and only if for every $g\in S\backslash(f)$, there exists $q=p^e$ such that $f^{q-1}g\notin\mathfrak{m}^{[q]}$.*

(2.3) Next we will review the above notions from a geometrical view point. Let X be a scheme over a perfect field k of characteristic $p>0$. Again we will use the letter F to denote the absolute Frobenius morphism $X\to X$. F will also denote the associated map $\mathcal{O}_X\to F_*\mathcal{O}_X$ etc., if there is no fear of confusion. For an invertible sheaf $\mathcal{L}$ on X and for an integer $n>0$, we denote by $\mathcal{L}^n$ (resp. $\mathcal{L}^{-n}$) the n-times tensor product (resp. the $\mathcal{O}_X$-dual of the n-times tensor product) of $\mathcal{L}$.

We say that X is *Frobenius split*, or *F-split* for short if the map $F\colon\mathcal{O}_X\to F_*\mathcal{O}_X$ is a split injective map of $\mathcal{O}_X$-modules (see [MR]). If X is a d-dimensional Gorenstein variety with canonical sheaf ω_X, then we have an isomorphism of $\mathcal{O}_X$-modules

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X,\mathcal{O}_X)&\cong\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X,\omega_X)\otimes\omega_X^{-1}\\&\cong F_*(\omega_X)\otimes\omega_X^{-1}\cong F_*(\omega_X^{1-p})_X\end{aligned}$$

by the adjunction formula. This isomorphism enables us to identify the natural map $\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X,\mathcal{O}_X)\to\mathcal{O}_X$ with a map $F_*(\omega_X^{1-p})\to\mathcal{O}_X$ which coincides with the one induced by the Cartier operator $F_*(\omega_X)\to\omega_X$ on the smooth locus of X [C, MR]. Thus X is F-split if and only if the map

$$H^0(X,\omega_X^{1-p})\to H^0(X,\mathcal{O}_X)$$

is surjective. By the duality it is equivalent to the injectivity of the Frobenius map $F\colon H^d(X,\omega_X)\to H^d(X,\omega_X^p)$.

If $Y\subset X$ is an effective Cartier divisor, then the maps considered above and the adjunction map $\omega_X(Y)\to\omega_Y$ give a commutative diagram

$$\begin{array}{ccccc}H^0(X,\omega_X(Y)^{1-p})&\hookrightarrow&H^0(X,\omega_X^{1-p})&\to&H^0(X,\mathcal{O}_X)\\\downarrow&&&&\downarrow\\H^0(Y,\omega_Y^{1-p})&&\longrightarrow&&H^0(Y,\mathcal{O}_Y).\end{array}\tag{2.3.1}$$

Now let X be a d-dimensional normal projective variety over a field k of characteristic $p>0$, and let K_X be its canonical divisor (we use the words "canonical divisor K_X" and "canonical sheaf $\omega_X=\mathcal{O}_X(K_X)$" interchangeably).

In [W2], Watanabe gave interesting criteria for F-regularity and F-purity for a normal graded ring R such that $\mathrm{Proj}(R)=X$ using its Demazure representation. Taking it into account we define the following notions.

DEFINITION (2.4). Let D be an effective $\mathbf{Q}$-Weil divisor on X as above, and let the coefficient of D in every irreducible component be less than 1.

(i) We say that the pair (X, D) is F-*split* if the Frobenius morphism,

$$F\colon H^d(X, \mathcal{O}_X(K_X)) = H^d(X, \mathcal{O}_X(K_X+D)) \\ \to H^d(X, \mathcal{O}_X(p(K_X+D))),$$

is injective. Even when X is not normal, we say that (X, D) satisfying this condition is F-split if X is Gorenstein and each irreducible component of D is a reduced Cartier divisor.

(ii) We say that (X, D) is *strongly F-split* if for every $n>0$ and for every nonzero $f \in H^0(X, \mathcal{O}_X(nH))$, there exists $e>0$ such that the map

$$H^d(X, \mathcal{O}_X(K_X)) \xrightarrow{F^e} H^d(X, \mathcal{O}_X(q(K_X+D))) \\ \xrightarrow{f\cdot} H^d(X, \mathcal{O}_X(q(K_X+D)+nH))$$

is injective, where H is an ample Cartier divisor on X. This definition makes sense since the injectivity of the above map does not depend on the choice of H (see [Ha1], Section 4).

Remark (2.5). (i) Definition (2.4) (i) is an extension of the notion of F-splitting for X, i.e., X is F-split if and only if the pair $(X, 0)$ is F-split.

(ii) Theorem 3.3 of [W2] asserts that if D is an ample $\mathbf{Q}$-Cartier divisor with the "fractional part" D', then the normal graded ring $R = R(X, D)$ is F-pure (resp. strongly F-regular) if and only if (X, D') is F-split (resp. strongly F-split). In particular, F-purity (resp. strong F-regularity) of R depends only on X and D', and not on individual D.

Using these terminologies, we have the following restatement of the classification of two-dimensional F-regular and F-pure normal *graded* rings [W2], which is verified by computing Čech cohomology.

PROPOSITION (2.6) [W2]. *Let D' be a $\mathbf{Q}$-divisor on a nonsingular rational curve $\mathbf{P}^1$ over an algebraically closed field k of characteristic $p>0$ of the form*

$$D' = \sum_{i=1}^{r} \frac{d_i - 1}{d_i} P_i.$$

We will denote D' by $(P_1, ..., P_r; d_1, ..., d_r)$ and if $r \leqslant 3$, we will denote D' simply by $(d_1, ..., d_r)$, since there is no need to distinguish the P_i's except for case (vi):

(1) $(\mathbf{P}^1, D')$ *is strongly F-split if and only if $r \leqslant 2$ or one of the following holds*:

(i) $D' = (2, 2, d)$, $d \geqslant 2$, *and* $p \neq 2$.

(ii) $D' = (2, 3, 3)$ *or* $(2, 3, 4)$ *and* $p > 3$.

(iii) $D' = (2, 3, 5)$ *and* $p > 5$.

(2) $(\mathbf{P}^1, D')$ *is F-split if and only if it is strongly F-split or one of the following holds*:

(iv) $D' = (3, 3, 3)$ *or* $(2,3,6)$ *and* $p \equiv 1 \pmod 3$.

(v) $D' = (2, 4, 4)$ *and* $p \equiv 1 \pmod 4$.

(vi) $D' = (\infty, 0, -1, \lambda; 2, 2, 2, 2)$ *with* $\lambda \in k$, $\lambda \neq 0, -1$, *and* $p = 2n+1$ *such that the coefficient of x^n in the expansion of $(x+1)^n (x-\lambda)^n$ is not zero.*

3. LEMMATA ON TWO-DIMENSIONAL SINGULARITIES

In this section we will review some elementary results on normal two-dimensional singularities.

(3.1) Throughout this section we fix the notations as follows: Let $(A, \mathfrak{m})$ be a two-dimensional Noetherian normal local ring containing an algebraically closed field k of any characteristic such that $A/\mathfrak{m} = k$, and (Y, y) be the associated singularity; i.e., $y = \mathfrak{m} \in Y = \mathrm{Spec}(A)$. Let $\pi: X \to Y$ be the minimal resolution of the singularity in the sense that the exceptional set $E = \pi^{-1}(y)$ contains no exceptional curve of the first kind. Let $E = E_1 \cup \cdots \cup E_n$ be the decomposition into irreducible components. We associate to the resolution π a graph with vertices v_i corresponding to irreducible components E_i of E, and with edges joining vertices v_i and v_j corresponding to intersection points of E_i and E_j. A vertex v_i (resp. E_i) is said to be a center (resp. a central curve) if either v_i is joined with at least three other vertices, or E_i is not a smooth rational curve. We say that the graph is *star-shaped* if it has at most one center.

A $\mathbf{Q}$-divisor on X is a linear combination $D = \sum a_i D_i$ of prime divisors with coefficients $a_i \in \mathbf{Q}$. The integral part $[D]$ and the roundup $\ulcorner D \urcorner$ of D are defined by

$$[D] = \sum [a_i] D_i, \qquad \ulcorner D \urcorner = \sum \ulcorner a_i \urcorner D_i,$$

where $[a]$ (resp. $\ulcorner a \urcorner$) denotes the greatest integer smaller than or equal to a (resp. the least integer greater than or equal to a).

We define a **Q**-divisor $\Delta = -\sum a_i E_i$ by the equations

$$\sum_{i=1}^{n} a_i (E_i \cdot E_j) = (K_X \cdot E_j) \qquad \text{for} \quad j = 1, ..., n. \tag{3.1.1}$$

Then $\Delta \geqslant 0$ since the resolution is minimal.

DEFINITION (3.2) [Z]. A *Zariski decomposition* of a **Q**-divisor D on X is a decomposition $D = P + N$ satisfying the following properties:

(i) P is a **Q**-divisor such that $P \cdot E_i \geqslant 0$ for all i.

(ii) N is an effective **Q**-divisor supported in E.

(iii) If E_i is a component of N, then $P \cdot E_i = 0$.

An integral divisor on X has a unique Zariski decomposition [Fu, Sa].

LEMMA (3.3) [Sa]. *Let D be an integral divisor on X, and $D = P + N$ be a decomposition satisfying properties* (i) *and* (ii) *in* (3.2). *Then*

$$R^1 \pi_* \mathcal{O}_X(K_X + D - [N]) = 0.$$

The proof of (3.3) can be found in [Sa, Theorem A.2], in which $D = P + N$ is assumed to be the Zariski decomposition, but condition (iii) is unnecessary.

(3.4) Now for a divisor D on X we construct a "computation sequence" of cycles $0 = Y_0 < Y_1 < \cdots < Y_\nu < \cdots$ as follows (cf. [La1]):

(a) $Y_0 := 0$.

(b) If $(D - Y_\nu \cdot E_i) < 0$ for some i, then set $Y_{\nu+1} = Y_\nu + E_i$.

(c) If $(D - Y_\nu \cdot E_i) \geqslant 0$ for all i, then stop the process.

The choice of E_i in (b) may not be unique, but we can easily see that any computation sequence terminates at the minimal integral effective cycle $Z = \sum r_i E_i$ satisfying the property

$$(D - Z \cdot E_i) \geqslant 0 \qquad \text{for all } i.$$

We call Z the *fundamental cycle* for D and denote it by Z_D.

LEMMA (3.5). *Let D be a divisor on X and Z_D be the fundamental cycle for D. If Z is an effective cycle such that $Z \leqslant Z_D$, then*

$$H^0(X, \mathcal{O}_X(D - Z)) \cong H^0(X, \mathcal{O}_X(D)).$$

In particular, if $D = P + N$ *is the Zariski decomposition, then*

$$H^0(X, \mathcal{O}_X([P])) \cong H^0(X, \mathcal{O}_X(D)).$$

Proof. Let $\{Y_\nu\}$ be a computation sequence for D. If $Y_\nu < Z$, then $\deg_{E_i} \mathcal{O}_{E_i}(D - Y_\nu) = (D - Y_\nu \cdot E_i) < 0$ and $Y_{\nu+1} = Y_\nu + E_i$ for some i. Hence by the exact sequence

$$0 \to \mathcal{O}_X(D - Y_{\nu+1}) \to \mathcal{O}_X(D - Y_\nu) \to \mathcal{O}_{E_i}(D - Y_\nu) \to 0,$$

we have $H^0(X, \mathcal{O}_X(D - Y_{\nu+1})) \cong H^0(X, \mathcal{O}_X(D - Y_\nu))$. By induction on ν, we obtain $H^0(X, \mathcal{O}_X(D - Z_D)) \cong H^0(X, \mathcal{O}_X(D))$. Now to prove $H^0(X, \mathcal{O}_X([P])) \cong H^0(X, \mathcal{O}_X(D))$, it is sufficient to show $D - [P] \leqslant Z_D$. We write $P - (D - Z_D) = Z_D - N$ as $\Gamma_+ - \Gamma_-$, where both Γ_+ and Γ_- are effective **Q**-divisors and have no common components. Then $\mathrm{Supp}(\Gamma_-) \subseteq \mathrm{Supp}(N)$, and $(\Gamma_+ - \Gamma_- \cdot E_i) = -(D - Z_D \cdot E_i) \leqslant 0$ for any $E_i \subseteq \mathrm{Supp}(N)$. Hence $(\Gamma_+ - \Gamma_- \cdot \Gamma_-) \leqslant 0$, so that $\Gamma_-^2 = \Gamma_+ \cdot \Gamma_- - (\Gamma_+ - \Gamma_- \cdot \Gamma_-) \geqslant 0$. Since the intersection matrix $((E_i \cdot E_j))$ is negative definite [M], we have $\Gamma_- = 0$ and $P \geqslant D - Z_D$. But $D - Z_D$ is an integral divisor, so that $[P] \geqslant D - Z_D$ as required.

COROLLARY (3.6). *Let* Δ *be the* **Q**-*divisor defined by* (3.1.1). *Then for* $m \geqslant 0$, *we have*

$$H^0(X, \mathcal{O}_X(-mK_X)) \cong H^0(X, \mathcal{O}_X(-mK_X - \ulcorner m\Delta \urcorner)).$$

Proof. Apply (3.5) to the Zariski decomposition $-mK_X = -m(K_X + \Delta) + m\Delta$.

(3.7) Next we will recall some characterization of several classes of singularities. Since our singularity (Y, y) is normal, we can consider its canonical divisor K_Y.

We say that (Y, y) is *rational* if $R^1\pi_* \mathcal{O}_X = 0$. If (Y, y) is rational, then its graph is a tree, and each irreducible component E_i of E is a smooth rational curve $\mathbf{P}^1$. Also, a rational (surface) singularity (Y, y) is **Q**-Gorenstein; i.e., mK_Y is linearly equivalent to 0 for some nonzero integer m. A *rational double point* (a rational singularity with multiplicity 2) is characterized as follows [A2]:

(Y, y) is a rational double point

$\Leftrightarrow$ (Y, y) is a Gorenstein rational singularity

$\Leftrightarrow$ the **Q**-divisor Δ defined by (3.1.1) is 0

$\Leftrightarrow$ $E_i \cong \mathbf{P}^1$ and $E_i^2 = -2$ for all i.

If (Y, y) is **Q**-Gorenstein, $mK_Y \sim 0$ for a nonzero integer m, then

$$mK_X = \pi^*(mK_Y) - m\Delta \sim -m\Delta,$$

where $\sim$ denotes the linear equivalence, and $\Delta = -\sum a_i E_i$ is the **Q**-divisor determined by (3.1.1). We usually write this equality as

$$K_X = \pi^*(K_Y) - \Delta = \pi^*(K_Y) + \sum a_i E_i. \tag{3.7.1}$$

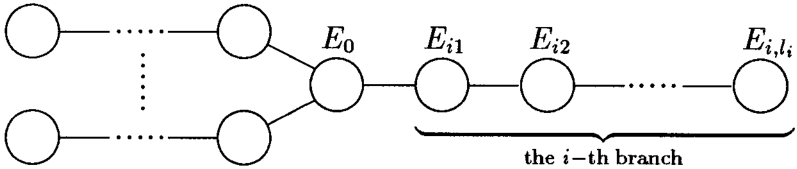
Now assume that π is the minimal good resolution in the sense that every E_i is smooth, E has only normal crossing. We say that (Y, y) is *log terminal* (resp. *log canonical*) if the following two conditions are satisfied [KMM]:

(1) (Y, y) is **Q**-Gorenstein.

(2) In (3.7.1), $a_i > -1$ (resp. $a_i \geqslant 1$) for all i.

A log terminal singularity is rational. In two-dimensional case, it is an easy consequence of (3.3). Two-dimensional log terminal and log canonical singularities are completely classified via graphs [Wk]. It is known that a normal two-dimensional singularity is log terminal (resp. log canonical) if and only if the graph associated to the minimal resolution is one of those listed in Fig. (A.1) (resp. Fig. (A.1) and (A.2)) in Appendix. In particular, every log terminal singularity has a star-shaped graph with at most three branches, and the graph associated to a rational log canonical singularity is star-shaped unless it is of type ${}_*\tilde{D}_{n+3}$ $(n \geqslant 2)$ (Fig. 1).

(3.8) We briefly review the observation in [Wk], which will be used in Section 4. Let $\pi\colon X \to Y$ be the minimal resolution of a two-dimensional rational singularity (Y, y), and assume that the graph associated to π is star-shaped with irreducible components E_0 and E_{ij} as follows:



(3.8.1)

Note that E_0 and E_{ij} are smooth rational curves since the singularity is rational.

For the ith branch, we take a $\mathbf{Q}$-divisor $D_i = \sum_{j=1}^{l_i} \alpha_{ij} E_{ij}$ on X determined by the equalities

$$\begin{aligned} -1-(K\cdot E_{i1}) &= (D_i \cdot E_{i1}), \\ -(K\cdot E_{ij}) &= (D_i \cdot E_{ij}) \qquad \text{for} \quad j \geqslant 2, \end{aligned} \tag{3.8.2}$$

where $K = K_X$ is the canonical divisor of X. Then we set

$$D := E_0 + \sum_i D_i.$$

Hereafter we employ the notation D in this sense. Note that this usage is different from that in (3.1)–(3.5).

Lemma (3.9) [Wk]. *In the above notation, let d_i be the natural number such that $(-1)^{l_i} d_i$ is the determinant of the intersection matrix of the ith branch. Then D is an effective $\mathbf{Q}$-divisor of the form*

$$D = E_0 + \sum_i \frac{d_i - 1}{d_i} E_{i1} + (\textit{terms of } E_{ij},\ j \geqslant 2)$$

and

$$\begin{aligned} (K+D\cdot E_0) &= \sum_i \frac{d_i-1}{d_i} - 2, \\ (K+D\cdot E_{ij}) &= 0 \qquad \textit{for all} \quad i,\ j \geqslant 1. \end{aligned} \tag{3.9.1}$$

Moreover, if (Y, y) is log terminal (resp. log canonical), then $(K+D\cdot E_0) < 0$ (resp. $(K+D\cdot E_0) \leqslant 0$).

Proof. Set $D_i = \sum_j \alpha_{ij} E_{ij}$ and $-b_{ij} = E_{ij}^2$. Applying the adjunction formula to (3.8.2), we have

$$\begin{pmatrix} -b_{i1} & 1 & & & \\ 1 & -b_{i2} & 1 & & \\ & 1 & -b_{i3} & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -b_{i,l_i} \end{pmatrix} \begin{pmatrix} \alpha_{i1} \\ \alpha_{i2} \\ \alpha_{i3} \\ \vdots \\ \alpha_{i,l_i} \end{pmatrix} = \begin{pmatrix} 1-b_{i1} \\ 2-b_{i2} \\ 2-b_{i3} \\ \vdots \\ 2-b_{i,l_i} \end{pmatrix}.$$

Note that the determinant of the matrix on the right-hand side is $(-1)^{l_i} d_i$. Then by Cramer's formula, it follows that

$$\alpha_{i1} = \frac{1}{d_i} \cdot \begin{vmatrix} b_{i1} & -1 & & & \\ -1 & b_{i2} & -1 & & \\ & -1 & b_{i3} & \ddots & \\ & & \ddots & \ddots & -1 \\ -1 & & & -1 & b_{i,l_i} \end{vmatrix} = \frac{d_i - 1}{d_i}.$$

To show $D \geqslant 0$, we write $D_i = D_i^+ - D_i^-$, where D_i^+ and D_i^- are effective **Q**-divisors having no common components. As $(K \cdot E_{ij}) = -2 + b_{ij} \geqslant 0$, the left-hand sides of equalities (3.8.2) are nonpositive, so that $(D_i \cdot D_i^-) \leqslant 0$. Hence, $(D_i^-)^2 = (D_i^+ \cdot D_i^-) - (D_i \cdot D_i^-) \geqslant 0$. Since the intersection matrix $((E_{ij} \cdot E_{ik}))_{1 \leqslant j, k \leqslant l_i}$ is negative definite [M], it follows that $D_i^- = 0$, and $D = E_0 + \sum_i D_i \geqslant 0$. Equalities (3.9.1) are easily checked by the adjunction formula. Now let Δ be the **Q**-divisor defined by equalities (3.1.1) and suppose $(\Delta - D \cdot E_0) = -(K + D \cdot E_0) \leqslant 0$ (resp. < 0). Then $\Delta - D$ is effective since its intersection numbers with all irreducible components of E are nonpositive, and so the coefficient of E_0 in Δ is not less than (resp. is greater than) 1, which is the coefficient of E_0 in D. This proves the last statement.

Remark (3.10). By (3.9), we can easily see that the graph of a log terminal singularity is a chain, or star-shaped of type $(2, 2, n)$ $(n \geqslant 0)$, (2, 3, 3), (2, 3, 4), or (2, 3, 5). Also, if the graph of a rational log canonical singularity is star-shaped, then it is one of those listed above, or of type (3, 3, 3), (2, 4, 4), (2, 3, 6), or (2, 2, 2, 2). This recovers the classification of graphs of those singularities (see the Appendix).

4. PROOF OF THEOREMS

To prove Theorems (1.1) and (1.2), we use the following results crucially:

(4.1) By [MS], a two-dimensional normal F-pure singularity is rational or minimally elliptic, whence **Q**-Gorenstein [A2, Li1, La2]. But it is shown by Watanabe [W3] that a strongly F-regular (resp. F-pure) normal singularity is log terminal (resp. log canonical) provided it is **Q**-Gorenstein. Thus, for two-dimensional normal singularities, we have the implications

$$\text{F-regular} \Rightarrow \text{log terminal}$$

and

$$\text{F-pure} \Rightarrow \text{log canonical.}$$

Here we used the word “F-regular” since strong and weak F-regularity coincide in two-dimensional case [Wi].

We divide the proofs of Theorems (1.1) and (1.2) into several steps.

Proof of Theorem (1.1). Since a log terminal singularity is rational with a star-shaped graph, we will assume that the singularity (Y, y) associated to A has the graph as (3.8.1). We use the notation in (3.8) and (3.9).

(4.2) We set $D' = D - E_0 = \sum_i D_i$ and define a $\mathbf{Q}$-divisor $\mathfrak{d}'$ on $E_0 \cong \mathbf{P}^1$ to be the restriction of the $\mathbf{Q}$-divisor D' on X to E_0. Then by (3.9), $\mathfrak{d}'$ can be written as

$$\mathfrak{d}' = \sum_i \frac{d_i - 1}{d_i} P_i,$$

where P_i is the point of intersection of E_0 and E_{i1}. Our first assertion is the following.

PROPOSITION (4.3). *If* $(E_0, \mathfrak{d}')$ *is strongly F-split, then* A *is F-regular.*

Proof. Let c be any nonzero element of A. We will show that the A-linear map $A \to A^{1/q}$ defined by $a \mapsto c^{1/q}a$ splits for some power $q = p^e$ of p. Let $\mathfrak{a}$ be an ample Cartier divisor on E_0 with $\mathcal{O}_{E_0}(\mathfrak{a}) = \mathcal{O}_X(-E_0) \otimes \mathcal{O}_{E_0}$, the conormal sheaf of E_0 in X, and take $n \geqslant 0$ such that $c \in H^0(X, \mathcal{O}_X(-nE_0)) \setminus H^0(X, \mathcal{O}_X(-(n+1)E_0))$ in $A = H^0(X, \mathcal{O}_X)$. Then the image $\bar{c}$ of c by the restriction map $H^0(X, \mathcal{O}_X(-nE_0)) \to H^0(E_0, \mathcal{O}_{E_0}(n\mathfrak{a}))$ is not zero, and we have the following commutative diagram for each $q = p^e$ (cf. (2.3.1)),

$$\begin{array}{ccc}
H^0(X, \omega_X(E_0)^{1-q}(-[qD'] + nE_0)) & \longrightarrow & H^0(E_0, \omega_{E_0}^{1-q}(-[q\mathfrak{d}'] - n\mathfrak{a})) \\
\downarrow & & \downarrow \\
H^0(X, \omega_X^{1-q}(nE_0)) & & H^0(E_0, \omega_{E_0}^{1-q}(-n\mathfrak{a})) \\
\downarrow{\scriptstyle c\cdot} & & \downarrow{\scriptstyle \bar{c}\cdot} \\
H^0(X, \omega_X^{1-q}) & & H^0(E_0, \omega_{E_0}^{1-q}) \\
\downarrow & & \downarrow \\
A \cong H^0(X, \mathcal{O}_X) & \longrightarrow & A/\mathfrak{m} \cong H^0(E_0, \mathcal{O}_{E_0}),
\end{array}$$

where the upper horizontal arrow is the adjunction map, and the maps $H^0(X, \omega_X^{1-q}) \to H^0(X, \mathcal{O}_X)$ and $H^0(E_0, \omega_{E_0}^{1-q}) \to H^0(E_0, \mathcal{O}_{E_0})$ are naturally induced by $H^0(X, \omega_X^{1-q}) \cong \operatorname{Hom}_{\mathcal{O}_X}(F_*^e \mathcal{O}_X, \mathcal{O}_X)$ and $H^0(E_0, \omega_{E_0}^{1-q}) \cong \operatorname{Hom}_{\mathcal{O}_{E_0}}(F_*^e \mathcal{O}_{E_0}, \mathcal{O}_{E_0})$, respectively. Then it is sufficient to show that the maps $H^0(X, \omega_X(E_0)^{1-q}(-[qD'] + nE_0)) \to H^0(E_0, \omega_{E_0}^{1-q}(-[q\mathfrak{d}'] - n\mathfrak{a}))$ and $H^0(E_0, \omega_{E_0}^{1-q}(-[q\mathfrak{d}'] - n\mathfrak{a})) \to H^0(E_0, \mathcal{O}_{E_0})$ in the above diagram are surjective for some $q=p^e$. Indeed, if so they are, then the composition map $H^0(X, \omega_X^{1-q}(nE_0)) \xrightarrow{c\cdot} H^0(X, \omega_X^{1-q}) \to H^0(X, \mathcal{O}_X)$ is also surjective, and since this surjective map can be identified with the map

$$\mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X(-nE_0), \mathcal{O}_X) \xrightarrow{c\cdot} \mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X),$$

the $\mathcal{O}_X$-homomorphism

$$\mathcal{O}_X \xrightarrow{F^e} F^e_* \mathcal{O}_X \xrightarrow{c\cdot} F^e_*(\mathcal{O}_X(-nE_0))$$

splits. Hence, by taking the sections $H^0(X-E, \ )$, we obtain a desired splitting of the map $A \xrightarrow{c\cdot} A^{1/q}$.

Concerning the surjectivity of the first map, in view of the exact sequence

$$0 \to \omega_X^{1-q}(-[qD]+nE_0) \to \omega_X(E_0)^{1-q}(-[qD']+nE_0) \\ \to \omega_{E_0}^{1-q}(-[q\mathfrak{d}']-n\mathfrak{a}) \to 0,$$

it suffices to show $H^1(X, \omega_X^{1-q}(-[qD]+nE_0))=0$. But by (3.9), inequalities $(q(K+D)-nE_0 \cdot E_0)<0$ and $(q(K+D)-nE_0 \cdot E_{ij}) \leqslant 0$ for all i, j hold if $q=p^e$ is sufficiently large, so that by applying (3.3) to the decomposition

$$-qK = (-q(K+D)+nE_0)+(qD-nE_0),$$

we get $R^1\pi_* \mathcal{O}_X((1-q)K-[qD]+nE_0) = H^1(X, \omega_X^{1-q}(-[qD]+nE_0))=0$.

To show the second surjectivity, we use the assumption that $(E_0, \mathfrak{d}')$ is strongly F-split: There exists $q=p^e$ such that the map

$$H^1(E_0, \omega_{E_0}) \xrightarrow{F^e} H^1(E_0, \omega_{E_0}^q(q\mathfrak{d}')) \xrightarrow{\bar{c}\cdot} H^1(E_0, \omega_{E_0}^q(q\mathfrak{d}'+n\mathfrak{a}))$$

is injective. Hence, the dual map

$$H^0(E_0, \omega_{E_0}^{1-q}(-[q\mathfrak{d}']-n\mathfrak{a})) \xrightarrow{\bar{c}\cdot} H^0(E_0, \omega_{E_0}^{1-q}(-[q\mathfrak{d}'])) \to H^0(E_0, \mathcal{O}_{E_0})$$

is surjective, proving the proposition.

(4.4) By (2.6), $(E_0, \mathfrak{d}')$ is strongly F-split if and only if one of conditions (1) and (i), (ii), (iii) of (2) in (1.1) holds. Considering (4.1) and Remark (3.10), it remains to show that A is *not* F-regular if the graph is star-shaped with three branches and one of the following conditions holds:

(i′) $(d_1, d_2, d_3)=(2, 2, n)$, $n \geqslant 2$, and $p=2$,

(ii′) $(d_1, d_2, d_3)=(2, 3, 3)$ or $(2, 3, 4)$ and $p=2$ or 3, or

(iii′) $(d_1, d_2, d_3)=(2, 3, 5)$ and $p=2$, 3, or 5.

Note that in the above cases, $(E_0, \mathfrak{d}')$ is not even F-split (2.6).

If (Y, y) is a rational double point, then it is a hypersurface singularity, and its defining equation is completely classified in [A3]. By applying (2.2) to each one of these defining equations, we see that all rational double points satisfying (i′), (ii′), or (iii′) are not F-regular (but they may be possibly F-pure and are listed in (1.2)).

Now suppose (Y, y) is not a rational double point and satisfies one of (i′), (ii′), and (iii′). In this case we can show that A is not even F-pure. To see this, we consider a commutative diagram,

$$\begin{array}{ccc} H^0(X, \omega_X(E_0)^{1-p}(-[pD'])) & \longrightarrow & H^0(E_0, \omega_{E_0}^{1-p}(-[p\mathfrak{d}'])) \\ \downarrow \iota & & \downarrow \\ H^0(X, \omega_X^{1-p}) & & H^0(E_0, \omega_{E_0}^{1-p}) \\ \downarrow & & \downarrow \\ A \cong H^0(X, \mathcal{O}_X) & \longrightarrow & A/\mathfrak{m} \cong H^0(E_0, \mathcal{O}_{E_0}), \end{array}$$

which is obtained by setting $n=0$ and $q=p$ in the diagram used in the proof (4.3). Then we have the following.

CLAIM (4.4.1). *The inclusion map* $\iota\colon H^0(X, \omega_X(E_0)^{1-p}(-[pD'])) \to H^0(X, \omega_X^{1-p})$ *in the above diagram is an isomorphism.*

Proof of Claim. Let Z be the fundamental cycle for $(1-p)K_X$ defined in (3.4). Then by (3.5), it suffices to show the inequality $(p-1)E_0 + [pD'] \leqslant Z$, the both sides of which are determined by the weighted dual graph of the singularity (Y, y) and the characteristic p (cf. (3.4), (3.8.2)).

First we consider case (i′). Since we are assuming that (Y, y) is not a rational double point, there is an irreducible component of $E = \pi^{-1}(y)$, say E_i, such that $E_i^2 < -2$. Then (3.4) tells us that the coefficient of Z in E_i is at least 1. This implies that so are the coefficients of Z in the neighboring components of E_i, and finally, it follows that all the coefficients of Z are at least 1. On the other hand, as one can check easily, every coefficient of D' is strictly between 0 and 1. Therefore, all the coefficients of $(p-1)E_0 + [pD'] = E_0 + [2D']$ is at most 1, whence the result.

For our remaining cases (ii′) and (iii′), we only need to consider finitely many combinations of dual graphs (see Fig. (A.1) in Appendix) and characteristics $p = 2, 3, 5$. By Computing D' and Z one by one for each of these combinations, we can verify $(p-1)E_0 + [pD'] \leqslant Z$.

Note that if (Y, y) is a rational double point, then $Z=0$, and Claim (4.4.1) does not hold.

Now suppose A is F-pure. Then the open set $U = X \backslash E \cong Y \backslash \{y\}$ of X is F-split, so that the map $H^0(U, \omega_U^{1-p}) \cong \mathrm{Hom}_{\mathcal{O}_U}(F_* \mathcal{O}_U, \mathcal{O}_U) \to H^0(U, \mathcal{O}_U)$ is surjective. On the other hand, we have $H^0(X, \omega_X^{1-p}) \cong H^0(U, \omega_U^{1-p})$, since the resolution π is minimal [Sa, Lemma 1.6, MS]. Hence the composition map $H^0(X, \omega_X^{1-p}) \to H^0(X, \mathcal{O}_X) \hookrightarrow H^0(U, \mathcal{O}_U)$ is surjective, and so is the map $H^0(X, \omega_X^{1-p}) \to H^0(X, \mathcal{O}_X)$, too. This surjectivity, together with (4.4.1), implies the surjectivity of the composition map $H^0(E_0, \omega_{E_0}^{1-p}(-[p\mathfrak{d}'])) \to H^0(E_0, \omega_{E_0}^{1-p}) \to H^0(E_0, \mathcal{O}_{E_0})$ in the above diagram, which contradicts the fact that $(E_0, \mathfrak{d}')$ is not F-split. Hence A is not F-pure.

Thus we have shown that A is not F-regular in cases (i′), (ii′), and (iii′), which completes the proof of Theorem (1.1). Q.E.D

Remark (4.5). The proof of Theorem (1.1) implies that if (Y, y) is a log terminal singularity which is not a rational double point, then $A = \mathcal{O}_{Y,y}$ is F-regular if and only if it is F-pure. But this is not true for rational double points. For example, if (Y, y) is a rational double point with dual graph E_8 (this means that the graph is star-shaped of type (2, 3, 5)), and if $p = 5$, then by [A3], (Y, y) is isomorphic to a hypersurface singularity whose defining equation is one of the following.

$$E_8^0 \colon z^2 + x^3 + y^5,$$

$$E_8^1 \colon z^2 + x^3 + y^5 + xy^4.$$

Both of the above singularities are not F-regular. But E_8^1 is F-pure while E_8^0 is not F-pure. Similarly, by using Fedder's criteria (2.2), we can determine whether each one of the rational double points in characteristic $p = 2, 3, 5$ listed in [A3, Section 3] is F-pure (resp. F-regular) or not.

Proof of Theorem (1.2). By Remark (4.5), we only need to consider the case that (Y, y) is log canonical, but is not log terminal. We keep this condition throughout the proof.

(4.6) We define divisors Δ, Δ_0, Δ', $\mathfrak{d}'$ as follows: Δ is the **Q**-divisor on X defined by the equalities (3.1.1), so that it is effective, and satisfies the equality

$$K_X = \pi^*(K_Y) - \Delta.$$

We define $\Delta_0 = [\Delta]$, $\Delta' = \Delta - \Delta_0$. By the assumption, Δ_0 is a nonzero reduced integral divisor on X. We also define $\mathfrak{d}'$ to be the restriction $\Delta'|_{\Delta_0}$ of the **Q**-divisor Δ' to Δ_0.

Note that if the graph of the singularity (Y, y) is star-shaped, then Δ, Δ_0, Δ', $\mathfrak{d}'$ coincide with the divisors D, E_0, D', $\mathfrak{d}'$ considered in the proof of Theorem (1.1), respectively. If the graph is not star-shaped, then it is of

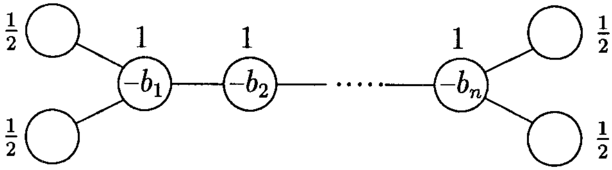


Fig. 1. The graph ${}_*\tilde{D}_{n+3}$.

type ${}_*\tilde{D}_{n+3}$, $n \geqslant 2$, and the coefficients of the irreducible components in Δ (the numbers outside the circles) are given as Fig. 1.

We will prove the following.

Proposition (4.7). *In our assumption, A is F-pure if and only if $(\Delta_0, \mathfrak{d}')$ is F-split.*

Proof. Recall the diagram in the proof of Theorem (1.1), and replace D, E_0, D', $\mathfrak{d}'$ by Δ, Δ_0, Δ', $\mathfrak{d}'$:

$$\begin{array}{ccc} H^0(X, \omega_X(\Delta_0)^{1-p}(-[p\Delta'])) & \xrightarrow{\varphi} & H^0(\Delta_0, \omega_{\Delta_0}^{1-p}(-[p\mathfrak{d}'])) \\ {\scriptstyle \iota}\downarrow & & \downarrow \\ H^0(X, \omega_X^{1-p}) & & H^0(\Delta_0, \omega_{\Delta_0}^{1-p}) \\ \downarrow & & \downarrow \\ A \cong H^0(X, \mathcal{O}_X) & \longrightarrow & A/\mathfrak{m} \cong H^0(\Delta_0, \mathcal{O}_{\Delta_0}). \end{array}$$

In this diagram, A is F-pure if and only if the map $H^0(X, \omega_X^{1-p}) \to H^0(X, \mathcal{O}_X)$ is surjective, as we have seen in the proof of Theorem (1.1), and $(\Delta_0, \mathfrak{d}')$ is F-split if and only if the composition map $H^0(\Delta_0, \omega_{\Delta_0}^{1-p}(-[p\mathfrak{d}'])) \hookrightarrow H^0(\Delta_0, \omega_{\Delta_0}^{1-p}) \to H^0(\Delta_0, \mathcal{O}_{\Delta_0})$ is surjective. For these two conditions to coincide, it is sufficient to show that the map $\varphi: H^0(X, \omega_X(\Delta_0)^{1-p}(-[p\Delta'])) \to H^0(\Delta_0, \omega_{\Delta_0}^{1-p}(-[p\mathfrak{d}']))$ is surjective and that the inclusion map ι: $H^0(X, \omega_X(\Delta_0)^{1-p}(-[p\Delta'])) \hookrightarrow H^0(X, \omega_X^{1-p})$ is an isomorphism.

Concerning the surjectivity of φ, by the exact sequence

$$0 \to \omega_X^{1-p}(-[p\Delta]) \to \omega_X(\Delta_0)^{1-p}(-[p\Delta']) \to \omega_{\Delta_0}^{1-p}(-[p\mathfrak{d}']) \to 0,$$

it suffices to show $H^1(X, \omega_X^{1-p}(-[p\Delta])) = 0$. But by the definition of Δ, $(K + \Delta \cdot E_i) = 0$ for any irreducible component E_i of E, so that the desired vanishing follows by applying (3.3) to the Zariski decomposition $-pK = -p(K + \Delta) + p\Delta$.

Concerning the surjectivity of ι, we have $H^0(X, \mathcal{O}_X((1-p)K)) \cong H^0(X, \mathcal{O}_X((1-p)K - \ulcorner (p-1)\Delta \urcorner))$ by (3.6). Hence, it suffices to show the inequality $(p-1)\Delta_0+[p\Delta']\leqslant\ulcorner(p-1)\Delta\urcorner$. By taking the integral parts of the both sides of $[p\Delta']-(p-1)\Delta'\leqslant p\Delta'-(p-1)\Delta'=\Delta'$, we have $[p\Delta']-\ulcorner(p-1)\Delta'\urcorner\leqslant[\Delta']=0$, which shows $(p-1)\Delta_0+[p\Delta']\leqslant(p-1)\Delta_0+\ulcorner(p-1)\Delta'\urcorner=\ulcorner(p-1)\Delta\urcorner$ as required.

(4.8) Thus the F-purity of A is reduced to the F-splitting of $(\Delta_0, \mathfrak{d}')$. If the graph is star-shaped, then $\Delta_0\cong\mathbf{P}^1$ is the central curve, and $\mathfrak{d}'$ is a $\mathbf{Q}$-divisor on $\mathbf{P}^1$ of the form $\sum_i((d_i-1)/d_i)P_i$ (3.9). In this case, the statements in Theorem (1.2) immediately follow from (2.6).

To complete the proof, it remains to show the following.

Claim (4.8.1). *If the graph is* $_*\tilde{D}_{n+3}$ *and* $n\geqslant 2$, *then* $(\Delta_0, \mathfrak{d}')$ *is F-split if and only if* $p\neq 2$.

Proof. First note that the degree map $\operatorname{Pic}(\Delta_0)\to\mathbf{Z}^n$ is isomorphic (n is the number of irreducible components of Δ_0), since our singularity is rational [A1].

If $p=2$, then the total degree of the invertible sheaf $\omega_{\Delta_0}^{1-p}(-[p\mathfrak{d}'])=\omega_{\Delta_0}^{-1}(-[2\mathfrak{d}'])$ is -2, whence $H^0(\Delta_0, \omega_{\Delta_0}^{1-p}(-[p\mathfrak{d}']))=0$, and $(\Delta_0, \mathfrak{d}')$ cannot be F-split.

Now suppose $p>2$, and let $E_0\cong\mathbf{P}^1$ be one of the two central curves of Δ. Then we have $\omega_{\Delta_0}^{1-p}(-[p\mathfrak{d}'])\cong\mathcal{O}_{\Delta_0}$, since the degree of the right-hand side is $(0, ..., 0)\in\mathbf{Z}^n$, so that the restriction map $H^0(\Delta_0, \omega_{\Delta_0}^{1-p}(-[p\mathfrak{d}']))\to H^0(E_0, \omega_{\Delta_0}^{1-p}(-[p\mathfrak{d}'])\otimes\mathcal{O}_{E_0})$ is an isomorphism as well as $H^0(\Delta_0, \mathcal{O}_{\Delta_0})\to H^0(E_0, \mathcal{O}_{E_0})=k$. Hence the map $H^0(\Delta_0, \omega_{\Delta_0}^{1-p}(-[p\mathfrak{d}']))\to H^0(\Delta_0, \mathcal{O}_{\Delta_0})$ can be identified with

$$H^0\left(\mathbf{P}^1, \omega_{\mathbf{P}}^{1-p}\left(-\left[p\left(\frac{p-1}{p}(\infty)+\frac{1}{2}(0)+\frac{1}{2}(-1)\right)\right]\right)\right)\to H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}}).$$

This map is surjective since $(\mathbf{P}^1, ((p-1)/p)(\infty)+\frac{1}{2}(0)+\frac{1}{2}(-1))$ is F-split by (2.6).

Thus $(\Delta_0, \mathfrak{d}')$ is F-split if and only if $p\neq 2$, and the proof of Theorem (1.2) is complete. Q.E.D

As corollaries of Theorems (1.1) and (1.2), we have the following.

Corollary (4.9). *Let* $A=\mathcal{O}_{Y,y}$ *be the local ring of a normal surface singularity* (Y, y) *of characteristic* $p>5$. *Then* A *is F-regular if and only if* (Y, y) *is log terminal.*

Corollary (4.10). *Let* $A=\mathcal{O}_{Y,y}$ *be the local ring of a normal surface singularity* (Y, y) *of characteristic* $p>0$ *with a star-shaped graph, and let* R *be a two-dimensional normal graded ring over a field of the same characteristic* *p such that the dual graph of the minimal resolution of* Spec(R) *is the same as that of* (Y, y). *Then*:

(1) *A is F-regular if and only if so is R.*

(2) *If R is F-pure, then so is A. If A is not a rational double point, the converse is also true.*

APPENDIX: DUAL GRAPHS OF LOG TERMINAL AND LOG CANONICAL SINGULARITIES

We will list up those which appear as the weighted dual graph of the exceptional divisor E of the minimal resolution of a log terminal (resp. log canonical) singularity (Fig. A.1) following [Wk]. In the figures below,

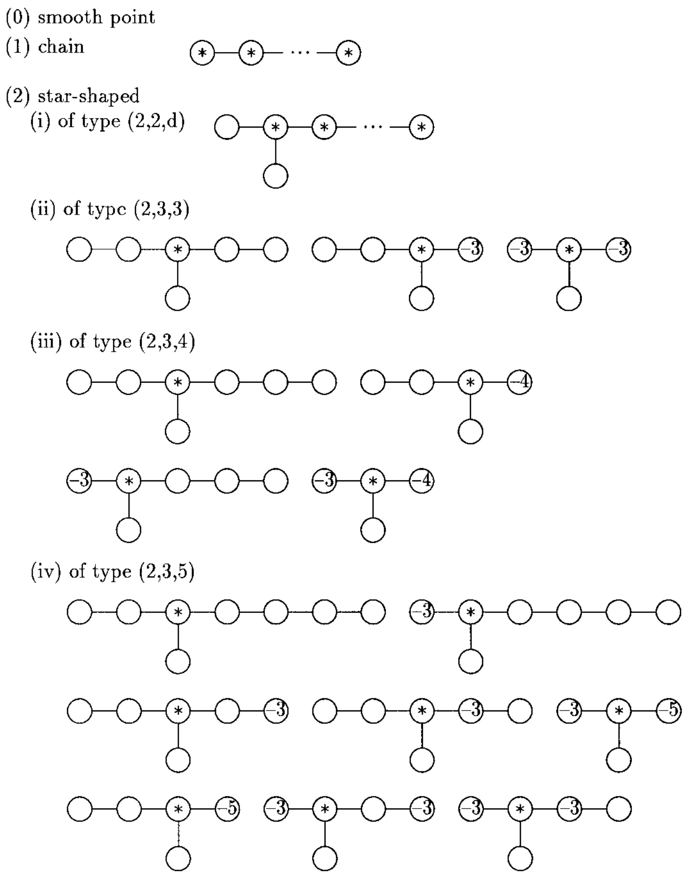


Fig. A.1. Log terminal singularities.

a circle will mean a vertex of a dual graph which corresponds to an irreducible component $E_i \cong \mathbf{P}^1$ of E, and a line joining vertices will mean normal crossing intersection at one point. The number in the circle is the self-intersection number E_i^2 of E_i. A (-2)-curve is denoted by a blank circle. Also the symbol $*$ in a circle will mean any self-intersection number $\leqslant -2$ which makes the intersection matrix $((E_i \cdot E_j))$ of E negative definite.

A surface singularity is log canonical (Fig. A.2) if and only if it is a simple elliptic singularity, a cusp singularity, or a rational singularity and its graph is one of the following.

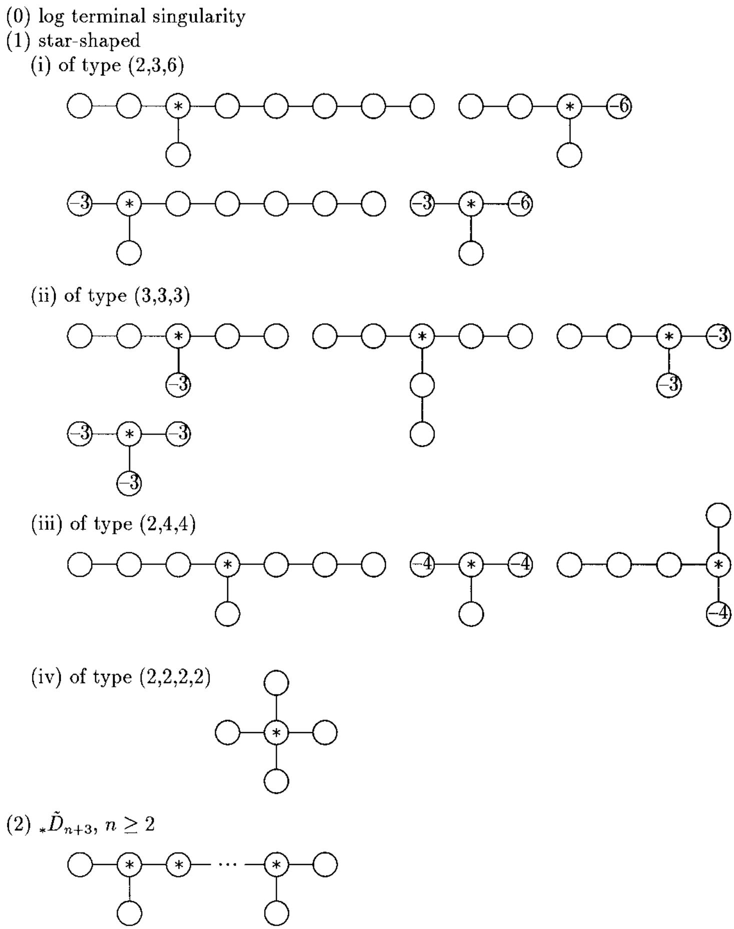


Fig. A.2. Rational log canonical singularities.

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