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Radial Bounds for Perturbations of Elliptic Operators

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Elliptic operators $A = \sum_{|\alpha| \leqslant m} b_\alpha(x) D^\alpha$, α a multi-index, with leading term positive and constant coefficient, and with lower order coefficients $b_\alpha(x) \in L^{r_\alpha} + L^\infty$ (with $(n/r_\alpha) + |\alpha| < m$) defined on $\mathbb{R}^n$ or a quotient space $\mathbb{R}^n/U_\alpha$, $U_\alpha \subset \mathbb{R}^n$ are considered. It is shown that the L^p-spectrum of A is contained in a "parabolic region" Ω of the complex plane enclosing the positive real axis, uniformly in p. Outside Ω, the kernel of the resolvent of A is shown to be uniformly bounded by an L^1 radial convolution kernel. Some consequences are: A can be closed in all L^p $(1 \leqslant p \leqslant \infty)$, and is essentially self-adjoint in L^2 if it is symmetric; A generates an analytic semigroup e^{-tA} in the right half plane, strongly L^p and pointwise continuous at $t = 0$. A priori estimates relating the leading term and remainder are obtained, and summability $\phi(\varepsilon A) f \to_{\varepsilon \to 0} \phi(0) f$, with ϕ analytic, is proved for $f \in L^p$, with convergence in L^p and on the Lebesgue set of f. More comprehensive summability results are obtained when A has constant coefficients.

INTRODUCTION

We study a functional calculus (multipliers and summability) for a class of operators on $\mathbb{R}^n$. By L^p-multiplier of a pseudodifferential operator A we mean a function $\varphi(\lambda)$ of a real or complex variable λ such that the operator $\varphi(A)$ is well defined and bounded on L^p. Similarly one can define multipliers on Sobolev spaces $\mathscr{L}^p_s$, Hölder spaces, etc.

A summability method is a family $\varphi_\varepsilon(\lambda)$ of multipliers depending on a real or complex parameter ε such that the family of operators $\{\varphi_\varepsilon(A)\}_\varepsilon$ forms an

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"approximate identity." Typically $\varphi_\varepsilon(\lambda)=\varphi(\varepsilon\lambda)$ are dilations of a function φ (the "summation method").

One aims at proving convergence

$$\varphi_\varepsilon(A)f(x)\to f(x), \tag{1}$$

as $\varepsilon\to 0$ (or ∞) in norm (L^p, Sobolev, Hölder, etc.), or pointwise, for instance, a.e., or on the Lebesgue set of f. Well-known examples of such families are modified resolvents $\{\lambda(\lambda-A)^{-1}\}$ and one-parameter semigroups $\{e^{-tA}\}$. The convergence in (1) is important in the theory of semigroups of operators (cf. [3, 5]).

We consider a class of operators on $\mathbb{R}^n$ of the form $A=A_0+B$ whose leading term A_0 is a constant coefficient positive elliptic operator. The perturbation $B=\sum_{|\alpha|<m} b_\alpha(x)\,D^\alpha$ has coefficients in certain L^r-spaces, $b_\alpha\in L^{r_\alpha}(\mathbb{R}^n)$. More generally, the coefficients b_α can be defined according to a decomposition $R^n=U_\alpha\oplus V_\alpha$, with $b_\alpha=b_\alpha(u_\alpha, v_\alpha)$ independent of u_α, and $b_\alpha\in(L^{r_\alpha}+L^\infty)(V_\alpha)$. Natural examples of such operators are Hamiltonians of atoms (see, e.g., [7, Chap. 10]) $A=-\Delta+(1/|x|)$, or

$$A=-\Delta+\sum_{i<j}^{N}\frac{b_{ij}}{|x_i-x_j|}+\sum_{i=1}^{N}\frac{b_i}{|x_i|}$$

on $\mathbb{R}^{3N}$, where $x_i\in\mathbb{R}^3$ $(i=1,\ldots,N)$ and Δ is the Laplacian on $\mathbb{R}^{3N}$.

The usual approach to functional calculus based on pseudodifferential operators and Fourier integrals (cf. [13]) does not apply directly to such A. First, A is not necessarily self-adjoint ($B\neq B^*$ in general). Second, a "ψDO-Fourier integral" approach requires certain smoothness conditions on the coefficients $b_\alpha(x)$, while we assume only $b_\alpha\in L^r+L^\infty$. The best way to approach the problem is to combine two techniques, namely, to use pseudodifferential operator calculus for the "nice" leading part A_0 of A, and then to apply perturbation theory and the Dunford functional calculus for A itself.

Let us describe this procedure in our setting. We start (Section 1) with a functional calculus for A_0, taking multipliers φ in so-called symbol classes $S^0_{1,0}$ and $S^{-\delta}_{1,0}$. For the 0-class the boundedness of $\varphi(A_0)$ in L^p $(p>1)$ and its weak (1, 1)-type follow from Hörmander's well-known multiplier theorem [4]. We then use the following fact: for symbols of strictly negative order, $\varphi\in S^{-\delta}_{1,0}$ $(\delta>0)$, the kernel $K(x,y)$ of an operator $\varphi(x,D)$ is bounded by translations of a radial decreasing L^1-function $h(|x|)$, i.e.,

$$|K(x,y)|\leqslant h(|x-y|). \tag{2}$$

As in Hörmander's theorem, the L^1-norm of h is estimated by symbol class seminorms of φ.

Motivated by this observation we introduce a class of integral operators on $\mathbb{R}^n$ whose kernels $K(x,y)$ satisfy (2) for some L^1 radial decreasing function h, and call this class [RB]. Natural examples of operators in [RB] are convolutions $K(f)=K*f$ with kernels $|K(x)|\leqslant h(|x|)$, and by well-known results pseudodifferential operators of any negative order belong in [RB].

Obviously, operators in [RB] are bounded in all L^p-spaces $(1\leqslant p\leqslant\infty)$. Moreover, suitable families of operators $\{K_\varepsilon\}_{\varepsilon>0}$ in [RB] form nice "approximate identities" (Proposition 2), i.e.,

$$(K_\varepsilon f)(x)\xrightarrow[\varepsilon\to 0]{} f(x)$$

in L^p-norm and on the Lebesgue set of $f\in L^p$ $(1\leqslant p<\infty)$. In this way we obtain summability for constant coefficient elliptic operators and suitable "summation families" $\{\varphi_\varepsilon\}_{\varepsilon>0}\subset S^{-\delta}_{1,0}$.

Some important examples of summation families are:

(I) modified resolvents $\{\zeta(\zeta-A)^{-1}\}$,

(II) one-parameter semigroups $\{e^{-tA}\}$,

(III) fractional powers $\{A^s\}$ $(\operatorname{Re} s<0)$,

(IV) semigroups generated by fractional powers $\{e^{-tA^s}\}$.

Note that all of these belong to negative order symbol classes $S^{-\delta}_{1,0}$. The first family (resolvent) is essential for the next step, the perturbation (Section 2). Its starting point is the following identity between the resolvents of A_0 and $A=A_0+B$,

$$(\zeta-A)^{-1}-(\zeta-A_0)^{-1}=(\zeta-A_0)^{-1}B(\zeta-A)^{-1} \tag{3}$$

which leads to the formal series expansion

$$(\zeta-A)^{-1}=\sum_{k=0}^{\infty}(\zeta-A_0)^{-1}[B(\zeta-A_0)^{-1}]^k. \tag{4}$$

We study (4) by two methods. First (Lemma 1), we show that under suitable restriction on the L^r-classes of its coefficients b_α, the operator B is small relative to A_0. Precisely, we estimate the norm of $B(\zeta-A_0)^{-1}$ by certain negative powers of the "large parameter" ζ.

As a corollary of Lemma 1 we get so-called "a priori estimates" for the pair (A_0,B). Along with the Kato–Rellich theorem [5, Chap. 5] they imply essential self-adjointness of a formally symmetric operator $A=A_0+B$ in L^2. Lemma 1 can also be used to obtain a variety of other results, including:

(1) uniform bounds on the spectrum of A in different L^p,

(2) maximum decrease of the resolvent,

(3) resolvent summability.

See [9] for an alternative treatment of the a priori bounds in Section 2.

To study the kernel of the resolvent of A we examine each term of the perturbation series (4). We show (Theorem 2) that under the assumptions of Lemma 1 all of them, and consequently the resolvent $(\zeta - A)^{-1}$, are in [RB]. Therefore, Proposition 2 applies to $\zeta(\zeta - A)^{-1}$ to get "resolvent summability" for A in all L^p-spaces (Theorem 3) as well as other results.

Finally we proceed to a wider class of multipliers and summation methods using resolvent summability and Cauchy integration. We show, in particular, that families $\{e^{-tA}\}_{\operatorname{Re} t>0}$ and $\{e^{-tA^s}\}_{\operatorname{Re} t>0}$ (whenever the latter exists) can be obtained in this way, i.e., the operator A and its fractional powers A^s ($s \in \mathbb{R}$) are generators of analytic semigroups in the right half plane. Moreover, $e^{-tA}f(x) \to f(x)$, as $t \to 0$ uniformly in any sector $|\arg t| \leqslant \theta < \pi/2$ in L^p-norm and on the Lebesgue set of f.

Let us note that our semigroups do not fall within the scope of "generalized heat-diffusion" semigroups, treated, for instance, in [12]. They are neither self-adjoint ($A \neq A^*$), nor positivity-preserving. However, in two points our results sharpen the general theory:

(1) Semigroups $\{e^{-tA}\}$ are shown to be analytic in the whole right half plane $\operatorname{Re} t > 0$, independent of L^p-space.

(2) Pointwise convergence is proved on the Lebesgue set of $f \in L^p$ rather than a.e., as in general.

1. Constant Coefficient Elliptic Operators

We consider classical symbol classes $S^m_{1,0} = S^m$, which consist of functions $\varphi(x, \xi)$ on $\mathbb{R}^n \times \{\mathbb{R}^n \backslash 0\}$ N times differentiable in $\xi \in \mathbb{R}^n \backslash 0$ and such that all seminorms

$$|\varphi|_\alpha = \sup_{|\xi| \geqslant 1} \| \xi|^{-m+|\alpha|} D^\alpha_\xi \varphi| \qquad (|\alpha| \leqslant N) \tag{1.1}$$

are finite. Here $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index on $\mathbb{R}^n$ and D^α the corresponding partial derivative. Denote by $|\varphi|_S$ the norm of φ in S^m,

$$|\varphi|_S = \sum_{|\alpha| \leqslant N} |\varphi|_\alpha. \tag{1.2}$$

Each symbol φ defines a pseudodifferential operator by the formula

$$\varphi(x, D) f = \int \varphi(x, \xi)\, e^{-ix \cdot \xi} \hat{f}(\xi)\, d\xi. \tag{1.3}$$

Here $\hat{f}$ denotes the Fourier transform of f

$$\hat{f}(\xi) = \mathscr{F}(f)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x)\, e^{ix\cdot\xi}\, dx.$$

One can rewrite (3) as

$$\varphi(x, D) f = \int K(x, y) f(y)\, dy$$

with a distribution kernel

$$K(x, y) = \int \varphi(x, \xi)\, e^{i\xi\cdot(y-x)}\, d\xi = \hat{\varphi}^{(2)}(x, y-x); \tag{1.4}$$

a superscript (2) indicates the Fourier transform in the second variable ξ. Constant coefficient operators $\varphi(D)$ correspond to symbols φ which depend only on ξ, i.e., $K(x, y)$ is a convolution kernel $K(x, y) = \hat{\varphi}(y-x)$.

In the study of certain naural convolution operators (e.g., homogeneous ones $(-\Delta)^s$, $s \notin \mathbb{Z}$), appropriate symbols must be allowed singularities at the origin. For this purpose we introduce symbol classes $S^{m;l}_{1,0}$ which consist of functions $\varphi(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n \backslash 0$ such that for constants c_α,

$$|D^\alpha_\xi \varphi| \leqslant c_\alpha \begin{cases} 1 + |\xi|^{l-|\alpha|}, & |\xi| \leqslant 1 \\ |\xi|^{m-|\alpha|}, & |\xi| \geqslant 1 \end{cases} \qquad (\alpha \leqslant N); \tag{1.5}$$

the corresponding local seminorms which measure the local (zero) singularity of symbols are

$$|\varphi|^\alpha = \|(1 + \xi^{l-|\alpha|})^{-1}\, \partial^\alpha_\xi \varphi\|_{L^\infty(|\xi|<1)}. \tag{1.6}$$

The symbol class $S^{m,l}_{1,0}$ contains $S^m_{1,0}$; the latter differs only in that its symbols are regular at the origin. An example of the above symbol class arises in symbols of the form $\varphi = \psi \circ a$, where $\psi \in S^{-m'}_{1,0}$, and $a(\xi)$ is homogeneous of degree $m > 0$. Using the chain rule

$$\partial^\alpha \varphi \circ a = \sum_{\substack{1 \leqslant k \leqslant |\alpha| \\ \alpha_1 + \cdots + \alpha_k = \alpha}} C_{\alpha_1, \ldots, \alpha_k} \phi^{(k)} \circ a \prod_{j=1}^{k} (\partial^{\alpha_j} a), \tag{1.7}$$

it can be shown that $\varphi \in S^{-m'm, m}_{1,0}$, i.e., φ has negative order $-mm'$ at infinity and positive order m at 0. Symbols such as these will arise in Theorem 1.

The kernels of classical pseudodifferential operators of nonpositive order have been studied quite thoroughly. It is well known, for example, that pseudodifferential operators of 0 order ($m=0$) are given by Calderon–Zygmund kernels (see [1, 4]) $K(x,y)$ satisfying

$$\int_{|x-y_0|>2|y-y_0|} |K(x,y)-K(x,y_0)|\,dx \leqslant C.$$

Thus they are bounded in all L^p-spaces ($1<p<\infty$) and are of (1, 1)-weak type.

For symbols of negative order at ∞ and positive order at $\{0\}$ we can say more about the "local" (near zero) and "global" (at ∞) singularities of the kernel K. They correspond to respectively "global" (1.1) and "local" (1.6) estimates of symbols. Let $\varphi \in S_{1,0}^{-m,l}$, and

$$K(z)=\int e^{iz\cdot\xi}\varphi(\xi)\,d\xi \qquad (z=x-y)$$

be its convolution kernel. We will analyze $K(z)$ using standard methods to separate local and nonlocal behavior. Let $\chi(\xi)\in C^\infty$ satisfy

$$\begin{aligned} \chi(\xi) &= 1, \qquad |\xi|\leqslant \tfrac{1}{2}, \\ &= 0, \qquad |\xi|\geqslant 1, \end{aligned}$$

and write

$$\varphi(\xi)=\chi\varphi+(1-\chi)\,\varphi. \tag{1.8}$$

To handle the second term, we need

PROPOSITION 1 (cf. [6]). *If* $\varphi\in S^{-m}$, *then*

$$|K(z)|\leqslant Ch_{s,t}(|z|),$$

where

$$\begin{aligned} h_{s,t}(|z|) &= |z|^{-s}\ (-\ln|z| \text{ if } s=0), \qquad &|z|\leqslant 1,\\ &= |z|^{-t}, &|z|\geqslant 1, \end{aligned} \tag{1.9}$$

where $s=\max(n-m,0)$, $t=N\geqslant n$, *and*

$$C=C'\sum_{|\alpha|\leqslant N}|\varphi|_\alpha,$$

where C' *is independent of* φ.

The first term of (1.8) is singular near 0; we thus need

PROPOSITION 2. *Let $\varphi \in S^{m;l}_{1,0}$ have compact support, with $l > 0$. Then*

(a) *$\hat{\varphi}$ is bounded near $\{0\}$ and bounded by $|x|^{-t}$ for some $t > n$ for large x.*

(b) *moreover, if φ is homogeneous of degree l at $\{0\}$, i.e.,*

$$\varphi(\xi) = |\xi|^l \, \psi(\xi') \, \chi(|\xi|), \tag{1.10}$$

where $\xi' = \xi/|\xi|$ and χ has compact support, then we may choose $t = -n - l$.

Both results are fairly standard in pseudodifferential operator calculus, especially the local singularity of the kernel. The global one can be studied in a similar fashion by writing the kernel $K(z)$ as the Fourier transform of its symbol (1.4) and "cutting off" a suitable small neighborhood of $\{0\}$ in ξ-space whose size depends on $|x|$. For symbols homogeneous near $\{0\}$, more precise asymptotics can be obtained by separating variables in polar coordinates and estimating the angular integral by the stationary phase method. We omit the details. We note the following consequence of Proposition 1, which will be needed for the proof of Lemma 1 in Section 2. Let $\varphi(D)$ ($\varphi \in S^{-m}_{1,0}$) be a constant coefficient pseudodifferential operator given by a convolution kernel $K(x-y)$, and $\mathbb{R}^n$ decompose into the direct sum $V \oplus U$ ($\dim V = n'$, $\dim U = n''$). Then the $L^{p,q}$-mixed norms of K

$$\|K\|_{p,q} = \left(\int_U \left(\int_V |K(x' + x'')|^p \, dx' \right)^{q/p} dx'' \right)^{1/q} \tag{1.11}$$

are bounded for all p, q such that

$$n - m < \frac{n'}{p} + \frac{n''}{q}, \text{ i.e., } \frac{n'}{p'} + \frac{n''}{q'} < m \quad \left(\frac{1}{p'} = 1 - \frac{1}{p};\ \frac{1}{q'} = 1 - \frac{1}{q} \right). \tag{1.12}$$

Moreover,

$$\|K\|_{p,q} \leqslant C_{p,q} \, |\varphi|_S, \tag{1.13}$$

and in particular

$$\|K\|_{p,1} \leqslant C_p \, |\varphi|_S$$

for all p such that $n'/p' < m$.

In applications, two important families of symbols are

(1) $\varphi_\varepsilon(\xi) = \varphi(\varepsilon a(\varepsilon^{-1/m} \xi))$ $\quad (\varepsilon > 0)$,

(2) $\varphi_\theta(\xi) = e^{i\theta}/(e^{i\theta} - a(\xi))$ $\quad (\theta \neq 0)$.

Here $a = a(\xi) \in S^m_{1,0}$ is a positive elliptic symbol ($a(\xi) > 0$) of order $m > 0$ and $\varphi \in S^{-m'}_{1,0}$ for some $m' > 0$.

Both of these appear in the summability theory of positive constant coefficient operators $A = a(D)$. The first family corresponds to dilations $\{\varphi(\varepsilon A)\}_{\varepsilon>0}$, while the second appears in the study of "resolvent summability," using $\{\zeta(\zeta - A)^{-1}\}$. Recall that a symbol $a \in S^m_{1,0}$ is called *elliptic* if $a(\xi)$ is invertible for sufficiently large ξ and $a^{-1}(\xi) = 1/a(\xi) \in S^{-m}_{1,0}$ modulo a function with compact support. This means that for large ξ, $a(\xi)$ is bounded from both sides by $|\xi|^m$,

$$c_2 |\xi|^m \leqslant |a(\xi)| \leqslant c_1 |\xi|^m, \tag{1.14}$$

where

$$c_1 = |a|_S, \qquad c_2 = |a^{-1}|_S^{-1}.$$

We can now formulate summability results for constant coefficient elliptic operators.

THEOREM 1. *If $A = a(D)$ is positive elliptic in $S^{m;l}_{1,0}$, with m, $l > 0$, and if $\varphi \in S^{-m'}_{1,0}$, then*

(a) *$\varphi(\varepsilon A)$ is given by a convolution kernel $K_\varepsilon(x-y)$ with radial bound*

$$K_\varepsilon(z) \leqslant c\varepsilon^{-n/m} h_{s,t}(\varepsilon^{-1/m} |z|),$$

with c depending on the seminorms $|a|_S$ and $|\varphi|_S$, where $s < n < t$.

(b) *The resolvent $(\zeta - A)^{-1}$ is given by a convolution kernel $R_\zeta(x-y)$, such that*

$$|R_\zeta(z)| \leqslant \frac{c}{|\sin(\theta/2)|^{n+1}} \rho^{n/m} h_{s,t}(\rho^{1/m} |z|), \tag{1.15}$$

where $\zeta = \rho e^{i\theta}$.

(c) *(summability) if $f \in L^p$, $1 \leqslant p < \infty$, the family of operators $\{\varphi(\varepsilon A) f(x)\}_{\varepsilon>0}$ converges to $\varphi(0) f(x)$ in L^p-norm and on the Lebesgue set of f. Similarly, as $|\zeta| \to \infty$, $(\zeta - A)^{-1} f \to f$, uniformly in each sector*

$$\Omega_{\theta_0} = \{\zeta \in \mathbb{C} : |\arg \zeta| \geqslant \theta_0 > 0\}.$$

Proof. Note that $K_\varepsilon(z) = \varepsilon^{-n/m} L_\varepsilon(\varepsilon^{-1/m} z)$, where L_ε is the kernel of $\varphi_\varepsilon(\xi) = \varphi(\varepsilon a(\varepsilon^{-1/m} \xi))$, and $R_\zeta(z) = \rho^{n/m} R_\theta(\rho^{1/m} z)$, where $R_\zeta(z)$ is the inverse Fourier transform of $\psi_\theta(\xi) \equiv e^{i\theta}/(e^{i\theta} - a(\xi))$. Accordingly, we study the families $L_\varepsilon(z)$ and $R_\theta(z)$ in appropriate symbol classes. The symbol $\varphi_\varepsilon(\xi)$ may become singular at $\xi = 0$ as ε approaches 0, so in a standard way we cut off the part of the symbol near 0, writing

$$\varphi_\varepsilon = \chi\varphi_\varepsilon + (1 - \chi)\varphi_\varepsilon,$$

where $\chi \in C^{\infty}$ and

$$\chi(\xi) = 1, \qquad |\xi| \leqslant \tfrac{1}{2},$$
$$= 0, \qquad |\xi| \geqslant 1.$$

For $|\xi| \geqslant 1$, the term $(1-\chi)\varphi_\varepsilon$ can be bounded uniformly in $S^m_{1,0}$, so that an appropriate function $ch_{s_1,t_1}(|x|)$ bounds the corresponding kernel.

To deal with the first term, we notice that the support of $\chi\varphi_\varepsilon$ is in $\{|\xi| \leqslant 1\}$, and for ξ in this region $\varphi^{(k)} \circ a_\varepsilon$ is bounded, where $a_\varepsilon(\xi) = \varepsilon a(\varepsilon^{-1/m}\xi)$. On the other hand, although $a_\varepsilon(\xi)$ is not uniformly bounded in $S^m_{1,0}$, it is in $S^{m,k}_{1,0}$, where $k = \min(m, l)$. Hence, according to the Leibniz rule, (1.5), and (1.7),

$$\begin{aligned} |D^\beta(\chi(\varphi \circ a_\varepsilon))| &\leqslant \sum_{\substack{\alpha_1 + \cdots + \alpha_i = \alpha \\ 1 \leqslant i \leqslant |\alpha| \\ |\alpha| \leqslant |\beta|}} C_\alpha \cdot C_{\alpha_1 \cdots \alpha_i} \prod_{j=1}^{i} |D^{\alpha_j} a_\varepsilon| \\ &\leqslant \sum C_\alpha C_{\alpha_1 \cdots \alpha_i} \prod_{j=1}^{i} |a_\varepsilon|^\alpha (1 + |\xi|^{k-|\alpha_j|}) \\ &\leqslant K_\beta(1 + |\xi|^{k-|\beta|}) \qquad (|\xi| \leqslant 1), \end{aligned}$$

where the norm $|a_\varepsilon|^\alpha$ is taken in $S^{\cdot,k}$; therefore $\chi(\varphi \circ a_\varepsilon)$ is uniformly bounded in $S^{\cdot,k}_{1,0}$. Thus, by Proposition 2, $\mathscr{F}(\chi(\varphi \circ a_\varepsilon))(x) \leqslant c_2 h_{s_2,t_2}(|x|)$, where $s_2 < n < t_2$. We conclude that

$$L_\varepsilon(z) \leqslant ch_{s,t}(|z|) \qquad (s < n < t),$$

uniformly for small ε.

To obtain a bound on $R_\theta(z)$, we note that $\psi_\theta(\xi) \in S^{-1}_{1,0}(\mathbb{R})$, with $|\psi_\theta|_k, \leqslant 1/|\sin(\theta/2)|^{k+1}$, and then use arguments similar to those for the previous case to show (1.15).

To prove (c), we need the following modification of a well-known result in Fourier analysis [10, Theorem 1.25].

Proposition 3. *Let a family of kernels $\{L_\varepsilon(x,y)\}_\varepsilon$ be bounded by L^1-dilations of an L^1 radial decreasing function h, i.e.,*

$$|L_\varepsilon(x,y)| \leqslant \varepsilon^{-n} h(\varepsilon^{-1}|x-y|). \tag{1.16}$$

Assume also that the family of functions

$$\tilde{L}_\varepsilon(x) \equiv \int L_\varepsilon(x,y)\, dy \to c \tag{1.17}$$

as $\varepsilon \to 0$ in L^{∞}-norm. Then the operators

$$L_\varepsilon(f) = \int L_\varepsilon(x, y) f(y)\, dy \to cf(x)$$

as $\varepsilon \to 0$ in L^p-norm $(1 \leqslant p < \infty)$ and on the Lebesgue set of $f \in L^p$ $(1 \leqslant p \leqslant \infty)$.

In [10, Chap. 1] this result is stated for dilations of a convolution kernel $L_\varepsilon = \varepsilon^{-n} L(\varepsilon^{-1}(x-y))$. A generalized version follows along the same lines. Note that for convolution kernels condition (1.17) means $\hat{L}_\varepsilon(0) \to c$ as $\varepsilon \to 0$.

Theorem 1 follows easily now. Indeed, in the first case $\hat{K}_\varepsilon(0) = \varphi(\varepsilon a(0)) \to \varphi(0)$; in the second

$$\zeta \hat{R}_\zeta(0) = \frac{\zeta}{\zeta - a(0)} \xrightarrow[\zeta \to \infty]{} 1.$$

Remark. It is possible to obtain bounds which are better than algebraic for the kernel of $\varphi(a(D))$ and $(\zeta - a(D))^{-1}$ at infinity, if a and φ are assumed smooth at 0. Indeed, $(\zeta + \Delta)^{-1}$ has the convolution kernel

$$K(x) = \int_0^\infty (4\pi t)^{-n/2}\, e^{-|x|^2/4t} e^{t\zeta}\, dt;$$

some calculations show

$$|K(x)| \leqslant C \begin{cases} |x|^{-n+2}, & |x| \leqslant 1 \\ e^{-C_1 |\operatorname{Im}(\zeta^{1/2})|\,|x|}, & |x| > 1 \end{cases} \qquad (n \geqslant 3).$$

An immediate corollary of Theorem 1 is

COROLLARY 1. *A positive (semibounded) constant coefficient elliptic pseudodifferential operator $A = a(D)$ of the class $S^m_{1,0}$ has the same spectrum in all L^p-spaces $(1 \leqslant p \leqslant \infty)$.*

2. Resolvent of the Perturbed Operator

We consider a class of operators on $\mathbb{R}^n$ of the form $A = A_0 + B$, where the leading term A_0 is a constant coefficient homogeneous positive elliptic operator of order m, and a perturbation

$$B = \sum_{|\alpha| < m} b_\alpha(x)\, D^\alpha$$

has coefficients in certain L^r-classes. More generally, b_α can be defined on the quotient space $V_\alpha = \mathbb{R}^n/U_\alpha$ and be in $L^r + L^\infty$ on V_α (cf. example (2) of the Introduction). We want to study the resolvent of A using the perturbation series

$$(\zeta - A)^{-1} = (\zeta - A_0)^{-1} \sum_{k=0}^{\infty} [B(\zeta - A_0)^{-1}]^k. \tag{2.1}$$

The convergence of (1) as well as other properties of $(\zeta - A)^{-1}$ are determined by the operator norm of $B(\zeta - A_0)^{-1}$. Lemma 1 gives an estimate of $\|B(\zeta - A_0)^{-1}\|$ as a function of ζ.

LEMMA 1. *Let A_0 be a constant coefficient positive homogeneous elliptic operator of order m and $B = \sum_{|\alpha| < m} b_\alpha(x) D^\alpha$ have coefficients $b_\alpha \in (L^{r_\alpha} + L^\infty)(V_\alpha)$ (dim $V_\alpha = n_\alpha$). If the L^r-classes of the coefficients b_α satisfy*

$$d = \max_\alpha \left\{ \frac{n_\alpha}{r_\alpha} + |\alpha| \right\} < m \qquad (|\alpha| < m), \tag{2.2}$$

the operator norm of $B(\zeta - A_0)^{-1}$ in all L^p-spaces $1 \leqslant p \leqslant \min\{r_\alpha\}$ is estimated as

$$\|B(\zeta - A_0)^{-1}\| \leqslant \frac{c_p}{|\sin(\theta/2)|^{n+1}} |\zeta|^{d/m - 1}, \tag{2.3}$$

where $\theta = \arg \zeta \neq 0$.

Proof. Each coefficient b_α is a sum of two terms $b'_\alpha \in L^{r_\alpha}$ and $b''_\alpha \in L^\infty$: it suffices to consider $B = b_\alpha(x) D^\alpha$, with $b_\alpha \in L^{r_\alpha}$ $(r_\alpha \leqslant \infty)$. Notice that $B(\zeta - A_0)^{-1} = b_\alpha D^\alpha(\zeta - a(D))^{-1}$ is composed of two operators: a convolution with kernel

$$K_\zeta(x) = \mathscr{F}^{-1}(\xi^\alpha(\zeta - a(\xi))^{-1}) \tag{2.4}$$

and a multiplication by $b_\alpha \in L^{r_\alpha}(\mathbb{R}^n)$ or $b_\alpha \in L^{r_\alpha}(V_\alpha)$ $(V_\alpha = \mathbb{R}^n/U_\alpha)$. Let us first study the kernel K_ζ. Using the homogeneity of $a(\xi)$ we can write

$$K_\zeta(x) = |\zeta|^{((n+|\alpha|)/m)-1} K_\theta(|\zeta|^{1/m} x), \tag{2.5}$$

where

$$\hat{K}_\theta = \varphi_\theta(\xi) = \xi^\alpha(e^{i\theta} - a(\xi))^{-1}.$$

By the ellipticity of $a(\xi)$, $\varphi_\theta \in S_{1,0}^{-m+|\alpha|}$, so Proposition 1 applies to φ_θ. In particular, for $s = n - m + |\alpha| < n$ and some $t > n$, $K_\theta(x)$ has a radial bound

$$|K_\theta(x)| \leqslant C \|\varphi_\theta\|_s h_{s,t}, \qquad h_{s,t} = \begin{cases} |x|^{-s}, & |x| \leqslant 1, \\ |x|^{-t}, & |x| > 1. \end{cases} \tag{2.6}$$

We have $K_\theta \in L^q$ for all $1 \leqslant q < n/(n-m+|\alpha|)$ and is also included in all mixed $L^{q,1}$-classes with respect to a decomposition $\mathbb{R}^n = V_\alpha + U_\alpha$, where q satisfies

$$\frac{n_\alpha}{q} + (n - n_\alpha) > n - m + |\alpha| \qquad (n_\alpha = \dim V_\alpha),$$

i.e.,

$$\frac{n_\alpha}{q'} + |\alpha| < m \qquad \left(\frac{1}{q'} = 1 - \frac{1}{q}\right) \tag{2.7}$$

(see Eq. (1.12)). Furthermore, the $L^q(L^{q,1})$-norm of K_θ is estimated as

$$\|K_\theta\|_q \leqslant c_q \|\varphi_\theta\|_S, \qquad \|K_\theta\|_{q,1} \leqslant c_q' \|\varphi_\theta\|_S. \tag{2.8}$$

Let us note that by the lemma's assumption $r_\alpha = q'$ satisfies (2.7). So K_θ and consequently K_ζ is always in the dual class $L^{r_\alpha'}(L^{r_\alpha',1})$ to b_α. By (2.5) the $L^{r_\alpha',1}$-norm of K_ζ is equal to

$$\|K_\zeta\|_{r_\alpha',1} = |\zeta|^{(1/m)((n_\alpha/r_\alpha)+|\alpha|)-1} \|K_\theta\|_{r_\alpha',1}. \tag{2.9}$$

We use two basic interpolation inequalities for multiplication and convolution,

$$\|bf\|_r \leqslant \|b\|_q \|f\|_p \qquad \left(\frac{1}{p} + \frac{1}{q} = \frac{1}{r}\right),$$

$$\|K * f\|_r \leqslant \|K\|_q \|f\|_p \qquad \left(\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1\right),$$

and write

$$\|b_\alpha K_\zeta * f\|_p \leqslant \|b_\alpha\|_{r_\alpha} \|K_\zeta\|_q \|f\|_{p''} \tag{2.10a}$$

when $b_\alpha \in L^{r_\alpha}$ on $\mathbb{R}^n$, and

$$\|b_\alpha K_\zeta * f\|_p \leqslant \|b_\alpha\|_{r_\alpha} \|K_\zeta\|_{q,1} \|f\|_{p'',p} \tag{2.10b}$$

when $b_\alpha \in L^{r_\alpha}(V_\alpha)$. In both cases

$$\frac{1}{p''} = \frac{1}{p} - \frac{1}{r_\alpha} + 1 - \frac{1}{q}. \tag{2.11}$$

As we mentioned, by the lemma's assumption and (2.7), q can be taken equal to r_α'. Hence $p'' = p$ in (2.11) and (2.10b) takes the form

$$\|b_\alpha K_\zeta * f\|_p \leqslant \|b_\alpha\|_{r_\alpha} \|K_\zeta\|_{r_\alpha',1} \|f\|_p. \tag{2.12}$$

Of course, in order that this procedure make sense (i.e., the convolution with $K_\zeta \in L^{r'_\alpha}$ does not "push" the L^p-class of f out of the scale $\{L^r : 1 \leqslant r \leqslant \infty\}$). we need $p \leqslant r_\alpha$.

The right-hand side of (2.12) is estimated by (2.9) and (2.8). So we get

$$\|b_\alpha K_\zeta * f\|_p \leqslant C\,\|b_\alpha\|_{r_\alpha}\,\|\varphi_\theta\|_S\,|\zeta|^{(1/m)((n_\alpha/r_\alpha)+|\alpha|)-1}\,\|f\|_p.$$

Finally, the symbol class norm of φ_θ is estimated as in the proof of Theorem 1,

$$\|\varphi_\theta\|_S \leqslant \frac{c}{|\sin(\theta/2)|^{n+1}} \times \text{polynomial}\{\|a\|_{S,m}, |\beta| \leqslant n+1; \|a^{-1}\|_{S,0}\},$$

which completes the proof of Lemma 1.

Remark 1. The above argument applies to more general pairs (A_0, B). Namely, $A_0 = a(D)$ can be any positive elliptic pseudodifferential operator of order $m > 0$ and $B = \sum b_j(x)\,q_j(D)$ with $q_j(\xi) \in S^{m_j}_{1,0}$ $(m_j < m)$ and $b_j \in L^{r_j}(V_j)$ such that $(\dim V_j)/r_j + m_j < m$.

As a corollary of Lemma 1 we get a priori estimates for the pair (A_0, B).

COROLLARY 2. *If a pair of operators* A_0, B *satisfies the assumptions of Lemma* 1, *then for any* $\varepsilon > 0$ *there exists* $\lambda = \lambda_\varepsilon > 0$ *such that*

$$\|Bf\|_p \leqslant \varepsilon\,\|A_0 f\|_p + \lambda_\varepsilon\,\|f\|_p \qquad (1 \leqslant p \leqslant \min r_\alpha) \tag{2.13}$$

for all f *in the domain* $\mathscr{D}_p(A_0)$ *of* A_0 *in* L^p.

Indeed, by Lemma 1,

$$\|B(\zeta - A_0)^{-1} g\|_p \leqslant C(\theta)|\zeta|^{(d/m)-1}\,\|g\|_p,$$

where $d = \max(n_\alpha/r_\alpha) + |\alpha|) < m$. Then for $\lambda > 0$, $\varepsilon > 0$

$$\|B(\varepsilon A_0 + \lambda)^{-1} g\|_p < c\varepsilon^{-1}\left(\frac{\lambda}{\varepsilon}\right)^{(d/m)-1}\,\|g\|_p. \tag{2.14}$$

Taking λ sufficiently large we can make the scalar factor on the right-hand side of (2.14) less than 1. Then for $f = (\varepsilon A_0 + \lambda)^{-1} g \in \mathscr{D}_p(A_0)$ we get

$$\|Bf\|_p \leqslant \varepsilon\,\|A_0 f\|_p + \lambda\,\|f\|_p. \qquad \text{Q.E.D.}$$

Note that for all L^p $(1 < p < \infty)$ the domain $\mathscr{D}_p(A_0)$ is the Sobolev space $\mathscr{L}^p_m = (1 - \Delta)^{-m/2}(L^p)$. Indeed, the operator $(1 + A_0)^{-1}(1 - \Delta)^{m/2}$ is bounded and invertible on all L^p $(1 < p < \infty)$ (see [10, Chap. 6]).

In the Hilbert space setting, a priori estimates (2.13) are used in order to prove essential self-adjointness of perturbations $A = A_0 + B$. The Kato–Rellich theorem (see [5, Chap. 5]) states: if B is small relative to A_0, i.e.,

$$\|Bf\| \leqslant a\,\|A_0 f\| + b\,\|f\|,$$

for some $a < 1$, $b > 0$, then $A = A_0 + B$ is essentially self-adjoint on $\mathscr{D}(A_0)$, or any essential domain of A_0. Thus we get

COROLLARY 3. *A formally self-adjoint operator $A = A_0 + B$ satisfying the assumptions of Lemma* 1 *and such that* $\min r_\alpha \geqslant 2$ *is essentially self-adjoint on $\mathscr{D}_2(A_0) = \mathscr{L}^2_m$, or any essential domain of A_0.*

Remark 2. Condition (2) of Lemma 1 can be sharpened by inclusion of the equality

$$\frac{n_\alpha}{r_\alpha} + |\alpha| = m. \tag{2.2'}$$

Most of the above results remain true in this case, with suitable modification.

In this case the lemma takes the following form: for all $1 < p \leqslant \min_\alpha\{r_\alpha\}$ ($p \neq \infty$), the operator $B(\zeta - A_0)^{-1}$ is L^p-bounded, and its norm is estimated as

$$\|B(\zeta - A_0)^{-1}\| \leqslant \frac{C_p}{|\sin(\theta/2)|^{n+1}} \sum_\alpha \|b_\alpha\|. \tag{2.15}$$

Notice the absence of the factor $\rho^{d/m-1}$.

The proof of (2.15) follows along the same lines, with one important distinction. The convolution kernel $K_\alpha = D^\alpha(\zeta - A_0)^{-1}$ is considered in the weak L^q-class with the "sharp" value of q, $q = n/(n - m + |\alpha|)$ and instead of Young's inequality for convolutions with "strong" L^q-kernels K_α one applies the Hardy–Littlewood–Sobolev inequality for the weak classes L^q_w (see [11, Chap. 2, Sect. 5). This excludes the limit value $p = 1$ and introduces the constant C_p which depends on $1 < p < \infty$.

Now we indicate how the corollaries of Lemma 1 change in this case.

COROLLARY 2′ (A priori estimates). *As before the L^p-domain of B, $\mathscr{D}_p(B) \subseteq \mathscr{D}_p(A_0) = \mathscr{L}^p_m$ for all $1 < p \leqslant \min_\alpha\{r_\alpha\}$ and for any $f \in \mathscr{D}_p(A_0)$ one has*

$$\|Bf\|_p \leqslant \gamma\,\|A_0\|_p + \rho\gamma\,\|f\|_p$$

with ρ a constant and $\gamma = c_p \sum \|b_\alpha\|$.

In order to have B "small relative to A_0," it is no longer enough for the coefficients of B to belong to suitable L^r-classes; their norms must also be small.

For instance, in order to apply the Kato–Rellich theorem (essential self-adjointness) it suffices that $C_2 \sum_\alpha \|b_\alpha\| < 1$. In this case the operator $A = A_0 + B$ is semibounded from below. Indeed, the geometric series,

$$\sum_0^\infty R^0_{-\lambda_0}(BR^0_{-\lambda_0})^k \qquad (\lambda_0 > 0)$$

sums to the resolvent kernel of A, which is therefore L^2- (L^p-) bounded.

The necessity to limit the coefficient norms in the perturbation B in order to have B "small relative to A_0" is not very unusual in itself. This is illustrated by the well-known example of the Schrödinger operator $A = -\Delta - (k/|x|^2)$ in $\mathbb{R}^3$ (see [7]). For $k \leqslant k_c$, A is semibounded from below, whereas for $k > k_c$ the whole negative axis is included in its spectrum ($k_c = \frac{1}{4}$ in one dimension). In other words, passing a certain magnitude of the perturbation norm completely changes the nature of the operator. Whether such a phenomenon occurs more generally for the class of perturbations considered in this paper is not known, and will require more precise analysis.

3. Radial Bounds for the Resolvent

In this section we shall establish radial bounds for the kernel of the resolvent $(\zeta - A)^{-1}$ of operators $A = A_0 + B$ discussed in Section 2. Namely,

THEOREM 2. *Let $A = A_0 + B$ satisfy the assumptions of Lemma* 1. *Denote by*

$$d = \max\left\{\frac{n_\alpha}{r_\alpha} + |\alpha| : |\alpha| < m\right\}.$$

There exists a constant $C > 0$ depending on A such that the L^p-resolvent set of A $(1 \leqslant p \leqslant \min r_\alpha)$ contains the domain

$$\Omega = \left\{\zeta = \rho e^{i\theta} : C\rho^{(d/m)-1} < \left|\sin\left(\frac{\theta}{2}\right)\right|^{n+1}\right\} \tag{3.1}$$

and for each $\zeta \in \Omega$ the resolvent $R_\zeta = (\zeta - A)^{-1}$ is in the class $|RB|$. Moreover, the kernel $L_\zeta(x, y)$ of the operator $(\zeta - A)^{-1}$ is estimated in Ω as

$$|L_\zeta(x, y)| \leqslant \frac{C}{|\sin(\theta/2)|^{n+1}}\left(1 - \frac{C\rho^{(d/m)-1}}{|\sin(\theta/2)|^{n+1}}\right)^{-1} \rho^{(n/m)-1} h_{s,t}(\rho^{1/m}|x - y|). \tag{3.2}$$

where $h_{s,t}$ is an L^1 radial decreasing bound of Section 1. *In the remaining L^p-spaces ($p > \min r_\alpha$) the operator R_ζ is a bounded left inverse of $\zeta - A$ for $\zeta \in \Omega$.*

Recall that the radial bounds we are dealing with are functions

$$\begin{aligned} h_{s,t}(x) &= |x|^{-s}, \qquad |x| \leqslant 1 \\ &= |x|^{-t}, \qquad |x| > 1 \end{aligned} \qquad (s < n, t > n). \tag{3.3}$$

We shall need two modifications of the notion of convolution on $\mathbb{R}^n$. One of them (let us call it the p-convolution) is defined as

$$f *_p g = (|f|^p * |g|^p)^{1/p} = \left(\int |f(x-y)\, g(y)|^p \, dy\right)^{1/p} \quad (1 \leqslant p < \infty). \tag{3.4}$$

A more general notion is the so-called (p, q)-mixed convolution. We decompose $\mathbb{R}^n$ into a direct sum of subspaces $V \oplus U$ ($\dim V = n'$, $\dim U = n''$) and write (x', x'') for the components of $x \in \mathbb{R}^n$. Then the (p, q)-mixed convolution of functions f and g is defined as

$$f *_{p,q} g = \left(\int_U \left(\int_V |f(y' + y'')\, g(x' + x'' - y' - y'')|^p \, dy'\right)^{q/p} dy''\right)^{1/q}. \tag{3.5}$$

The convolution $f *_{p,q} g$ is simply the $L^{p,q}$-mixed norm of the function $F(y) = F(y', y'') = f(y)\, g(x-y)$; this allows extension of the definition to infinite p or q.

LEMMA 2. *Let $h_{s,t}$ and $h_{s',t'}$ be a pair of functions of the type* (3). *If t, $t' > n$ and s, s' satisfy*

$$\max\{s, s'\} < \frac{n'}{p} + \frac{n''}{q},$$

then the (p, q)-mixed convolution of $h_{s,t}$ and $h_{s',t'}$ is bounded by $ch_{s'',t''}$, where $s'' = \min\{s, s'\}$, $t'' = \min\{t, t'\}$.

The proof of Lemma 2 involves standard computational techniques: "dilations," "truncating integrals to balls," etc., and we omit it. Notice that the condition of Lemma 2 is equivalent to finiteness of the (p, q)-mixed norms $\|h_{s,t}\|_{p,q} < \infty$ and $\|h_{s',t'}\|_{p,q} < \infty$.

Lemma 2 extends by induction to a sequence of mixed convolutions.

COROLLARY 4. *Let $\mathbb{R}^n = V_i \oplus U_i$ ($i = 1,..., k$) be a sequence of partitions of $\mathbb{R}^n$, $\dim V_i = n_i$, $\dim U_i = n_i'$. If sequences of reals $s_0,..., s_k$; $t_0,..., t_k$; $p_1,..., p_k$; $q_1,..., q_k$ satisfy*

$$t_i > n;\ \max\{s_i;\ \min\{s_1, \ldots, s_{i-1}\}\} < \left(\frac{n_i}{p_i} + \frac{n_i'}{q_i}\right) \tag{3.6}$$

for $1 \leqslant i \leqslant k$, *then a sequence of mixed convolutions of* $\{h_{s_i, t_i}\}_{i=0}^{k}$ *is estimated by*

$$(h_{s_0, t_0} * h_{s_1, t_1}) * \cdots * h_{s_k, t_k} \leqslant c^k h_{s', t'}, \tag{3.7}$$

where

$$s' = \min\{s_0, \ldots, s_k\}, \qquad t' = \min\{t_0, \ldots, t_k\}.$$

The ith $*$ in (3.7) means a (p_i, q_i)-mixed convolution with respect to the ith splitting $\mathbb{R}^n = V_i \oplus U_i$.

Proof of Theorem 2. Denote by R_ζ^0 and R_ζ the resolvents (when they exist) of the unperturbed operator A_0 and the perturbation $A = A_0 + B$, respectively. Recall that R_ζ is given by a series

$$R_\zeta = \sum_{k=0}^{\infty} R_\zeta^0 (B R_\zeta^0)^k. \tag{3.8}$$

We shall study each term $L_k = R_\zeta^0 (B R_\zeta^0)^k$ of the series (3.8) and show that L_k is in |RB|. Furthermore we obtain an estimate on the radial bound h of $L_k(x, y)$ as a function of the large parameter ζ (cf. Lemma 1).

As before denote by $a(\xi) = \sum_{|\xi| = m} b_\alpha \xi^\alpha$ the leading symbol of A (a homogeneous positive elliptic polynomial of degree m) and by $\varphi(\xi)$, $\varphi_\alpha(\xi)$ the Fourier multipliers

$$\varphi(\xi) = (\zeta - a(\xi))^{-1}, \qquad \varphi_\alpha(\xi) = \xi^\alpha (\zeta - a(\xi))^{-1} \quad (|\alpha| < m).$$

Let $K = K_\zeta(x)$ and $K_{\alpha, \zeta}(x)$ be corresponding convolution kernels (inverse Fourier transforms of φ and φ_α). Obviously R^0 is a convolution with $K = K_0$ while $L_k = R^0 (B R^0)^k$ consists of combinations of convolutions and multiplications

$$L_k = \sum L_{\alpha_1 \cdots \alpha_k}, \tag{3.9}$$

where

$$L_{\alpha_1 \cdots \alpha_k} f = K_0 * (b_1 (K_1 * \cdots * (b_k (K_k * f) \cdots).$$

For simplicity we write b_i for b_{α_i}, K_i for K_{α_i}, etc. The multi-indices $\alpha_1 \cdots \alpha_k$ in (3.9) vary over the set of all multi-indices which appear in B. If their number is N, there are N^k terms in (3.9).

It suffices to estimate each term $L_{\alpha_1 \cdots \alpha_k}$ of L_k. Write the kernel $L_{\alpha_1 \cdots \alpha_k}(x, y)$ as a multiple integral

$$L_{\alpha_1 \cdots \alpha_k}(x, y) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} K_0(x - z_1)\, b_1(z_1) \cdots b_k(z_k)\, K_k(z_k - y)\, dz_1 \cdots dz_k \tag{3.10}$$

for $(z_i \in \mathbb{R}^n)$. Decomposing b_α into the sum $b_\alpha = b'_\alpha + b''_\alpha$ $(b'_\alpha \in L^{r_\alpha};\ b''_\alpha \in L^\infty)$ we can always assume that $b_\alpha \in L^{r_\alpha}$ $(1 \leqslant r_\alpha \leqslant \infty)$. If each function b_α belongs to L^{r_α} on all of $\mathbb{R}^n$ we use a multiple Hölder inequality

$$\begin{aligned} |L_{\alpha_1 \cdots \alpha_k}(x, y)| \leqslant \left(\prod_{j=1}^{k} \|b_j\|_{r_j}\right) \| K_0(x - z_1)\, K_1(z_1 - z_2) \\ \cdots K_k(z_k - y)\|_{p_1 \cdots p_k} \end{aligned} \tag{3.11}$$

p_j being the dual Hölder index to r_j $((1/p_j) + (1/r_j) = 1)$ and $\| \cdots \|_{p_1 \cdots p_k}$ a $(p_1 \cdots p_k)$-mixed norm

$$\|F\|_{p_1 \cdots p_k} = \left(\int dz_k \left(\int dz_{k-1} \cdots \left(\int dz_1\, |F(z_1 \cdots z_k)|^{p_1}\right)^{p_2/p_1}\right) \cdots\right)^{1/p_k}. \tag{3.12}$$

For $F = K_0(x - z_1)\, K_1(z_1 - z_2) \cdots K_k(z_k - y)$, (3.12) becomes a sequence of p-convolutions

$$(\cdots ((|K_0| *_{p_1} |K_1|) *_{p_2} |K_2|) \cdots) *_{p_k} |K_k|,$$

with $-y$ understood to be in the argument of K_k.

If the coefficients b_α of B are defined on quotients $V_\alpha = \mathbb{R}^n / U_\alpha$ $(\dim V_\alpha = n_\alpha)$, i.e., $b_\alpha \in L^{r_\alpha}(V_\alpha)$, we apply the Hölder inequality step by step in the variables of b_α. Let us illustrate this procedure for the first term $L_\alpha(x, y) = K_0 * (b_\alpha K_\alpha)(x - y)$. We separate variables for b_α, i.e., write $\mathbb{R}^n = V_\alpha \oplus U_\alpha$ $(\dim V_\alpha = n_\alpha)$, where $x = (x', x'')$, $x' \in V_\alpha$, $x'' \in U_\alpha$. Then

$$L_\alpha(x, y) = \iint K_0(x' - z'; x'' - z'')\, b_\alpha(z')\, K_\alpha(z' - y'; z'' - y'')\, dz'\, dz''.$$

Therefore

$$\begin{aligned} |L_\alpha(x, y)| \leqslant \|b_\alpha\|_{r_\alpha} \bigg(\int \bigg(\int & |K_0(x' - z'; x'' - z'') \\ & \times K_\alpha(z' - y'; z'' - y'')|^{p_\alpha}\, dz'\bigg)^{1/p_\alpha} dz''\bigg), \end{aligned} \tag{3.13}$$

which is by definition the $(p_\alpha, 1)$-mixed convolution of K_0 and K_α. Repeating this procedure k times, we get an estimate similar to (3.11)

$$|L_{\alpha_1 \cdots \alpha_k}(x, y)| \leqslant \left(\prod_1^k \|b_j\|_{r_j}\right)(\cdots((|K_0| * |K_1|) * \cdots) * |K_k|, \quad (3.14)$$

with $-y$ in the argument of K_k.

The ith $*$ in (3.14) means, of course, the $(p_i, 1)$-mixed convolution, but for notational convenience we omit the subscripts. As before b_i denotes b_{α_i}, $n_i = n_{\alpha_i}$ and p_i is the dual Hölder index to r_i. The functions $K_\alpha(x)$ were studied in Lemma 1 of Section 2. We showed that

$$K_\alpha = K_{\alpha,\zeta}(x) = |\zeta|^{(1/m)(n+|\alpha|)-1} K_{\alpha,\theta}(|\zeta|^{1/m} x), \quad (3.15)$$

where $\hat{K}_{\alpha,\theta} = \xi^\alpha (e^{i\theta} - a(\xi))^{-1}$, and $K_{\alpha,\theta}$ is estimated by the radial function $h_{s,t}$,

$$|K_{\alpha,\theta}(x)| \leqslant \frac{c'}{|\sin(\theta/2)|^{n+1}} h_{s,t}(|x|) \quad (3.16)$$

with $t > n$ (one can take $t = n + 1$) and $s = n - (m - |\alpha|)$. The constant c' depends on s, n, m and the symbol-class seminorms of $a(\xi)$ and $a(\xi)^{-1}$.

We now make another simple observation concerning mixed convolutions of dilations $f_\varepsilon(x) = f(\varepsilon x)$,

$$f_\varepsilon *_{p,q} g_\varepsilon = \varepsilon^{-((n'/p)+(n''/q))}(f *_{p,q} g)_\varepsilon.$$

Here $\mathbb{R}^n = V \oplus U$ and $\dim V = n'$, $\dim U = n''$. Using (3.15) and the last remark we have

$$\begin{aligned}(\cdots((K_0 * K_1) * \cdots) * K_k(x) = {} & |\zeta|^{(n/m)-1+\sum_\alpha[(1/m)(|\alpha|+(n_\alpha/r_\alpha))-1]} \quad (3.17)\\ & \times ((K_{0,\theta} * K_{1,\theta}) * \cdots * K_{k,\theta})(|\zeta|^{1/m} x);\end{aligned}$$

the summation is taken over the set $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. The sequence of convolutions in (3.17) is the same $(p_i, 1)$-mixed convolutions as in (3.14).

Next we apply inequality (3.16) to each term on the right-hand side of (3.17) and estimate it by the function

$$\begin{aligned}&\frac{c'}{|\sin(\theta/2)|^{n+1}} |\zeta|^{(n/m)-1} \left(\frac{c'}{|\sin(\theta/2)|^{n+1}} |\zeta|^{(d/m)-1}\right)^k\\ &\qquad \times (\cdots(h_{s_0,t} * h_{s_1,t}) * \cdots) * h_{s_k,t})(|\zeta|^{1/m} x), \quad (3.18)\end{aligned}$$

where $d = \max((n_\alpha/r_\alpha) + |\alpha|)$, and $h_{s_j,t}$ is a suitable radial bound of $K_{j,\theta}$ $(j = 1, \ldots, k)$.

Now we use Corollary 4 of Lemma 2 for the sequence of mixed convolutions (3.18). For this we have only to check conditions (3.6) of the corollary. Let us take exponentials $s_\alpha = n - (m - |\alpha|)$ according to (3.16). Then for a sequence $\{s_0, s_1, ..., s_k\}$, where $(s_i = s_{\alpha_i})$ $\min\{s_0, ..., s_{i-1}\} = s_0$ and $\max\{s_i; \min\{s_0, ..., s_{i-1}\}\} = s_i$ $(i \geqslant 1)$.

If $\mathbb{R}^n = V_i \oplus U_i$ denotes the α_i-splitting ($\dim V_i = n_i = n_{\alpha_i}$), then (3.6) is satisfied if

$$n - (m - |\alpha_i|) < \left(\frac{n_i}{p_i} + n - n_i\right). \tag{3.19}$$

Recalling that $1/p_i = 1 - (1/r_i)$ $(r_i = r_{\alpha_i})$, (3.19) becomes $(n_i/r_i) + |\alpha_i| < m$ (the condition of the theorem). Thus Corollary 4 applies to $\{h_{s_i t_i}\}_{i=0}^k$, i.e.,

$$(\cdots(h_{s_0,t} * h_{s_1,t}) * \cdots) * h_{s_k,t} \leqslant C^k h_{s,t},$$

where $s = \min s_j = s_0$. Hence each term $L_{\alpha_1 \cdots \alpha_k}$ of L_k in (9) is estimated as

$$|L_{\alpha_1 \cdots \alpha_k}(x, y)| \leqslant \left(\prod_\alpha \|b_\alpha\|_{r_\alpha}\right) H_{\zeta,k}(|x-y|),$$

where

$$H_{\zeta,k}(x) = \frac{c'' |\zeta|^{(n/m)-1}}{|\sin(\theta/2)|^{n+1}} \left(\frac{c''}{|\sin(\theta/2)|^{n+1}} |\zeta|^{(d/m)-1}\right)^k h_{s,t}(|\zeta|^{1/m} |x|).$$

From here

$$|L_k(x, y)| \leqslant \sum |L_{\alpha_1 \cdots \alpha_k}| \leqslant \left(\sum_\alpha \|b_\alpha\|\right)^k H_{\zeta,k}(|x-y|).$$

We denote by C a new constant $\max(c''(\sum_\alpha \|b_\alpha\|), c'')$ which depends only on A_0 and B. The geometric series of bounds $(\sum_\alpha \|b_\alpha\|)^k H_{\zeta,k}$ of the kernels $\{L_k(x, y)\}_k$ converges absolutely if

$$\frac{C}{|\sin(\theta/2)|^{n+1}} |\zeta|^{(d/m)-1} < 1;$$

this condition defines the domain $\Omega \subset C$ of the Theorem. Moreover, the sum is estimated by the sum of a geometric series

$$|L_\zeta(x, y)| \leqslant \frac{C}{|\sin(\theta/2)|^{n+1}} \left(1 - \frac{C}{|\sin(\theta/2)|^{n+1}} |\zeta|^{(d/m)-1}\right)^{-1} \times |\zeta|^{(n/m)-1} h_{s,t}(|\zeta|^{1/m} |x-y|).$$

Hence if $\zeta \in \Omega$, the operator R_ζ defined by this kernel is in [RB].

If $1 \leqslant p \leqslant \min_\alpha r_\alpha$ and $\zeta \in \Omega$ then the series

$$R_\zeta(\zeta - A) f = R_\zeta^0 \sum_{k=0}^{\infty} (BR_\zeta^0)^k (\zeta - A_0 - B) f \tag{3.20}$$

is well defined by Lemma 1, and a cancellation may be performed on the right side of (3.20) to yield the identity times f. Thus $R_\zeta(\zeta - A) = I$ on the domain of A_0 (i.e., $\mathscr{L}_m^p$ if $p \neq 1, \infty$) and similarly $(\zeta - A) R_\zeta = I$ on L^p for $\zeta \in \Omega$, so that R_ζ is the resolvent of A.

If $p > \min r_\alpha$, $\zeta \in \Omega$, and

$$f \in \mathscr{D}(A) = \bigcap_{|\alpha| < m} \mathscr{D}(b_\alpha D^\alpha) \cap \mathscr{D}(A_0),$$

then (3.20) still holds, since the operators R_ζ and $R_\zeta^0(BR_\zeta^0)^k$ are in [RB] and hence bounded. However, in execution of the same cancellation there are steps wherein functions under consideration are no longer in the scale of L^p spaces, but only in the sum $L^1 + L^\infty$. The identities become distributional ones until the return to L^p before the final summation, which yields $R_\zeta(\zeta - A) f = f$. This procedure shows that R_ζ is a left inverse of $\zeta - A$, and the theorem is proved.

Remark. The question of whether R_ζ is the resolvent of A in L^p for $p > \min r_\alpha$ is generally less than well defined, since such an operator may have trivial domain. This is the case for $A = -\Delta + V(x)$, where $V \in L^1$ is a function with the property $V(x)|_U \notin L^p$ for all open rectangles $U \subset \mathbb{R}^n$.

We note that the remark subsequent to Lemma 1 applies also to Theorem 2. Some further corollaries of Theorem 2 and Lemma 1 are

COROLLARY 5. *The spectrum of the operator $A = A_0 + B$ in L^p $(1 \leqslant p \leqslant \min r_\alpha)$ is included in the set*

$$\Omega' = \left\{ \zeta = \rho e^{i\theta} : \rho^{(d/m)-1} \geqslant \frac{|\sin(\theta/2)|^{n+1}}{C} \right\},$$

(a parabolic domain about positive real axis). The resolvent of A has maximum decrease in all nonzero directions, i.e.,

$$\|R_\zeta\| \leqslant \frac{c(\theta)}{|\zeta|}, \qquad \forall \theta \neq 0.$$

The norm of the resolvent is easily bounded via the L^1-norm of the kernel (3.2).

COROLLARY 6. *The operator A is closeable in L^p $(1 \leqslant p \leqslant \min r_\alpha)$.*

Another application of Theorem 2 is to resolvent summability for A. We will henceforth use the operators R_ζ and $(\zeta - A)^{-1}$ interchangeably, even when $p > \min r_\alpha$.

THEOREM 3. *If an operator* $A = A_0 + B$ *satisfies the assumptions of Lemma* 1, *then*

$$\zeta(\zeta - A)^{-1} f(x) \to f(x)$$

as $\zeta \to \infty$ *uniformly in each sector* $\Omega_\theta = \{\zeta : |\arg \zeta| \geqslant \theta > 0\}$, *in* L^p*-norm* $(1 \leqslant p < \infty)$ *and on the Lebesgue set of* $f \in L^p$ $(1 \leqslant p \leqslant \infty)$.

Proof. By Proposition 2 we have only to check that the kernel $\zeta L_\zeta(x, y)$ of ζR_ζ satisfies $\zeta \tilde{L}_\zeta(x) = \int \zeta L_\zeta(x, y)\, dy \to 1$ as $\zeta \to \infty$, uniformly in each sector $\Omega_\theta = \{\zeta : |\arg \zeta| \geqslant \theta\}$ $(\theta > 0)$.

We write

$$\int L_\zeta(x, y)\, dy = \sum_{k=0}^{\infty} \int L_k(x, y)\, dy = \sum_{k=0}^{\infty} \sum_{\alpha_1 \cdots \alpha_k} \int L_{\alpha_1 \cdots \alpha_k}(x, y)\, dy.$$

By (3.10)

$$\int L_{\alpha_1 \cdots \alpha_k}(x, y)\, dy = \hat{K}_k(0) \cdot K_0 * (b_1(K_1 * \cdots (K_{k-1} * b_k))) \cdots). \quad (3.21)$$

As before K_j stands for K_{α_j}, $b_j = b_{\alpha_j}$ $(j = 1, \ldots, k)$ and $\alpha = \alpha_j$ varies over the set of multi-indices which appear in B. Notice that

$$\begin{aligned} \hat{K}_k(0) = \xi^{\alpha_k}(\zeta - a(\xi))^{-1}|_{\xi=0} &= 0, \qquad &&\text{if } \alpha_k \neq 0, \\ &= \zeta^{-1}, &&\text{if } \alpha_k = 0. \end{aligned}$$

Thus the only nonvanishing terms in (3.21) are those which start and end with K_0. With this observation we can write the integral of the difference between the kernels of R_ζ and R_ζ^0 as

$$\begin{aligned} &\int (L_\zeta(x, y) - K_0(x - y))\, dy \\ &= \frac{1}{\zeta} \sum_{k=1}^{\infty} \sum_{\alpha_1 \cdots \alpha_{k-1}} K_0 * (b_1(K_1 * \cdots (K_{k-1} * b_0) \cdots)). \end{aligned} \quad (3.22)$$

The argument of Theorem 2 yields for each term in (3.22),

$$\|K_0 * (\cdots (K_{k-1} * b_k) \cdots)\|_\infty \leqslant |\zeta|^{((d/m)-1)k} \prod_{j=1}^{k} \|K_{\alpha_{j-1}, \theta}\|_{p_{\alpha_j}} \|b_{\alpha_j}\|_{r_{\alpha_j}},$$

where $\alpha_0 = \alpha_k = 0$. Hence

$$\zeta\,|\tilde{L}_\zeta(x) - 1| \leqslant \frac{c_\theta\,|\zeta|^{(d/m)-1}}{1 - c_\theta\,|\zeta|^{(d/m)-1}}$$

with the constant $c_\theta = \sum_{j=1}^{k} \|K_{\alpha_{j-1},\theta}\|_{p_{\alpha_j}} \|b_{\alpha_j}\|_{r_{\alpha_j}}$. The uniform convergence $\zeta\hat{L}_\zeta(x) \to 1$ as $\zeta \to \infty$ in Ω_θ is now clear. The theorem is proved.

Finally, we can use Cauchy integration of R_ζ over suitable contours Γ embracing Ω' to obtain a variety of other multipliers $\varphi(A)$ and summation methods $\{\varphi_\varepsilon(A)\}$. Indeed, in order that the Cauchy integral

$$\frac{1}{2\pi i} \int_\Gamma \varphi(\zeta)\, R_\zeta\, d\zeta$$

define an operator $\varphi(A)$ in the class $|\mathrm{RB}|$ it suffices that φ be integrable over Γ with respect to the measure (here $\rho = |\zeta|$)

$$d\mu = \frac{C}{\rho\,|\sin(\theta/2)|^{n+1}} \left(1 - \frac{C\rho^{(d/m)-1}}{|\sin(\theta/2)|^{n+1}}\right)^{-1} d\zeta.$$

Then the radial bound of $\varphi(A)$ is given by

$$h(|x|) = \frac{1}{2\pi} \int_\Gamma |\varphi(\zeta)|\,|\zeta|^{n/m}\, h_{s,t}(|\zeta|^{1/m}\,|x|)\, d\mu(\zeta),$$

where $h_{s,t}$ is the radial bound of Theorem 1.

We will mention one example of summation families obtained in this way, namely, the one-parameter semigroup $\{e^{-tA}\}$ generated by A. Here a contour Γ is formed by two rays $\{\rho e^{\pm i\theta} : \rho \geqslant \rho_0\}$, with small angle θ, and a finite arc $\{\rho_0 e^{i\psi} : |\psi| \geqslant \theta\}$; ρ_0 must be sufficiently large so that $\Gamma \subset \Omega$. Take the analytic function $\varphi = \varphi_t(\zeta) = e^{-t\zeta}$ and let $t = |t|\, e^{i\psi}$; $\zeta = |\zeta|\, e^{i\theta}$. In order that $\varphi_t(\zeta) = e^{-t\zeta} \in L^1(\Gamma, d\mu)$ it suffices to have

$$|\psi + \theta| < \frac{\pi}{2}. \tag{3.23}$$

Since θ can be chosen to be arbitrarily small it follows that for each t in the right half plane ($\operatorname{Re} t > 0$) the operator e^{-tA} is in the class $|\mathrm{RB}|$. Moreover, its kernel $M_t(x, y)$ satisfies

$$\tilde{M}_t(x) = \frac{1}{2\pi i} \int_\Gamma e^{-t\zeta} \tilde{L}_\zeta(x)\, d\zeta.$$

Using the argument of Theorem 3 one can easily show that $\tilde{M}_t(x) \to 1$ as

$t \to 0$ uniformly in each sector $\Omega'_\theta = \{t: |\arg t| \leqslant \theta < \pi/2\}$. Therefore Proposition 2 applies to the family $\{e^{-tA}\}_{\mathrm{Re}\, t > 0}$ and we obtain

THEOREM 4. *An operator A of Lemma* 1 *is the generator of an analytic semigroup* $\{e^{-tA}\}_{\mathrm{Re}\, t>0}$ *in* L^p $(1 \leqslant p \leqslant \min r_\alpha)$. *Furthermore, for* $f \in L^p$ *we have* L^p $(1 \leqslant p < \infty)$ *and Lebesgue* $(1 \leqslant p \leqslant \infty)$ *convergence*

$$e^{-tA} f(x) \to f(x)$$

as $t \to 0$ *uniformly in each sector* $\Omega'_0 = \{t: |\arg t| \leqslant \theta < \pi/2\}$.

A similar result holds for one-parameter semigroups generated by fractional powers A^s $(s \in \mathbb{R})$, whenever the latter exist (for instance, if the spectrum of A is in the right half plane). The corresponding multiplier is $\varphi_{s,t}(\zeta) = e^{-t\zeta^s}$ and condition (3.23) becomes

$$|\psi + s\theta| < \frac{\pi}{2},$$

which can always be satisfied for sufficiently small θ.

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REFERENCES

1. A. P. CALDERON AND A. ZYGMUND, On singular integrals, *Acta Math.* **88** (1952), 289–309.
2. M. CANTOR, Spaces of functions with asymptotic conditions on $\mathbb{R}^n$, *Indiana J. Math.* **24** (1975), 897–902.
3. E. HILLE AND R. S. PHILLIPS, Functional analysis and semigroups, Amer. Math. Soc. Colloq. Publ., Vol. 31, 1957.
4. L. HÖRMANDER, Estimates for translation invariant operators in L^p-spaces, *Acta Math.* **104** (1960), 93–139.
5. T. KATO, "Perturbation Theory for Linear Operators," Springer, New York, 1975.
6. U. NERI, The integral kernels of certain pseudo-differential operators, *Math. Ann.* **186** (1970), 155–162.
7. M. REED AND B. SIMON, "Methods of Modern Mathematical Physics," Vol. II, Academic Press, New York, 1975.
8. L. A. RAPHAEL, A generalized stieltjes transform summability method and summing sturm–Liouville expansions, preprint.
9. M. SCHECHTER, "Spectra of Partial Differential Operators," North–Holland, Amsterdam, 1971.

10. E. Stein and G. Weiss, "Introduction to Fourier Analysis on Euclidean Spaces," Princeton Univ. Press, Princeton, N. J., 1971.
11. E. Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, N. J., 1970.
12. E. Stein, "Topics in Harmonic Analysis Related to the Littlewood–Paley Theory," Princeton Univ. Press, Princeton, N. J., 1970.
13. M. Taylor, Fourier integral operators and harmonic analysis on compact manifolds, *Proc. Symp. Pure Math.* **32** (2) (1979), 115–136.