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Oscillation Results for Emden–Fowler Type Differential Equations

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The third order nonlinear differential equation

$$x''' + a(t)x' + b(t)f(x) = 0 \qquad (*)$$

is considered. We present oscillation and nonoscillation criteria which extend and improve previous results existing in the literature, in particular some results recently stated by M. Greguš and M. Greguš, Jr., (*J. Math. Anal. Appl.* **181**, 1994, 575–585). In addition, contributions to the classification of solutions are given. The techniques used are based on a transformation which reduces $(*)$ to a suitable disconjugate form. To this aim auxiliary results on the asymptotic behavior of solutions of a second order linear differential equation associated to $(*)$ are stated. They are presented in an independent form because they may be applied also to simplify and improve other qualitative problems concerning differential equations with quasiderivatives.

INTRODUCTION

The aim of this paper is to study the oscillatory and nonoscillatory behavior of the nonlinear differential equation

$$x''' + a(t)x' + b(t)f(x) = 0, \qquad (1)$$

where

$$a, b \in C(J),\ J = [0, \infty),\ b(t) > 0 \text{ except at isolated points},$$
$$f \in C(\mathbb{R}),\ u \cdot f(u) > 0 \text{ for } u \neq 0.$$

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In some cases, concerning the nonlinearity, the following hypotheses will be also assumed (not necessarily all together):

$$f \text{ nondecreasing for } |u| \text{ large enough;} \tag{H_1}$$

$$\lim_{u \to 0} \frac{f(u)}{u} = \theta, \qquad 0 \le \theta < \infty. \tag{H_2}$$

We recall that a nontrivial continuable solution of Eq. (1) is said to be *oscillatory* if it has infinitely large zeros, *nonoscillatory* otherwise. Equation (1) is said to be *nonoscillatory* if all its solutions are nonoscillatory, *oscillatory* otherwise. A prototype of Eq. (1) is the Emden–Fowler equation

$$x''' + a(t)x' + b(t)|x|^{\alpha} \operatorname{sgn} x = 0 \qquad (\alpha > 0). \tag{2}$$

Oscillation and nonoscillation of Eq. (1) or (2) has been considered by many authors with some additional assumptions on the function f [2–4, 10, 11, 14–17, 21, 23–25]. By a suitable transformation preserving zeros of solutions (see, e.g., [27]), the complete equation

$$z''' + a_1(t)z'' + a_2(t)z' + a_3(t)f(z) = 0 \tag{3}$$

may be written in the form of Eq. (1). Hence Eq. (1) is not much less general than Eq. (3) as regards the oscillation and nonoscillation.

A classical approach in the study of the qualitative behavior of solutions of (1) is based on a suitable transformation, associated to a disconjugate differential operator, which reduces (1) to an equation of the type

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'\right)'\right)' + b(t)f(x) = 0, \tag{4}$$

where $p \in C^1(J)$, $r \in C^2(J)$, $p(t) > 0$, $r(t) > 0$. If

$$\int_{t_0}^{\infty} p(t)\,dt = \int_{t_0}^{\infty} r(t)\,dt = \infty, \tag{5}$$

then (4) is said to be in the canonical form [28]. The divergence of the integrals of the functions p and r plays an important role in the study of nonoscillation of Eq. (4). Indeed if (5) is satisfied, then Eq. (4) has very interesting properties. For example it is possible to classify the nonoscillatory solutions of (1) in a very simple way. In the linear case we can also give necessary and sufficient criteria for the nonoscillation which are useful in the study of the nonlinear oscillation via a linearization device. A discussion on these topics is given at the beginning of Section 3.

Equation (1) may be written in the disconjugate form (4) if the second order comparison equation

$$y'' + a(t)y = 0 \tag{6}$$

is nonoscillatory. Nevertheless, the question whether (1) may be written in the canonical form is still open. A partial answer is given in [3] by assuming, in addition to other assumptions, that $\lim_{t\to\infty} a(t) = c < 0$. In this paper we give other sufficient conditions in order for (1) to be written in the canonical form, which extend those quoted in [3]. Such a result is employed to improve and generalize some recent oscillatory and nonoscillatory results in [17]. Indeed in [17] Greguš and Greguš, using a technique already employed in [4], have given some oscillation and nonoscillation results for Eqs. (1) and (2). For example, the following results are proved:

THEOREM A [17, THEOREM 5]. *Assume* (H_2) *and*

(i) $a \in C^1((0,\infty))$, $a(t) \geq 0$, $a'(t) \leq 0$; $b \in C((0,\infty))$, $b(t) > 0$, $\int^\infty tb(t)\,dt = \infty$;

(ii) *the linear differential Eq.* (6) *is disconjugate on* $(0,\infty)$, *that is, every nontrivial solution of Eq.* (6) *has at most one zero on* $(0,\infty)$.

Then every bounded continuable solution of Eq. (1) *with a zero at some point* $t_1 > 0$ *is oscillatory.*

THEOREM B [17, THEOREM 3]. *Assume*

(i) $a \in C^1((0,\infty))$, $a(t) \geq 0$, $a'(t) \leq 0$; $b \in C((0,\infty))$, $b(t) > 0$, $\int^\infty t^2[b(t) - a'(t)]\,dt < \infty$;

(ii) *the linear differential Eq.* (6) *is disconjugate on* $(0,\infty)$.

Then each bounded continuable solution of Eq. (2) *with* $\alpha > 1$ *is nonoscillatory.*

In this paper we consider Eqs. (1) and (2) without sign or monotonicity or regularity conditions on the function a as assumed in [17]. We obtain oscillatory and nonoscillatory criteria that extend and improve previous ones stated in [17], in particular Theorems A and B quoted above. In addition, contributions to the classification of solutions are given, which are related with some results contained in the book [20], as well as in the papers [2, 24, 26]. For a wide bibliography on this last argument we refer the reader in particular to the quoted paper [2]. Finally the obtained results are extended to the nonlinear general equation (4). We also note that our results do not require that the perturbation f satisfies hypotheses on superlinearity and/or sublinearity in the whole domain $\mathbb{R}$. Relationships and comparisons with other known results (in particular [2–4, 23]) will be pointed out throughout the paper.

The approach used is based on a suitable transformation and a linearization device. In particular Eq. (1) is transformed into the equation

$$\left(h^2(t)\left(\frac{1}{h(t)}x'\right)'\right)' + h(t)b(t)f(x) = 0, \tag{1$'$}$$

where h is a positive nonoscillatory solution of Eq. (6) satisfying

$$\int_0^\infty \frac{1}{h^2(t)}\,dt = \infty, \qquad \int_0^\infty h(t)\,dt = \infty, \lim_{t\to\infty} h(t) > 0. \tag{7}$$

To prove the existence of such a solution of (6), we need to state some auxiliary results on the asymptotic behavior of solutions of Eq. (6) when the function a does not exhibit fixed sign. Such results are given in Section 1. They are related to certain asymptotic properties of the principal solutions of Eq. (6), and are presented as independent results because, in our opinion, they may be applied also to problems different from those considered in this paper, in particular to ones concerning differential equations with quasiderivatives.

1. NOTATION AND AUXILIARY RESULTS

In this section we present some preliminary results on second order linear differential equations that we will use in the proof of the main results. Consider the equation

$$\left(\frac{1}{r(t)}y'\right)' + q(t)y = 0, \tag{8}$$

where $r \in C^1(J)$, $q \in C(J)$, $r(t) > 0$. In the study of the nonoscillation of Eq. (8), an important role (see, e.g., [18, 27]) is played by principal solutions (at infinity), that is, by solutions y_0 of Eq. (8) such that

$$\int^\infty \frac{r(t)}{y_0^2(t)}\,dt = \infty. \tag{9}$$

If Eq. (8) is nonoscillatory, then Eq. (8) has a solution y_0 satisfying (9) which is uniquely determined up to a constant factor (see, e.g., [18]). In addition an arbitrary solution y_1 of Eq. (8), linearly independent of y_0, satisfies

$$\int^{\infty} \frac{r(t)}{y_1^2(t)}\, dt < \infty,$$

and it is called nonprincipal (at infinity). Henceforward, for sake of simplicity, we will denote by a principal [nonprincipal] solution a principal [nonprincipal] solution at infinity.

We recall also that if Eq. (8) is nonoscillatory, then Eq. (8) is said to be *disconjugate* on J if each nontrivial solution of Eq. (8) has at most one zero on J. Moreover Eq. (8) is disconjugate on J if and only if Eq. (8) has a positive solution on $(0, \infty)$ (see, e.g., again [18]).

Consider now the linear binomial differential equation (6) where $a \in C(J)$, and let $a^+(t) = \max_{t \in J}\{a(t), 0\}$, $a^-(t) = \min_{t \in J}\{a(t), 0\}$. Clearly $a(t) = a^+(t) + a^-(t)$. The following holds:

PROPOSITION 1. *Assume the following conditions*

(i) $\int_0^\infty t a^-(t)\, dt = -K > -\infty$;

(ii) *the equation*

$$y'' + e^{-2K} a^+(t) y = 0 \tag{10}$$

is disconjugate on J.

Then Eq. (6) *is disconjugate on J and there exists a* (*principal*) *solution h of Eq.* (6), $h(t) > 0$ *on* $(0, \infty)$, *such that*

$$\int_0^{\infty} \frac{1}{h^2(t)}\, dt = \infty, \qquad \int_0^{\infty} h(t)\, dt = \infty, \quad \lim_{t \to \infty} h(t) > 0. \tag{7}$$

If, in addition, the following

(iii) $\int_0^\infty t a^+(t)\, dt < \infty$,

holds, then the principal solution h satisfying (7) *is bounded on* J.

Roughly speaking Proposition 1 guarantees the nonoscillatoriness of Eq. (6) when the negative part of a is small in some sense and the corresponding equation associated to the positive part of a is nonoscillatory. The crucial point of this proposition is the fact that Eq. (6) has principal solutions which verify conditions (7), and are bounded if (iii) is verified.

It is easy to give an example which illustrates the necessity of assumption (i). To this aim it is sufficient to consider the equation $y'' - y = 0$. Clearly (i) is not satisfied, and there is no solution satisfying (7) since $h(t) = e^{-t}$.

Remark 1. Assumption (iii) of Proposition 1 implies that Eq. (10) is eventually disconjugate. Indeed, from Theorem 1 in [5], Eq. (10) is nonoscillatory. Then, from a result in [8], Eq. (10) is eventually disconjugate. We remark that assumption (ii) requires, in addition, that Eq. (10) be disconjugate on the whole half real line.

The proof of Proposition 1 depends on the following lemmas concerning Eq. (8) with "$q(t) \leq 0$ on J" and "$q(t) \geq 0$ on J," respectively. As it is well known, "$q(t) \leq 0$ on J" is sufficient for Eq. (8) to be disconjugate on J. The following hold:

LEMMA 1. *Consider Eq.* (8) *with* $q(t) \leq 0$. *If the following conditions*

(i) $\int_0^\infty r(t)\,dt = \infty$,

(ii) $\int_0^\infty |q(t)| \int_0^t r(s)\,ds\,dt < \infty$,

are satisfied, then any positive principal solution u_0 *of Eq.* (8) *is nonincreasing and such that*

$$u_0(\infty) = \lim_{t\to\infty} u_0(t) > 0 \tag{11_1}$$

$$\frac{u(0)}{u_0(\infty)} \leq \exp\left(\int_0^\infty |q(t)| \int_0^t r(s)\,ds\,dt \right). \tag{11_2}$$

Proof. A classical result of Kneser [18, Exercise 6.7, p. 352] states the existence of positive nonincreasing principal solutions u_0 of Eq. (8) approaching a nonzero limit as $t \to \infty$.

For these solutions, it is easy to prove that $\lim_{t\to\infty}(1/r(t))u_0'(t) = 0$ (see, e.g., [22]). Hence integrating Eq. (8) in (t, ∞) we obtain

$$\frac{1}{r(t)} u_0'(t) = \int_t^\infty q(s) u_0(s)\,ds;$$

because u_0 is a positive nonincreasing function, we get

$$u_0'(t) \geq r(t) u_0(t) \int_t^\infty q(s)\,ds.$$

Dividing by u_0 and integrating again in J we obtain

$$\log \frac{u_0(\infty)}{u_0(0)} \geq \int_0^\infty r(t) \int_t^\infty q(s)\,ds\,dt = \int_0^\infty q(t) \int_0^t r(s)\,ds\,dt > -\infty$$

which implies (11_2). ∎

LEMMA 2. *Consider Eq.* (8) *with* $q(t) \geq 0$. *If the following conditions*

(i) $\int_0^\infty r(t)\,dt = \infty$,

(ii) $\limsup_{t\to\infty} r(t) < \infty$,

(iii) *Eq.* (8) *is disconjugate on* J,

are satisfied, then Eq. (8) *has principal solutions* v_0 *satisfying*

$$v_0(t) > 0, \qquad v_0'(t) \geq 0 \text{ on } (0,\infty), \tag{12_1}$$

$$\int_0^\infty \frac{1}{v_0^2(t)}\,dt = \infty. \tag{12_2}$$

If, in addition, the following

(iv) $\int_0^\infty q(t) \int_0^t r(s)\,ds\,dt < \infty$,

holds, then v_0 *is bounded on* J.

Proof. With Eq. (8) being disconjugate on J, Eq. (8) has positive solutions on $(0,\infty)$. Then the existence of a principal solution v_0 satisfying (12_1) follows from two results of Hartman and Potter (see, e.g., [18, Chap. XI, Corollary 6.3; 27, Theorem 2.39]). Hence

$$\int_0^\infty \frac{r(t)}{v_0^2(t)}\,dt = \infty,$$

and, taking into account (ii), also (12_2) is satisfied.

In order to complete the proof it remains to show that if (i)–(iv) holds, then v_0 is bounded on J. Two cases are possible: (a) q is eventually positive; (b) there exists a sequence $\{t_k\}$, $t_k \to \infty$, such that $q(t_k) = 0$.

Case (a). The assertion follows from Proposition 2 in [5].

Case (b). In this case the assertion follows by using the Sturm Comparison Theorem. Consider the linear perturbed equation

$$\left(\frac{1}{r(t)}y'\right)' + \big(q(t) + q_1(t)\big)y = 0, \tag{13}$$

where $q_1 \in C(J)$, $q_1(t) > 0$ for $t \in J$ and $\int_0^\infty q_1(t)\int_0^t r(s)\,ds\,dt < \infty$. Thus Eq. (13) is nonoscillatory and has bounded solutions v_b (see, e.g., again [5, Proposition 2]). In addition from the quoted result in [8], Eq. (13) is also eventually disconjugate. With Eq. (13) being a Sturm majorant of Eq. (8), from a result of Hartman and Wintner (see, e.g., [18, Chap. XI, Corollary 6.5]) we have for t large enough

$$\frac{1}{r(t)}\frac{v_0'(t)}{v_0(t)} \leq \frac{1}{r(t)}\frac{v_b'(t)}{v_b(t)}$$

or

$$v_0(t) \leq \frac{v_0(t_0)}{v_b(t_0)} v_b(t).$$

This implies that v_0 is bounded. The proof is now complete. ∎

We remark that as already noted in Remark 1, assumption (iv) of Lemma 2 implies that Eq. (8) is eventually disconjugate.

Proof of Proposition 1. Consider the differential equation

$$y'' + a^-(t)y = 0. \tag{14}$$

From Lemma 1, Eq. (14) has a positive principal nonincreasing solution $\tilde{u}_0$ satisfying

$$\tilde{u}_0(\infty) > 0, \qquad \int_0^\infty \frac{1}{\tilde{u}_0^2(t)}\,dt = \infty, \qquad \frac{\tilde{u}_0(0)}{\tilde{u}_0(\infty)} \leq e^{-K}, \tag{15}$$

where $K = -\int_0^\infty ta^-(t)\,dt$. Let $m = \tilde{u}_0^2(\infty)$, $M = \tilde{u}_0^2(0)$, and consider the linear differential equation

$$(my')' + Ma^+(t)y = 0 \tag{16_1}$$

or

$$y'' + \frac{M}{m}a^+(t)y = 0. \tag{16_2}$$

With $M/m < e^{-2K}$, the Sturm Comparison Theorem implies that Eq. (16_2) is disconjugate on J. Consider now the linear differential equation

$$\left(\tilde{u}_0^2(t)y'\right)' + \tilde{u}_0^2(t)a^+(t)y = 0. \tag{17}$$

With $M \geq \tilde{u}_0^2(t) \geq m$, again from the Sturm Comparison Theorem we get that Eq. (17) is disconjugate on J. Hence assumptions (i), (ii), (iii) of Lemma 2 are satisfied and so Eq. (17) has a principal solution $\tilde{v}_0$ verifying on $(0, \infty)$

$$\tilde{v}_0(t) > 0, \qquad \tilde{v}_0'(t) \geq 0, \qquad \int_0^\infty \frac{1}{\tilde{v}_0^2(t)}\,dt = \infty. \tag{18}$$

In order to complete the proof it is sufficient to consider the function h given by

$$h(t) = \tilde{u}_0(t)\tilde{v}_0(t).$$

It is easy to show, by standard calculations, that h is a solution of Eq. (6). With h being positive on $(0, \infty)$ Eq. (6) is then disconjugate on J. Taking into account (15) and (18), we obtain that h satisfies also conditions (7). Finally if also condition (iii) holds, then we have

$$\int_0^\infty \tilde{u}_0^2(t) a^+(t) \int_0^t \frac{1}{\tilde{u}_0^2(s)}\, ds\, dt \le \frac{M}{m} \int_0^\infty t a^+(t)\, dt < \infty,$$

which implies that condition (iv) of Lemma 2 is satisfied. Then, from Lemma 2, $\tilde{v}_0$ is bounded on J. With $\tilde{u}_0$ being positive nonincreasing, also h is bounded on J. This completes the proof of Proposition 1. ▮

2. MAIN RESULTS

Consider the nonlinear differential Eq. (1). If the second order linear Eq. (6) is nonoscillatory, that is, Eq. (6) does not have oscillatory solutions, then, by standard computations, Eq. (1) may be transformed, for $t \ge t_0 > 0$, in the disconjugate form

$$\left(h^2(t)\left(\frac{1}{h(t)} x'\right)'\right)' + h(t) b(t) f(x) = 0, \tag{1'}$$

where h is a solution of Eq. (6). This equation is a prototype of the more general equation (4). The divergence of the integrals of the functions p and r plays an important role in the study of nonoscillation of Eq. (4). Indeed in [26] Švec, generalizing a Lemma of Kiguradze and Elias (see, e.g., [8; 9; 20, Lemma 1.1, p. 2, Lemma 2.1, p. 43]) states that if the conditions (5) are satisfied, then every nonoscillatory solution x of Eq. (4) satisfies, for t large enough, either

$$|x(t)| > 0,\ x(t) \cdot x^{[1]}(t) < 0,\ x(t) \cdot x^{[2]}(t) > 0 \tag{19}$$

or

$$|x(t)| > 0,\ x(t) \cdot x^{[1]}(t) > 0,\ x(t) \cdot x^{[2]}(t) > 0, \tag{20}$$

where $x^{[1]}, x^{[2]}$ are the *quasiderivatives* of x, that is, $x^{[1]}(t) = x'(t)/r(t)$, $x^{[2]}(t) = (x^{[1]}(t))'/p(t)$. Solutions satisfying (19) [(20)] are said to be *solutions of degree zero* [*two*] (see, e.g., [12]). Solutions satisfying (19) are known also as *Kneser solutions* (see, e.g., [20]).

If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of Eq. (4) and by $\mathcal{N}_0[\mathcal{N}_2]$ the set of solutions of degree zero [two], then (5) implies that $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2$. A simple consequence of Proposition 1 gives us sufficient conditions in order for the same classification of nonoscillatory solutions to occur for Eq. (1′). The following holds:

PROPOSITION 2. *Assume*

(i) $\int_0^\infty ta^-(t)\,dt = -K > -\infty$;

(ii) *Equation* (10) *is disconjugate on* J.

Then Eq. (1) *can be written for* $t > 0$ *in the disconjugate form* (1′) *and every nonoscillatory solution of Eq.* (1′) *is either in the class* $\mathcal{N}_0$ *or in the class* $\mathcal{N}_2$.

Proof. The assertion follows immediately from Proposition 1 choosing as function h a principal solution of Eq. (6). ∎

We can state now our theorems which improve the quoted results in [17].

THEOREM 1. *Assume*

(i) $\int_0^\infty ta^-(t)\,dt = -K > -\infty$;

(ii) *Equation* (10) *is disconjugate on* J.

Then every bounded continuable solution of Eq. (1) *with a zero at some point* $t_1 \geq 0$ *is oscillatory.*

Proof. Let x be a continuable solution of Eq. (1) such that $x(t_1) = 0$, $t_1 \geq 0$. Without loss of generality we may suppose $x(t) > 0$ for t large enough. From Proposition 2, Eq. (1) may be transformed, for $t > 0$ into (1′) where h satisfies (7). Assume x nonoscillatory. Then, from Proposition 2, x is either in the class $\mathcal{N}_0$ or in the class $\mathcal{N}_2$.

Assume that $x \in \mathcal{N}_0$. Without loss of generality suppose $x(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$ for $t > T$. We assert first that x does not have positive maxima. Let t_1, $t_x < t_1 < T$, be the last point of maximum for x. Then $x(t) > 0$ for $t \geq t_1$, which implies that the quasiderivative $x^{[2]}$ is decreasing on $[t_1, \infty)$. Hence we have

$$x^{[2]}(t) = h^2(t)\left(\frac{1}{r(t)}x'(t)\right)' < h^2(t_1)\left(\frac{1}{h(t)}x'(t)\right)'_{t=t_1} = \frac{1}{h(t_1)}x''(t_1) \leq 0,$$

which is a contradiction. Then x does not have positive maxima, and so x does not have zeros on its existence interval. This is again a contradiction since $x(t_1) = 0$. Hence $x \notin \mathcal{N}_0$.

Assume now that $x \in \mathcal{N}_2$ and $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for $t \geq T > t_1$. With $x^{[1]}$ being an increasing function, we have for $t \geq T$, $x'(t) > h(t)(x'(T)/h(T))$ or $x(t) > x(T) + (x'(T)/h(T))\int_T^t h(s)\,ds$. Since $\int^\infty h(t)\,dt = \infty$ we get that x is unbounded, which is a contradiction. The proof is now complete. ∎

Remark 2. In the first part of the above proof we have shown that solutions of Eq. (1′) in the class $\mathscr{N}_0$ cannot have zeros on its existence interval. An alternative proof of this assertion is given in [2].

Remark 3. Theorem 1, as well as Theorems 2–4 below, requires the continuability of solutions of Eq. (1) with a zero. On this topic we refer the reader to the books [1, 20] and to the papers [2, 7, 19].

Theorem 1 improves the quoted Theorem A. Observe that Theorem 1 does not require monotonicity and regularity assumptions on the function a nor does it require hypotheses on the function f of superlinearity and/or sublinearity at zero or at infinity.

With an additional assumption on the nonlinearity in a neighborhood of infinity, we may state the following result which guarantees the oscillatory behavior of all continuable solutions, possibly unbounded, vanishing at some point $t_1 \geq 0$.

THEOREM 2. *Assume condition* (H_1) *and*

(i) $\int_0^\infty t a^-(t)\,dt = -K > -\infty$;

(ii) *Equation* (10) *is disconjugate on* J;

(iii) $\int_0^\infty f(kt)b(t)\,dt = -\int_0^\infty f(-kt)b(t)\,ds = \infty$ *for every* $k \in (0,1)$.

Then every continuable solution of Eq. (1) *with a zero at some point* $t_1 \geq 0$ *is oscillatory.*

Proof. Let x be a continuable solution of Eq. (1) such that $x(t_1) = 0$, $t_1 \geq 0$. Without loss of generality we may suppose $x(t) > 0$ for t large enough. From Proposition 2, Eq. (1) may be transformed, for $t > 0$, into (1′) where h satisfies (7). Assume x is nonoscillatory. From Proposition 2, we have that $x \in \mathscr{N}_0 \cup \mathscr{N}_2$. Reasoning again as in the proof of Theorem 1, we get that x is not in the class $\mathscr{N}_0$.

Suppose that $x \in \mathscr{N}_2$ and $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for $t \geq T > t_1$. Denote $m_h = \inf_{t \in [T,\infty)} h(t)$. With $T > 0$, from Proposition 1 we get that $m_h > 0$. Integrating Eq. (1) in (T,t), $t > T$, we obtain

$$x^{[2]}(t) - x^{[2]}(T) + \int_T^t h(s)b(s)f(x(s))\,ds = 0,$$

which implies

$$x^{[2]}(T) > \int_T^t h(s)b(s)f(x(s))\,ds. \tag{21}$$

With $x^{[1]}$ being a positive increasing function, we have for $t \geq T$

$$x(t) > x(T) + x^{[1]}(T)\int_T^t h(s)\,ds > x^{[1]}(T)\int_T^t h(s)\,ds$$
$$> x^{[1]}(T)\cdot m_h\cdot(t-T). \tag{22}$$

Let k be a constant such that $0 < k < \min[1, x^{[1]}(T)\cdot m_h]$. Then from (22) we have for all t sufficiently large, $x(t) > k\cdot t$. With f eventually increasing, there exists T_1 such that for $t > T_1$, $f(x(t)) > f(k\cdot t)$, and, from (21), we obtain

$$x^{[2]}(T) > \int_T^{T_1} h(s)b(s)f(x(s))\,ds + \int_{T_1}^t h(s)b(s)f(k\cdot s)\,ds$$
$$\geq \int_T^{T_1} h(s)b(s)f(x(s))\,ds + m_h\int_{T_1}^t b(s)f(k\cdot s)\,ds. \tag{23}$$

Taking into account (iii), the right side of (23) tends to infinity as $t\to\infty$, which is a contradiction. Then x is oscillatory and the proof is complete. ∎

Theorem 1 and 2 are related to a result in [3, Corollary 1] in which the case $a^+(t)\equiv 0$, $\int_0^\infty a(t)\,dt = -\infty$, is considered.

Remark 4. When the perturbation f is superlinear at infinity, that is, $\liminf_{|u|\to\infty}(f(u)/u) > 0$, condition (iii) of Theorem 2 is satisfied if $\int_0^\infty t b(t)\,dt = \infty$. Moreover in this case it is easy to prove that monotonicity assumption (H_1) is unnecessary.

For the Emden–Fowler equation (N_α), assumption (iii) becomes $\int_0^\infty s^\alpha b(s)\,ds = \infty$. The following examples show that assumptions (i) and (iii) cannot be dropped without violating the validity of Theorem 2.

EXAMPLE 1. Consider the sublinear differential equation

$$x''' - (t-1)^3 x' + 2(t-1)^3|x|^{1/2}\operatorname{sgn} x = 0. \tag{E_1}$$

The function x given by $x(t) = (t-1)^2$ is a solution of Eq. (E_1) with a zero at $t_1 = 1$. For this equation the assumption (i) does not hold, since $a^-(t) = a(t) = -2(t-1)^3$, while condition (iii) is verified. As regards assumption (ii), equation $y'' + e^{-2K}a^+(t)y = 0$ is not defined since $K=\infty$, but (ii) is satisfied for the ‘‘limit equation,’’ that is, for the equation $y'' = 0$.

EXAMPLE 2. Consider the sublinear differential equation

$$x''' + \frac{1}{(t+1)^3}|x|^\alpha\operatorname{sgn} x = 0 \qquad 0<\alpha<1. \tag{E_2}$$

Since $a^- \equiv a^+ \equiv 0$, the assumptions (i) and (ii) are satisfied, but (iii) does not hold since $\int_0^\infty s^\alpha b(s)\,ds < \infty$. Moreover every solution of Eq. (E_2) is nonoscillatory as it follows from a result in [21, Corollary 5].

Assuming that the function f is superlinear in a neighborhood of zero, we can give nonoscillatory results which generalize some criteria obtained in [4, 17].

THEOREM 3. *Assume* (H_2) *and*

(i) $\int_0^\infty ta^-(t)\,dt = -K > -\infty$;

(ii) $\int_0^\infty ta^+(t)\,dt < \infty$.

Let ψ *be a positive function defined on J such that* $\int_0^\infty t^2 b(t)\psi(t)\,dt < \infty$. *If x is a continuable solution of Eq.* (1) *such that, for t large enough,*

$$|f(x(t))| \le |x(t)|\psi(t), \tag{24}$$

then x is nonoscillatory.

Proof. Assume there exists an oscillatory solution x of Eq. (1), defined on $[t_x, \infty)$, $t_x \ge 0$, satisfying (24) for $t \ge T > t_x$. Consider the linearized equation

$$w''' + a(t)w' + b(t)F(t)w = 0 \qquad (t \ge T), \tag{25}$$

where

$$F(t) = \begin{cases} \dfrac{f(x(t))}{x(t)} & \text{if } x(t) \ne 0 \\ \theta & \text{if } x(t) = 0. \end{cases}$$

From Remark 1 we have that Eq. (10) is eventually disconjugate, that is, there exists $t_0 \ge 0$ such that Eq. (10) is disconjugate on (t_0, ∞). Hence, from Proposition 2, we have that Eq. (25) may be transformed, for $t \ge T > t_0$, into

$$\left(h^2(t)\left(\frac{1}{h(t)}w'\right)'\right)' + h(t)b(t)F(t)w = 0,$$

where h satisfies (7). Define

$$m_h = \inf_{t\in[T,\infty)} h(t), \qquad M_h = \sup_{t\in[T,\infty)} h(t).$$

As already denoted in the proof of Theorem 2, because $T > 0$, we have $m_h > 0$. With h bounded, we get also $M_h < \infty$. Thus

$$\int_T^\infty h(t)b(t)F(t)\left(\int_T^t h(s)\left(\int_T^s \frac{1}{h^2(\lambda)}\,d\lambda\right)ds\right)dt$$

$$\le C_h \int_T^\infty b(t)F(t)(t-T)^2\,dt,$$

where $C_h = (1/2)(M_h/m_h)^2$. Taking into account (24), we obtain

$$\int_T^\infty h(t)b(t)F(t)\left(\int_T^t h(s)\left(\int_T^s \frac{1}{h^2(\lambda)}\,d\lambda\right) ds\right) dt$$

$$\leq C_h \int_T^\infty b(t)\psi(t)(t-T)^2\,dt < \infty.$$

By a slight modification of a result in [6, Theorem 5] we obtain that Eq. (25) is nonoscillatory, which is a contradiction because x is an oscillatory solution. ∎

Remark 5. It is easy to show that Theorem 3 improves Theorem B given in [17], for Eq. (2) with $\alpha > 1$. To this end choose $a(t) \geq 0$ and $\psi \equiv 1$: it is sufficient to prove that the assumptions of Theorem B imply $\int_0^\infty ta^+(t)\,dt = \int_0^\infty ta(t)\,dt < \infty$.

Assume $\lim_{t\to\infty}\int_T^t sa(s)\,ds = \infty$. With $\int_T^t sa(s)\,ds = (t^2/2)a(t) - (T^2/2)a(T) - \int_T^t s^2 a'(s)\,ds$, we obtain $\lim_{t\to\infty}(t^2/2)a(t) = \infty$. Hence a classical result of Kneser (see, e.g., [27, p. 45]) implies that Eq. (6) is oscillatory, which is a contradiction.

A suitable choice of ψ gives the following:

COROLLARY 1. *Assume condition* (H_2) *and suppose that conditions* (i), (ii) *of Theorem* 3 *hold. If* $\int_0^\infty t^2 b(t)\,dt < \infty$, *then Eq.* (1) *does not have bounded oscillatory solutions.*

Proof. The assertion follows from Theorem 3 choosing $\psi(t) = c$, c constant. ∎

When Eq. (1) is sublinear in a neighborhood of infinity, we have the following (see also [4, Corollary 3; 13, Theorem 2]).

COROLLARY 2. *Assume condition* (H_2) *and suppose that conditions* (i), (ii) *of Theorem* 3 *hold. If* $\int_0^\infty t^2 b(t)\,dt < \infty$ *and*

(iii) $\limsup_{|u|\to\infty}(f(u)/u) < \infty$,

then Eq. (1) *does not have oscillatory solutions.*

Proof. Taking into account (H_2) and (iii), there exists a constant k such that $0 \leq f(u)/u \leq k$. Then the assertion follows by reasoning as in the proof of Corollary 1. ∎

We conclude this section with some applications of the previous results to the nonlinear equation

$$x''' + a(t)x' + b(t)|x|^\alpha \operatorname{sgn} x = 0 \qquad (\alpha > 0). \tag{2}$$

The following holds:

THEOREM 4. (A) *Assume*

(i) $\int_0^\infty ta^-(t)\,dt = -K > -\infty$;

(ii) *Equation* (10) *is disconjugate on* J;

(iii) $\int_0^\infty t^\alpha b(t)\,dt = \infty$.

Then every continuable solution of Eq. (2) *with a zero at some point* $t_1 \geq 0$ *is oscillatory.*

(B) *Let* $\alpha \geq 1$ *and assume* (i), $\int_0^\infty ta^+(t)\,dt < \infty$ *and* $\int_0^\infty t^{n(\alpha-1)+2}b(t)\,dt < \infty$. *Then Eq.* (2) *does not have continuable oscillatory solutions* x *such that* $|x(t)| \leq t^n$. *In particular* $(n = 2)$ *if* $\int_0^\infty t^{2\alpha}b(t)\,dt < \infty$, *then Eq.* (2) *does not have continuable oscillatory solutions* x *such that* $|x(t)| \leq t^2$.

Proof. Claim (A) follows from Theorem 2. Claim (B) follows from Theorem 3 with $\psi(t) = t^{n(\alpha-1)}$. ∎

Theorem 4 extends to Eq. (2) an analogous result stated in [4] for the binomial equation

$$x''' + b(t)|x|^\alpha \operatorname{sgn} x = 0 \qquad (\alpha > 0). \tag{26}$$

Part (A) of Theorem 4 is also related with some results in [23], in which the oscillation of solutions with a zero is considered. Part (B) of Theorem 4 is related with a problem settled in [11]. Indeed in [11] the authors conjecture that if $\int_0^\infty t^{2\alpha}b(t)\,dt < \infty$, then the binomial Eq. (26) with $\alpha > 1$ does not have oscillatory solutions. Other conditions assuring that Eq. (26) does not have oscillatory solutions have been given recently in [7].

3. SOME EXTENSIONS

Consider now Eq. (4). It is easy to extend to Eq. (4) all results stated in the previous section by assuming the condition (5). For example the following holds:

THEOREM 5. *Assume condition* (5). *Then every bounded continuable solution of Eq.* (4) *with a zero is oscillatory.*

Proof. (Sketch). The argument is similar to that given in the proof of Theorem 1. Let x be a continuable solution of Eq. (4) defined on $[t_x, \infty)$, $t_x \geq 0$, such that $x(t_1) = 0$, $t_1 \geq t_x$. Assume $x \in \mathcal{N}$, that is, x nonoscillatory. From the quoted result [26], x is either in the class $\mathcal{N}_0$ or in the class $\mathcal{N}_2$. The assertion follows by showing that: (a) solutions in $\mathcal{N}_0$ cannot have zeros in their existence interval; (b) $\int_0^\infty r(t)\,dt = \infty$ implies that solutions in $\mathcal{N}_2$ are unbounded. ∎

THEOREM 6. *Assume conditions* (H_1), (5), *and*

$$\int_0^\infty b(t) f\left(k \cdot \int_0^t r(s)\, ds\right) dt = -\int_0^\infty b(t) f\left(-k \cdot \int_0^t r(s)\, ds\right) dt = \infty$$

$$\text{for every } k \in (0,1). \quad (27)$$

Then every continuable solution of Eq. (4) *with a zero is oscillatory.*

THEOREM 7. *Assume conditions* (H_2) *and* (5). *Let* ψ *be a positive function defined on* J *such that*

$$\int_0^\infty b(t) \psi(t) \int_0^t r(s) \int_0^s p(\lambda)\, d\lambda\, ds\, dt < \infty.$$

If x *is a continuable solution of Eq.* (4) *such that, for* t *large enough,*

$$|f(x(t))| \le |x(t)| \psi(t), \quad (24)$$

then x *is nonoscillatory.*

The proofs of Theorem 6 and 7 are similar to those of Theorems 2 and 3, respectively, and are omitted.

As already noted in Remark 4, when the perturbation f is superlinear at infinity, that is, $\liminf_{|u| \to \infty} (f(u)/u) > 0$, if

$$\int_0^\infty b(t) \int_0^t r(s)\, ds\, dt = \infty,$$

then (27) is satisfied. Also in this case the monotonicity assumption (H_1) is unnecessary. Finally, extensions of the above results to the equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)} x'\right)'\right)' + b(t)|x|^\alpha \operatorname{sgn} x = 0 \qquad (\alpha > 0).$$

are left to the reader.

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