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Pseudoholomorphic discs attached to *CR*-submanifolds of almost complex spaces

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Abstract

Let E be a generic real submanifold of an almost complex manifold. The geometry of Bishop discs attached to E is studied in terms of the Levi form of E.

Résumé

Nous étudions la géométrie des disques de Bishop attachés à une sous-variété réelle générique d'une variété presque complexe.

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1. Introduction

Attachment of holomorphic discs to a prescribed real submanifold of a complex manifold is a well-known and powerfull method of geometric complex analysis developed by many authors. Recently, in view of deep connections with symplectic geometry discovered by M. Gromov [4], this method found its way to the almost complex case. Authors usually consider the attachment of pseudoholomorphic discs to totally real submanifolds of almost complex manifolds, for instance, near points admitting a non-trivial holomorphic tangent space.

In the present paper we consider pseudoholomorphic discs (Bishop discs) attached to generic real submanifolds, of a positive complex dimension, of almost complex manifolds. In Section 2 we prove the existence of Bishop discs for a generic submanifold E of an almost complex manifold (M, J). We show that, roughly speaking, these discs can be parametrized quite similarly to the case of the standard complex structure. Our proof is based on isotropic dilations of local coordinates in a similar way to Sikorav's proof of the Nijenhuis–Woolf theorem on the existence of local pseudoholomorphic discs in a prescribed direction [7].

Our main aim is to study the geometry of pseudoholomorphic Bishop discs in terms of the Levi form of a *CR*-submanifold. We begin with the Levi flat case. In Section 3 we prove that each (sufficiently small) pseudoholomorphic Bishop disc attached to a real hypersurface with identically vanishing Levi form lies in this hypersurface. Hence such a hypersurface contains pseudoholomorphic discs passing in an arbitrary prescribed complex tangent direction (Theorem 3.1). This gives an affirmative answer to a question raised by Ivashkovich and Rosay [6]. They also constructed in [6] an example of a real hypersurface in an almost complex manifold of complex dimension 3 that has an identically vanishing Levi form, but contains no complex hypersurfaces. In particular, this hypersurface is minimal in the sense of Tumanov [9]. Recall that the well-known result of Trépreau [8] and Tumanov [9] claims that in the case of an integrable complex structure, Bishop discs of a minimal hypersurface fill its one-sided neighborhood. Thus, Theorem 3.1 in combination with the example of Ivashkovich–Rosay shows that the Trépreau–Tumanov theorem has no straightforward generalisation to the almost complex case.

In Section 4 we consider the case of a *CR*-submanifold E with Levi form distinct from zero. We prove that in this case the corresponding Bishop discs sweep out a submanifold containing E as an open piece of the boundary (Theorem 4.1). This is an almost complex analog of results due to Hill and Taiani [5] and Boggess [1], but our proof in the almost complex setting requires a new idea because the Nijenhuis tensor (the torsion) of an almost complex structure has a strong influence on the geometry of the Levi form of a real submanifold: it is not even always possible to take a *CR*-submanifold of the standard complex space for a local model of a *CR*-submanifold of an almost complex space. To overcome arising difficulties, we use in Section 4 non-isotropic dilations in a suitable coordinate system (a similar idea is used in [3] in order to study boundary behavior of the Kobayashi metric in almost complex manifolds). It turns out that if E has CR dimension 1, then the non-isotropic dilations allow one to represent the pair (E, J) as a small deformation of the pair (E_0, J_{st}), where E_0 is the quadric manifold in $\mathbb{C}^n$ of which the Levi form with respect to J_{st} coincides with the Levi form of E with respect to J. This results in the existence of pseudoholomorphic Bishop discs with a certain special geometry (Theorem 4.1). The general case could in principle be treated by considering a foliation of E by submanifolds of *CR*-dimension 1. However, we put forward a method allowing us to give a more straightforward description of the Bishop discs involved in our construction in the general case.

We point out that our methods allow one to deal only with the first Levi form. For instance, suitable almost complex analogs of highly precise results of Trépreau [8] and Tumanov [9] require another approach.

2. Existence and local parametrization of Bishop discs

2.1. Almost complex manifolds

Let (M, J) be an almost complex manifold with operator of complex structure J. Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and J_{st} the standard (operator of) complex structure on $\mathbb{C}^n$ for arbitrary n. Let f be a smooth map from D into M. We say that f is *J-holomorphic* if $df \circ J' = J \circ df$. We call such a map f a J-holomorphic disc and denote by $\mathcal{O}_J(\mathbb{D}, M)$ the set of *J-holomorphic discs* in M. We denote by $\mathcal{O}(\mathbb{D})$ the space of usual holomorphic functions on $\mathbb{D}$.

The following lemma shows that an almost complex manifold (M, J) can be locally viewed as the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ equipped with a small almost complex deformation of J_{st}. We shall repeatedly use this observation in what follows.

Lemma 2.1. *Let (M, J) be an almost complex manifold. Then for each $p \in M$, each $\delta_0 > 0$, and each $k \geqslant 0$ there exist a neighborhood U of p and a smooth coordinate chart $z : U \to \mathbb{B}$ such that $z(p) = 0$, $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$, and the direct image $z_*(J) := dz \circ J \circ dz^{-1}$ satisfies the inequality $\|z_*(J) - J_{st}\|_{\mathcal{C}^k(\bar{\mathbb{B}})} \leqslant \delta_0$.*

Proof. There exists a diffeomorphism z from a neighborhood U' of $p \in M$ onto $\mathbb{B}$ such that $z(p) = 0$ and $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$. For $\delta > 0$ consider the isotropic dilation $d_\delta : t \mapsto \delta^{-1} t$ in $\mathbb{C}^n$ and the composite $z_\delta = d_\delta \circ z$. Then $\lim_{\delta \to 0} \|(z_\delta)_*(J) - J_{st}\|_{\mathcal{C}^k(\bar{\mathbb{B}})} = 0$. Setting $U = z_\delta^{-1}(\mathbb{B})$ for sufficiently small positive δ we obtain the required result. □

The operators ∂_J and $\bar\partial_J$. Let (M, J) be an almost complex manifold. We denote by TM the real tangent bundle of M and by $T_{\mathbb{C}}M$ its complexification. Recall that $T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$ where $T^{(1,0)}M := \{X \in T_{\mathbb{C}}M\colon\ JX = iX\} = \{\zeta - iJ\zeta, \zeta \in TM\}$, and $T^{(0,1)}M := \{X \in T_{\mathbb{C}}M\colon\ JX = -iX\} = \{\zeta + iJ\zeta, \zeta \in TM\}$. Let T^*M be the cotangent bundle of M. Identifying $\mathbb{C} \otimes T^*M$ with $T^*_{\mathbb{C}}M := \mathrm{Hom}(T_{\mathbb{C}}M, \mathbb{C})$ we define the set of complex forms of type $(1, 0)$ on M as $T_{(1,0)}M = \{w \in T^*_{\mathbb{C}}M\colon\ w(X) = 0, \forall X \in T^{(0,1)}M\}$ and we denote the set of complex forms of type $(0, 1)$ on M by $T_{(0,1)}M = \{w \in T^*_{\mathbb{C}}M\colon\ w(X) = 0, \forall X \in T^{(1,0)}M\}$. Then $T^*_{\mathbb{C}}M = T_{(1,0)}M \oplus T_{(0,1)}M$. This allows us to define the operators ∂_J and $\bar\partial_J$ on the space of smooth functions on M: for a smooth complex function u on M we set $\partial_J u = du_{(1,0)} \in T_{(1,0)}M$ and $\bar\partial_J u = du_{(0,1)} \in T_{(0,1)}M$. As usual, differential forms of any bidegree (p, q) on (M, J) are defined by exterior multiplication.

Plurisubharmonic functions. We say that an upper semicontinuous function u on (M, J) is *J-plurisubharmonic* on M if the composition $u \circ f$ is subharmonic on Δ for every $f \in \mathcal{O}_J(\mathbb{D}, M)$.

Let u be a $\mathcal{C}^2$ function on M, let $p \in M$ and $v \in T_pM$. *The Levi form* of u at p evaluated on v is defined by the equality $L^J(u)(p)(v) := -d(J^*du)(X, JX)(p)$ where X is an arbitrary vector field on TM such that $X(p) = v$ (of course, this definition is independent of one's choice of X).

The following result is well known (see, for instance, [6]).

Proposition 2.2. *Let u be a $\mathcal{C}^2$ real valued function on M, let $p \in M$ and $v \in T_pM$. Then $L^J(u)(p)(v) = \Delta(u \circ f)(0)$ where f is an arbitrary J-holomorphic disc in M such that $f(0) = p$ and $df(0)(\partial/\partial \operatorname{Re} \zeta) = v$ (here ζ is the standard complex coordinate variable in $\mathbb{C}$).*

The Levi form is obviously invariant with respect to biholomorphisms. More precisely, let u be a $\mathcal{C}^2$ real valued function on M, let $p \in M$ and $v \in T_pM$. If Φ is a (J, J')-holomorphic diffeomorphism from (M, J) into (M', J'), then $L^J(u)(p)(v) = L^{J'}(u \circ \Phi^{-1})(\Phi(p))(d\Phi(p)(v))$.

Finally, it follows from Proposition 2.2 that a C^2-smooth real function u is J-plurisubharmonic on M if and only if $L^J(u)(p)(v) \geqslant 0$ for all $p \in M$, $v \in T_pM$. Thus, similarly to the case of an integrable structure one arrives in a natural way to the following definition: a $\mathcal{C}^2$ real valued function u on M is *strictly J-plurisubharmonic* on M if $L^J(u)(p)(v)$ is positive for every $p \in M$, $v \in T_pM \backslash \{0\}$.

It follows easily from Lemma 2.1 that for every point $p \in (M, J)$ there exists a neighborhood U of p and a diffeomorphism $z : U \to \mathbb{B}$ with center at p (in the sense that $z(p) = 0$) such that the function $|z|^2$ is J-plurisubharmonic on U and $z_*(J) = J_{st} + O(|z|)$.

Let u be a $\mathcal{C}^2$ function in a neighborhood of a point p of (M, J) that is strictly J-plurisubharmonic. Then there exists a neighborhood U of p with local complex coordinates $z : U \to \mathbb{B}$ such that the function $u - c|z|^2$ is J-plurisubharmonic on U for some constant $c > 0$.

Real submanifolds in almost complex manifolds. Let E be a real submanifold of codimension m in an almost complex manifold (M, J) of complex dimension n. For every p we denote by $H^J_p(E)$ the maximal complex (with respect to $J(p)$) subspace of the tangent space $T_p(E)$. Similarly to the integrable case, E is said to be a CR manifold if the (complex) dimension of $H^J_p(E)$ is independent on p; it is called the CR dimension of E and is denoted by $\operatorname{CRdim} E$.

In complex analysis by a *generic* submanifold of a complex manifold one usually means a submanifold E such that at every point $p \in E$ the complex linear span of $T_p(E)$ coincides with the tangent space of the ambient manifold. We think that the use of this term in precisely that sense outside the framework of complex analysis proper can sometimes be misleading. For this reason we shall provisionally call submanifolds with similar properties of an almost complex manifold M (that is, submanifolds such that the complex linear span of $T_p(E)$ at each point coincides with TM) *generating* submanifolds. Of course, every generating submanifold is CR.

If E is defined as the common zero level of functions $r_1, \dots, r_m$, then after the standard identification of TM and $T^{(1,0)}M$ $H_p^J(E)$ can be defined as the zero subspace of the forms $\partial_J r_1, \dots, \partial_J r_m$. In particular, let E be a smooth real hypersurface in an almost complex manifold (M, J) defined by an equation $r = 0$. We say that E is strictly pseudoconvex (for J) if the Levi form of r is strictly positive definite on each holomorphic tangent space $H_p^J(E)$, $p \in \Gamma$. Of course, this definition does not depend on one's choice of the defining function r. We shall require the following result, which is well-known in the case of an integrable structure.

Lemma 2.3. *Let E be a strictly pseudoconvex hypersurface in an almost complex manifold (M, J). Then E admits a strictly plurisubharmonic defining function in a neighborhood of each its point p.*

The proof is quite similar to the case of the standard structure. Selecting suitable local coordinates $Z = (z, w_1, \dots, w_{n-1})$ in $\mathbb{C}^n$ we can assume that M is a neighborhood of the origin in $\mathbb{C}^n$ and $p = 0$; as usual, we assume that $J(0) = J_{st}$. Furthermore, we may suppose that $r(Z) = z + \bar{z} + O(|Z|^2)$ and therefore $H_0^J(E) = \{z = 0\}$. Then $L_0^J(r^2)(v) = |v_1|^2$ for complex tangent vectors $v \in T_0(M)$. Since $L_0^J(r)$ is strictly positive definite on $H_0^J(E)$, the Levi form $L_0^J(r + Cr^2)$ is strictly positive on $T_0(M)$ for a sufficiently large positive constant C.

2.2. Bishop discs and Bishop's equation

Let (M, J) be a smooth almost complex manifold of real dimension $2n$ and E a generating submanifold of M of real codimension m. A J-holomorphic disc $f : \mathbb{D} \to M$ continuous on $\overline{\mathbb{D}}$ is called a *Bishop disc* if $f(b\mathbb{D}) \subset E$ (where $b\mathbb{D}$ denotes the boundary of $\mathbb{D}$). Our aim is to prove the existence and to describe certain classes of Bishop discs attached to E.

Consider the case where E is defined as the zero set of an $\mathbb{R}^m$-valued function $r = (r^1, \dots, r^m)$ on M. Then a smooth map f defined on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ is a Bishop disc if and only if it satisfies the following non-linear boundary problem of the Riemann–Hilbert type for the quasi-linear operator $\bar\partial_J$:

$$\text{(RH):}\quad \begin{cases} \bar\partial_J f(\zeta) = 0, & \zeta \in \mathbb{D}, \\ r(f)(\zeta) = 0, & \zeta \in b\mathbb{D}. \end{cases}$$

To describe solutions of this problem we fix a chart $U \subset M$ and a coordinate diffeomorphism $z : U \to \mathbb{B}^n$ where $\mathbb{B}^n$ is the unit ball of $\mathbb{C}^n$. Identifying M with $\mathbb{B}^n$ we may assume that in these coordinates $J = J_{st} + O(|z|)$ and the norm $\|J - J_{st}\|_{C^k(\overline{\mathbb{B}}^n)}$ is small enough for some positive real k in accordance with Lemma 2.1. (Here k can be arbitrary, but we assume it for convenience to be non-integer and fix it throughout what follows.) More precisely, using the notation $Z = (z, w)$, $z = (z_1, \dots, z_m)$, $w = (w_1, \dots, w_{n-m})$ for the standard coordinates in $\mathbb{C}^n$, we may also assume that $E \cap U$ is described by the equations

$$r(Z) = \operatorname{Re} z - h(\operatorname{Im} z, w) = 0 \tag{1}$$

with vector-valued C^∞-function $h : \mathbb{B} \to \mathbb{R}^m$ such that $h(0) = 0$ and $\nabla h(0) = 0$.

Similarly to the proof of Lemma 2.1 consider the isotropic dilations $d_\delta : Z \mapsto Z' = \delta^{-1}Z$. In the new Z-variables (we drop the primes) the image $E_\delta = d_\delta(E)$ is defined by the equation $r_\delta(Z) := \delta^{-1}r(\delta Z) = 0$. Since the function r_δ approaches $\operatorname{Re} z$ as $\delta \to 0$, the manifolds E_δ approach the flat manifold $E_0 = \{\operatorname{Re} z = 0\}$, which, of course, may be identified with the real tangent space to E at the origin. Furthermore, as seen in the proof of Lemma 2.1, the structures $J_\delta := (d_\delta)_*(J)$ converge to J_{st} in the C^k-norm as $\delta \to 0$. This allows us to find explicitly the $\bar\partial_J$-operator in the Z variables.

Consider a J_δ-holomorphic disc $f : \mathbb{D} \to (\mathbb{B}^n, J_\delta)$. The J_δ-holomorphy condition $J_\delta(f) \circ f_* = f_* \circ J_{st}$ can be written in the following form:

$$\frac{\partial f}{\partial \bar\zeta} + Q_{J,\delta}(f)\left(\frac{\partial \overline{f}}{\partial \bar\zeta}\right) = 0, \tag{2}$$

where $Q_{J,\delta(Z)}$ is the complex $n \times n$ matrix of an operator the composite of which with complex conjugation is equal to the endomorphism $-(J_{st} + J_\delta(Z))^{-1}(J_{st} - J_\delta(Z))$ (which is an anti-linear operator with respect to the standard structure J_{st}). Hence the entries of the matrix $Q_{J,\delta}(z)$ are smooth functions of δ, z vanishing identically in z for $\delta = 0$.

Using the Cauchy–Green transform

$$T_{\mathrm{CG}}(g) = \frac{1}{2\pi i} \iint\limits_{\mathbb{D}} \frac{g(\tau)}{\zeta - \tau} d\tau \wedge d\bar\tau$$

we may write $\bar\partial_J$-equation (2) as follows:

$$\frac{\partial}{\partial \bar\zeta}\left(f + T_{\mathrm{CG}}\left(Q_{J,\delta}(f)\left(\frac{\partial \overline{f}}{\partial \bar\zeta}\right)\right)\right) = 0.$$

According to classical results [10], the Cauchy–Green transform is a continuous linear operator from $C^k(\overline{\mathbb{D}})$ into $C^{k+1}(\overline{\mathbb{D}})$ (recall that k is non-integer). Hence the operator

$$\Phi_{J,\delta} : f \to g = f + T_{\mathrm{CG}}\left(Q_{J,\delta}(f)\left(\frac{\partial \overline{f}}{\partial \bar\zeta}\right)\right)$$

takes the space $C^k(\overline{\mathbb{D}})$ into itself. Thus, f is J_δ-holomorphic if and only if $\Phi_{J,\delta}(f)$ is holomorphic (in the usual sense) on $\mathbb{D}$. For sufficiently small positive δ this is an invertible operator on a neighborhood of zero in $C^k(\overline{D})$ which establishes a one-to-one correspondence between the sets of J_δ-holomorphic and holomorphic discs in $\mathbb{B}^n$.

These considerations allow us to replace the non-linear Riemann–Hilbert problem (RH) by *generalized Bishop's equation*

$$r_\delta\big(\Phi_{J,\delta}^{-1}(g)\big)(\zeta) = 0, \quad \zeta \in b\mathbb{D}, \tag{3}$$

for an unknown *holomorphic* function g in the disc (with respect to the standard complex structure).

If g is a solution of the boundary problem (3), then $f = \Phi_{J,\delta}^{-1}(g)$ is a Bishop disc with boundary attached to E_δ. Since the manifold E_δ is biholomorphic via isotropic dilations to the initial manifold E, the solutions of Eq. (3) allow to describe Bishop's discs attached to E. Of course, this gives just the discs close enough (in the C^k-norm) to the trivial solution $f \equiv 0$ of the problem (RH).

2.3. Solution of generalized Bishop's equation

Let $\mathcal{U}$ be a neighborhood of the origin in $\mathbb{R}$, X' a sufficiently small neighborhood of the origin in the Banach space $(\mathcal{O}(\mathbb{D}) \cap C^k(\bar{\mathbb{D}}))^m$ (with positive non-integer k), X'' a neighborhood of the origin in the Banach space $(\mathcal{O}(\mathbb{D}) \cap C^k(\bar{\mathbb{D}}))^{n-m}$, and Y the Banach space $(C^k(b\mathbb{D}))^m$. If $z \in X'$, $z : \zeta \mapsto z(\zeta)$ and $w \in X''$, $w : \zeta \mapsto w(\zeta)$ are holomorphic discs, then we denote by Z the holomorphic disc $Z = (z, w)$. We may also assume that J is a C^{2k}-smooth real $(2n \times 2n)$-matrix valued function and denote by W the Banach space of these functions.

Consider the map of Banach spaces $R : W \times X' \times X'' \times \mathcal{U} \to Y$ defined as follows:

$$R : (J, z, w, \delta) \mapsto r_\delta\big(\Phi_{J,\delta}^{-1}(z, w)\big)(\bullet)|b\mathbb{D}.$$

Let ϕ be a C^{2k}-map between two domains in $\mathbb{R}^n$ and $\mathbb{R}^m$; it determines a map ω_ϕ acting by composition on C^k-smooth maps g into the source domain: $\omega_\phi : g \mapsto \phi(g)$. The well-known fact is that ω_ϕ is a C^k-smooth map between the corresponding spaces of C^k-maps. In our case this means that the map R is of class C^k. For a holomorphic disc $Z = (z, w) \in X' \times X''$ and $J \in W$ the tangent map $D_{X'}R(J, z, w, 0) : X' \to Y$ (the partial derivative with respect to the space X') is defined by the equality $D_{X'}R(J, z, w, 0)(q) = (\operatorname{Re} q_1, \ldots, \operatorname{Re} q_m)$. Clearly, it is surjective and has the kernel of real dimension m consisting of constant functions $h = i(c_1, \ldots, c_m)$, $c_j \in \mathbb{R}$. (We point out that the map $D_{X'}R$ coincides with the tangent map arising after the linearization of Bishop's equation corresponding to the standard structure J_{st}.) Therefore, by the implicit function theorem [11] there exists $\delta_0 > 0$, a neighborhood V_1 of the origin in X', a neighborhood V_2 of the origin in X'', a neighborhood V_3 of the origin in $\mathbb{R}^m$, a neighborhood W_1 of J_{st} in W and a C^k-smooth map $G : W_1 \times V_2 \times V_3 \times [0, \delta_0] \to V_1$ such that for every $(J, w, c, \delta) \in W_1 \times V_2 \times V_3 \times [0, \delta_0]$ the function $g = (G(J, w, c, \delta)(\bullet), w(\bullet))$ is the unique solution of generalized Bishop's equation (3) belonging to $V_1 \times V_2$.

Now, the pullback $f = \Phi_{J,\delta}^{-1}(g)$ gives us a J_δ-holomorphic disc attached to E_δ. Thus, the initial data consisting of $J \in W_1$, and a set $(c_1, \ldots, c_m, w)$, $c_j \in \mathbb{R}$, $w \in V_2$ define for each small δ a unique J_δ-holomorphic disc f attached to E_δ. Since the almost complex structures J and J_δ are biholomorphic via isotropic dilations, we can give the following description of local solutions of Bishop's equation (the problem (RH)).

Theorem 2.4. *Let E be a smooth submanifold of $\mathbb{C}^n$ defined as the zero set of a smooth $\mathbb{R}^m$-valued function r of the form* (1). *Then there exists a neighborhood U of the origin in $(C^k(\bar{\mathbb{D}}))^n$, a neighborhood W_1 of J_{st} in the space W, a neighborhood V_2 of the origin in X'', and a neighborhood V_3 of the origin in $\mathbb{R}^m$ such that for each $J \in W_1$ the set of maps in U that are Bishop discs attached to E with respect to J is a Banach submanifold of class C^k in U with local chart defined by a* (*smooth*) *map $F : V_2 \times V_3 \to U$, which depends smoothly on J.*

The proof follows from the above analysis of the Bishop equation. One merely fixes some value of δ, $0 < \delta \leqslant \delta_0$, and observes that the families of Bishop discs corresponding to distinct values of $\delta \neq 0$ are taken into one another by the corresponding dilations.

One important consequence of this statement is as follows: if E_1 and E_2 are C^{2k}-close submanifolds of $\mathbb{C}^n$ defined by equations of the form (1) and J_1 and J_2 are C^{2k} close almost complex structures, then there exists a (locally defined) diffeomorphism between the corresponding C^k Banach submanifolds of Bishop discs that depends smoothly on the pairs E_j, J_j, $j = 1, 2$, and is the identity in the case of equal pairs.

3. Hypersurfaces with vanishing Levi form

Here we prove the following result.

Theorem 3.1. *Let Γ be a real hypersurface in an almost complex manifold (M, J) with Levi form vanishing identically at the points of Γ. Then at each point $p \in \Gamma$ and for each direction $v \in H_p(\Gamma)$ there exists a J-holomorphic disc $f : \mathbb{D} \to M$ such that $f(0) = p$, $df(0)(\partial/\partial \operatorname{Re}\zeta) = v$ and $f(\mathbb{D}) \subset \Gamma$.*

If M has real dimension 4, then this result can be proved in the same fashion as in complex analysis, by the application of Frobenius's theorem to complex tangent spaces to Γ. However, this does not work in higher dimensions, when in the case of an almost complex structure distinct from the standard one the distribution of the planes $H_p(\Gamma)$ is not necessarily involutive. The idea of our proof is to show that *each* (sufficiently small) Bishop disc for Γ lies in Γ.

As before, passing to local coordinates $Z = (z, w_1, \dots, w_{n-1})$ we may assume that Γ is a real hypersurface in a neighborhood Ω of the origin in $\mathbb{C}^n$ and $J(0) = J_{st}$. Let r be a local defining function of Γ in Ω. Denote by Ω^+ (resp. Ω^-) the domain $\{Z \in \Omega\colon r(Z) > 0\}$ (resp. $\{Z \in \Omega\colon r(Z) < 0\}$).

A neighborhood Ω is supposed to be small enough; in particular, we may assume that

(a) the function $|Z|^2$ is strictly J-plurisubharmonic on Ω and there exists a constant $\varepsilon_0 > 0$ such that the value of the Levi form (with respect to J) of the function $|Z|^2$ on a vector v at a point $p \in \Omega$ is minorated by $(\varepsilon_0/2)\|v\|^2$.

For a constant $N > 1$, which will be chosen later, and sufficiently small $\varepsilon > 0$ consider the function $r_\varepsilon(Z) = r(Z) + \varepsilon|Z|^2 - \varepsilon/N$ and the hypersurface $\Gamma_\varepsilon := \{Z \in \Omega\colon r_\varepsilon(Z) = 0\}$. Recall that $\mathbb{B}^n$ is the unit ball in $\mathbb{C}^n$; we may assume that

(b) the ball $\mathbb{B}^n$ lies in Ω.

We shall make our choice of a coordinate system more precise. Namely, after a $\mathbb{C}$-linear change of coordinates preserving the previous assumptions, we may assume that

(c) $r(Z) = x - h(y, w)$ on Ω (as usual, $z = x + iy$ and $w = (w_1, \dots, w_{n-1})$); in particular $H_0^J(\Gamma) = \{Z\colon z = 0\}$, and

(d) for every $Z \in \Omega$ and for $\varepsilon < 1$ the kernel of the form $\partial_J r_\varepsilon(Z)$ is in a one-to-one correspondence with $H_0^J(\Gamma)$ via the projection $(z, w) \mapsto (0, w)$.

Furthermore, there exists a constant $c > 0$ such that for every $z \in \Omega$ one has $c^{-1}\operatorname{dist}(Z, \Gamma) \leqslant |r(Z)| \leqslant c \operatorname{dist}(Z, \Gamma)$ (where dist is the Euclidean distance). For $Z \in (1/2N)\mathbb{B}^n \cap \Gamma_\varepsilon$ we have $|r(Z)| = \varepsilon/N - \varepsilon|Z|^2$ so that

$$\varepsilon/2N \leqslant \left|r(Z)\right| \leqslant \varepsilon/N. \tag{4}$$

Throughout the rest of the proof we shall stay in $(1/2N)\mathbb{B}^n$. Note that
(e) pieces of the hypersurfaces Γ_ε and $\Gamma_{-\varepsilon}$ form a foliation of $(1/2N)\mathbb{B}^n$.

Lemma 3.2. *If N is a sufficiently large fixed constant, then for sufficiently small $\varepsilon > 0$ the hypersurface Γ_ε is strictly J-pseudoconvex at all its point lying in the ball $(1/4N)\mathbb{B}^n$.*

Proof. For every $Z \in \Omega^+ \cap (1/2N)\mathbb{B}^n$ there exists unique $\varepsilon = \varepsilon(Z)$ such that $Z \in \Gamma_\varepsilon$; clearly, the function $Z \mapsto \varepsilon(Z)$ is smooth on $(1/2N)\mathbb{B}^n$. Therefore for every $Z \in \Gamma_\varepsilon \cap (1/2N)\mathbb{B}^n$ the value of the Levi form $L^J_Z(r)(v)$ of r at Z at a vector $v \in H^J_Z(\Gamma_\varepsilon)$ has the estimate

$$\bigl|L^J_Z(r)(v)\bigr| < c_1(d_1 + d_2)\|v\|^2, \tag{5}$$

where d_1 is the distance from Z to the closest point Z_0 on the hypersurface $\Gamma = \Gamma_0$ and d_2 is the distance between $H^J_Z(\Gamma_\varepsilon)$ and $H^J_{Z_0}(\Gamma_0)$ measured in some smooth metric on the corresponding Grassmanian. Indeed, since $|L^J_Z(r)(v)| = |L^J_Z(r)(v/\|v\|)|\|v\|^2$, it is sufficient to find an estimate of $\sup\{|L^J_Z(r)(u)|\colon u \in H^J_Z(\Gamma_\varepsilon), \|u\| = 1\}$. Consider the real unit spheres $S(Z) = \{u \in H^J_Z(\Gamma_\varepsilon)\colon \|u\| = 1\}$ and $S(Z_0) = \{u' \in H^J_{Z_0}(\Gamma)\colon \|u'\| = 1\}$ in the tangent spaces $H^J_Z(\Gamma_\varepsilon)$ and $H^J_{Z_0}(\Gamma)$ respectively. In what follows the Levi forms $L^J_Z(r)$ and $L^J_{Z_0}(r)$ are viewed as quadratic forms on $\mathbb{R}^{2n}$ since the local coordinates are fixed; the tangent spaces $H^J_Z(\Gamma_\varepsilon)$ and $H^J_{Z_0}(\Gamma)$ are identified with subspaces in $\mathbb{R}^{2n}$. Denote by $\hat{L}^J_Z(r)$ the polarization of $L^J_Z(r)$, that is, the corresponding bilinear form on $\mathbb{R}^{2n}$.

For any vector $u \in S(Z)$ we have

$$\bigl|L^J_Z(r)(u)\bigr| \leqslant \inf_{u' \in S(Z_0)} \bigl(\bigl|L^J_Z(r)(u')\bigr| + \bigl|L^J_Z(r)(u - u')\bigr| + 2\bigl|\hat{L}^J_Z(r)(u', u - u')\bigr|\bigr).$$

When $Z \in \Gamma$ the form $L^J_Z(r)(u')$ vanishes for any $u' \in H^J_Z(\Gamma)$; so there exists a constant c_1 such that

$$\sup\bigl\{\bigl|L^J_Z(r)(u')\bigr|\colon u' \in S(Z_0)\bigr\} \leqslant Cd_1.$$

Furthermore, there exist constants c'_1 and c'_2 such that

$$\sup_{u \in S(Z)} \Bigl(\inf_{u' \in S(Z_0)} \bigl|L^J_Z(r)(u - u')\bigr|\Bigr) \leqslant c'_1 \sup_{u \in S(Z)} \Bigl(\inf_{u' \in S(Z_0)} \|u - u'\|^2\Bigr) \leqslant c'_2 d_2^2.$$

In a similar way

$$\sup_{u \in S(Z)} \Bigl(\inf_{u' \in S(Z_0)} 2\bigl|\hat{L}^J_Z(r)(u', u - u')\bigr|\Bigr) \leqslant c''_1 \sup_{u \in S(Z)} \Bigl(\inf_{u' \in S(Z_0)} \|u - u'\|\Bigr) \leqslant c''_2 d_2$$

for some positive constants c''_1 and c''_2. Obviously, $d_1 \sim |r|$. Moreover, a direct estimate from above of the quantities $|\partial r_\varepsilon/\partial z_k(Z) - \partial r/\partial z_k(Z_0)|$ and $|\partial r_\varepsilon/\partial \bar{z}_k(Z) - \partial r/\partial \bar{z}_k(Z_0)|$ shows that $d_2 < c_2(d_1 + \varepsilon|Z|)$.

Observing that if $Z \in \Gamma_\varepsilon$, then $\varepsilon = r(Z)/(\frac{1}{N} - |Z|^2)$, and also that $|Z| < 1/2N$ we see that

$$\bigl|L^J_Z(r)(v)\bigr| \leqslant \bigl(c_3\varepsilon(Z)/N\bigr)\|v\|^2. \tag{6}$$

We point out that the constant c_3 is independent of ε and N. Fix $N \geqslant \max\{2, 4c_3/\varepsilon_0\}$. Then, in view of condition (a) and (6), the Levi form of r_ε is strictly positive on $H^J_Z(\Gamma_\varepsilon)$ which proves the lemma. □

By Theorem 2.4, in each sufficiently small neighborhood of the origin there exists a family of J-holomorphic Bishop discs with boundaries in Γ. Fix a such a neighborhood $U \subset (1/2N\mathbb{B}^n)$.

Lemma 3.3. *Let $f : \mathbb{D} \to U$ be a J-holomorphic Bishop disc, that is, let f be a pseudo-holomorphic map continuous on $\overline{\mathbb{D}}$ such that $f(b\mathbb{D}) \subset \Gamma$. Then $f(\overline{\mathbb{D}})$ lies in Γ.*

Proof. Assume by contradiction that $f(\mathbb{D})$ does not lie in Γ. Recall that $U^+ := U \cap \{r > 0\}$ is filled by strictly pseudoconvex hypersurfaces Γ_ε, $0 \leqslant \varepsilon \leqslant \varepsilon_0$. We may assume that there exists a connected open subset G of $\mathbb{D}$ such that $f(G)$ lies in U^+ (otherwise we replace r by $-r$) and $f(bG) \subset \Gamma$. Consider the set $A = \{\varepsilon > 0\colon (r_\varepsilon \circ f)|_G < 0\}$. This set is not empty if the disc f is small enough. Let $\varepsilon_1 = \inf A$. Then $\varepsilon_1 > 0$, the hypersurface Γ_{ε_1} is strictly pseudoconvex, $(r_{\varepsilon_1} \circ f)|_{bG} < 0$ and $r_{\varepsilon_1} \circ f|_G \leqslant 0$. Moreover, there exists an interior point $\zeta \in G$ such that $r_{\varepsilon_1} \circ f(\zeta) = 0$.

On the other hand by Lemma 2.3 the hypersurface Γ_{ε_1} admits a strictly plurisubharmonic defining function in a neighborhood of the point $f(\zeta)$. This contradicts the maximum principle and proves the lemma. □

We now can prove Theorem 3.1. Similarly to the previous sections consider the isotropic dilations $d_\delta : Z \mapsto Z' = \delta^{-1} Z$. The image $\Gamma_\delta := (d_\delta)_*(\Gamma)$ of the hypersurface Γ approaches the hyperplane $\Gamma_0 = \{\operatorname{Re} z = 0\}$ as $\delta \to 0$. Let U_j, $j = 1, 2$, be neighborhoods of the origin in $\mathbb{C}^{n-1}$ and U_3 a neighborhood of the origin in $\mathbb{R}$; we assume that these neighborhoods are *sufficiently small*. For $p \in U_1$, $v \in U_2$ and $c \in U_3$ consider a J_{st}-holomorphic disc $f(p, v, c)(\zeta) = (ic, p + v\zeta)$ that is a Bishop disc lying in the hyperplane Γ_0. The centers $f(p, v, c)(0)$ of such discs fill a neighborhood of the origin in Γ_0 and their tangent vectors (at centers) $df(p, v, c)(0)(\partial/\partial \operatorname{Re} \zeta)$ fill a neighborhood of the origin in the holomorphic tangent space $H_q(\Gamma_0)$ for any $q \in \Gamma_0$ in a neighborhood of the origin. By Theorem 2.4 for any $\delta > 0$ there exists a family of discs $F(\delta, p, v, c)(\bullet)$ smoothly depending on parameters $\delta, p, v, c)$ such that

(a) every disc $F(\delta, p, v, c)(\bullet)$ is J_δ holomorphic (where as usual J_δ denotes the direct image $(d_\delta)_*(J)$);
(b) for every sufficiently small positive δ every disc $F(\delta, p, v, c)(\bullet)$ is a Bishop disc for Γ_δ, that is, $F(\delta, p, v, c)(b\mathbb{D}) \subset \Gamma_\delta$;
(c) we have $F(0, p, v, c)(\bullet) = f(p, v, c)(\bullet)$, so that the family $\{F(\delta, p, v, c)(\bullet)\}$ of J_δ-holomorphic discs is a small deformation of the family $\{f(p, v, c)(\bullet)\}$.

By Lemma 3.3, for small $\delta > 0$ every disc $F(\delta, p, v, c)(\mathbb{D})$ lies in Γ_δ. By standard arguments their centers fill a neighborhood U of the origin on Γ_δ and at every point $z \in U$ their tangent vectors fill a neighborhood of the origin in the tangent space $H^{J_\delta}_z(\Gamma_\delta)$. Since the structures J_δ and J are biholomorphic, the proof of the theorem is complete.

4. Manifolds with non-trivial Levi form

In this section we prove the following result

Theorem 4.1. *Let $E = \{r := (r^1, \ldots, r^m) = 0, j = 1, \ldots, m\}$ be a (germ of a) smooth generating submanifold passing through a point p in an almost complex manifold (M, J). Suppose that there exists j and a vector $v \in H_p^J(E)$ such that the Levi form $L_p^J(r^j)(v)$ does not vanish. Then for fixed non-integer $k > 2$ there exists in a neighborhood of p a C^k smooth generating manifold $\tilde{E}$ of dimension* $\dim E + 1$ *with boundary such that every point of $\tilde{E}$ belongs to a J-holomorphic disc with boundary on E and E is the boundary of $\tilde{E}$.*

Our proof is based on non-isotropic scaling. Isotropic dilations used in the previous section cannot be applied here since they do not give one the control over the Levi form of E. The crucial technical point here is a choice of a suitable coordinate system "normalizing" an almost complex structure. Indeed, the following elementary example shows the basic difficulty in dealing with the almost complex case if a coordinate system is not good enough. Consider in $\mathbb{C}^2$ the real hyperplane $\Pi : \operatorname{Re} z_2 = 0$, which is Levi flat in the standard complex structure J_{st} of $\mathbb{C}^2$. (Throughout, we identify an almost complex structure on a manifold with the corresponding field of operators on the tangent space.) Consider the diffeomorphism $\Phi : (z_1, z_2) \mapsto (z_1, z_2 - |z_1|^2)$. The image $\Phi(\Pi)$ is the hypersurface $\Gamma : \operatorname{Re} z_2 + |z_1|^2 = 0$ and the direct image of the standard structure is the almost complex structure $J(\Phi(z)) = d\Phi(z) \circ J(z) \circ d\Phi^{-1}(z)$. The structure J coincides with J_{st} at the origin, so that $J(z) = J_{st} + \mathrm{O}(|z|)$ and the hypersurface Γ is strictly pseudoconvex with respect to J_{st}, but Levi flat with respect to J!

4.1. *The case* CRdim $E = 1$

We begin with this case since it is particularly convenient for non-isotropic dilations. Passing to suitable local coordinates (similarly to the previous section we use the notation $Z = (z_1, \ldots, z_{n-1}, w)$) we may assume that M is a neighborhood of the origin in $\mathbb{C}^n$ and J is a smooth matrix valued function of the form $J = J_{st} + \mathrm{O}(|Z|)$. Moreover, we may assume that the holomorphic tangent space $H_0^J(E)$ coincides with the line $l = (0, \ldots, 0, \zeta)$, $\zeta \in \mathbb{C}$ and $E = \{r^j(Z) = 0, j = 1, \ldots, n-1\}$, where $r^j = z_j + \bar{z}_j + \mathrm{O}(|Z|^2)$.

Consider a J-holomorphic disc tangent to $H_0^J(E)$ at the center. Performing if necessary an appropriate diffeomorphism with linear part identity at the origin we can assume that this disc lies on l. Thus, we shall assume that l is J-holomorphic.

Lemma 4.2. *In the above variables, for every j the Levi form $L_0^J(r^j)$ coincides on $H_0^J(E)$ with the Levi form $L_0^{J_{st}}(r^j)$ with respect to J_{st}.*

Proof. This follows from Proposition 2.2 if in its setting we take the line l for a J-holomorphic disc f. □

For $\delta > 0$ consider now the *non-isotropic* dilations $\Lambda_\delta : (z, w) \mapsto (\delta^{-1}z, \delta^{-1/2}w)$ and the induced structures $J_\delta := (\Lambda_\delta)_*(J)$.

Lemma 4.3. *For any positive real k one has $\|J_\delta - J_{st}\|_{\mathcal{C}^k(K)} \to 0$ as $\delta \to 0$ on each compact subset K of $\mathbb{C}^n$.*

Proof. Consider the Taylor expansion of $J(Z)$ near the origin: $J(Z) = J_{st} + L(Z) + R(Z)$ where $L(Z)$ is the linear part of the expansion and $R(Z) = O(|Z|^2)$. Clearly, $\Lambda_\delta \circ R(\Lambda_\delta^{-1}(Z)) \circ \Lambda_\delta^{-1}$ converges to 0 as $\delta \to 0$. Denote by $L_{kj}^\delta(Z)$ (respectively, by $L_{kj}(Z)$) an entry of the real matrix $\Lambda_\delta \circ L(\Lambda_\delta^{-1}(Z)) \circ \Lambda_\delta^{-1}$ (respectively, of $L(Z)$). Then $L_{kj}^\delta(z, w) = L_{kj}(\delta z, \delta^{1/2}w) \to 0$ for $k, j = 1, \dots, 2n-2$ and $k, j = 2n-1, 2n$, $L_{kj}^\delta(z, w) = \delta^{1/2}L_{kj}(\delta z, \delta^{1/2}w) \to 0$ for $k = 2n-1, 2n$, $j = 1, \dots, 2n-2$. For $k = 1, \dots, 2n-2$ and $j = 2n-1, 2n$ we have $J_{kj}^\delta(z, w) = \delta^{-1/2}L_{kj}(\delta z, \delta^{1/2}w) \to L_{kj}(0, w)$. However, in the coordinate system fixed above the line l is J-holomorphic, that is, $J(l(\zeta)) \circ dl = dl \circ J_{st}$. This shows that $L(0, w) \equiv 0$. Thus, L_{kj}^δ approaches 0 for all k, j. This gives us the result of the lemma. □

We point out that this result fails for CRdim $E > 1$. For this reason we begin our construction with the case CRdim $E = 1$.

We may assume that E is defined by equations $r^j(z, w) = 0$, $j = 1, \dots, n-1$, with $r^j(z, w) = 2\,\mathrm{Re}\, z_j + 2\,\mathrm{Re}\, Q^j(z, w) + H^j(z, w) + O(|Z|^2)$. Here $Q^j(Z) = \sum q_{ks}^j Z_k Z_s$ and $H^j(Z) = \sum_{ks} h_{ks}^j Z_k \overline{Z}_s$ are complex and Hermitian quadratic forms, respectively. Then the manifold $E_\delta := \Lambda_\delta(E)$ is given by the equations $r_\delta^j(Z) := \delta^{-1} r_j((\delta^{1/2})z, \delta w) = 0$ and $r_\delta^j(Z) \to r_0^j(Z) := 2\,\mathrm{Re}\, z_j + 2\,\mathrm{Re}\, Q^j(0, w) + H^j(0, w)$ (in the C^k norm for any k) as δ approaches 0. Since the quadratic map

$$\big(H^1('0, w), \dots, H^{n-1}('0, w)\big)$$

can be identified with the Levi form of E at the origin, one of the forms $H^j(0, \bullet)$ does not vanish on $\mathbb{C}$. Replacing the functions r^j by their linear combinations if necessary one can assume that $H^j(0, w) \equiv 0$, $j = 1, \dots, n-2$, and $H_j(0, w) = -|w|^2$.

Consider the limit manifold $E_0 = \{r^j(Z) = 0, j = 1, \dots, m\}$. After a biholomorphic (with respect to J_{st}) change of the variables $(z, w) \mapsto (z', w') = (z + Q(0, w), w)$ (here $Q = (Q_1, \dots, Q_{n-1})$) we obtain a manifold E_0' defined by the equations $\mathrm{Re}\, z_j = 0$, $j = 1, \dots, n-2$, $2\,\mathrm{Re}\, z_{n-1} = |w|^2$ (we drop the primes).

Following Boggess and Pitts [2] we consider now the family $f : \zeta \mapsto (z(\zeta), w(\zeta)$ of holomorphic Bishop discs attached to E_0' and defined by the formulae

$$z_j(\zeta) = iy_j, \quad j = 1, \dots, n-2,$$

$$z_{n-1}(\zeta) = (1/2)\left(c\bar{c} + \frac{t^2}{(1+\lambda)^2}(\lambda^2 + 1)\right) + \frac{t\lambda}{1+\lambda}\bar{c} + iy_{n-1} + \left(\frac{t\bar{c}}{1+\lambda} + \frac{t^2\lambda}{(1+\lambda)^2}\right)\zeta,$$

$$w(\zeta) = c + \frac{t(\lambda+\zeta)}{1+\lambda}.$$

This family depends on parameters $y = (y_1, \ldots, y_{n-1})$ ranging in some neighborhood of the origin in $\mathbb{R}^{n-1}$, real parameters $t > 0$ and $\lambda \in [0, 1]$, and a complex parameter c ranging in a neighborhood of the origin in $\mathbb{C}$. We shall write $f(t, \lambda, y, c)(\bullet)$ for discs in this family. We are interested in the maps $f(t, \lambda, y, c)(-\lambda)$. They have the following properties:

(a) for any $t > 0$ one has $\lim_{\lambda\to 1} f(t, \lambda, y, c)(-\lambda) = (iy_1, \ldots, iy_{n-2}, (1/2)c\bar{c} + iy_{n-1}, c)$, so that the points $f(t, 1, y, c)(-1)$ fill a neighborhood of the origin in E_0' as (y, c) ranges over a neighborhood of the origin in $\mathbb{R}^{n-1} \times \mathbb{C}$ and the corresponding map is a diffeomorphism;
(b) for any fixed $t > 0$ the differential of the map $(\lambda, y, c) \mapsto f(t, \lambda, y, c)(-\lambda)$ evaluated at $(1, 0, 0)$ has the maximum possible rank $n + 2$.

Now fix sufficiently small positive $t = t_0 > 0$. By Theorem 2.4 for small $\delta > 0$ there exist J_δ-holomorphic discs $F_\delta(\lambda, y, \omega)(\bullet)$ C^k-smoothly depending on $\delta, y, \lambda, \omega$ such that $F_\delta(\lambda, y, \omega)(b\mathbb{D}) \subset E_\delta$ and $F_0(\lambda, y, \omega) = f(t_0, \lambda, y, \omega)$. It follows by continuity from (a) and (b) that the range $\tilde{E}_\delta$ of the map $(\lambda, y, \omega) \mapsto F_\delta(\lambda, y, \omega)(-\lambda)$ considered for λ close to 1 is an $n+2$-manifold with boundary that is the range of the map $(y, \omega) \mapsto F_\delta(1, y, \omega)(-1)$ and therefore lies in E_δ. Since this map is close to $f(t, 1, y, \omega)(-1)$ and so has the maximum possible rank $n + 1$, its range is entire E_δ.

Remark. Our proof allows one to 'control' in a certain measure the direction in which the manifold $\tilde{E}$ is attached to E. Indeed, differentiating the map $f(t, \lambda, y, \omega)(-\lambda)$ with respect to λ at the point $(t, 1, 0, 0)$ we see that the tangent space to $\tilde{E}_0$ at the origin is spanned by $T_0(E_0')$ and the vector $\nu = (0, \ldots, 1, 0)$. Hence the tangent space to $\tilde{E}$ at the origin is spanned by $T_0(E)$ and a vector close to ν.

4.2. The case CRdim $E > 1$

Let E be a generating submanifold in an almost complex manifold (M, J). In this section we are particularly interesting in the case CRdim $E > 1$, but our considerations are also meaningful for CRdim $E = 1$. As before, we assume that M is a neighborhood of the origin in $\mathbb{C}^n$, J is a smooth matrix valued function, $J = J_{st} + O(|Z|)$, and $E = \{r^j(Z) = 0, j = 1, \ldots, m\}$, where $r^j = z_j + \bar{z}_j + O(|Z|^2)$, $Z = (z, w) \in \mathbb{C}^m \times \mathbb{C}^{n-m}$.

Let $v \in H_0^J(E)$ be a vector such that the Levi form of r^m does not vanish on v. Consider a J-holomorphic disc tangent to v at the center. After a suitable diffeomorphism with linear part at the origin that is $\mathbb{C}$-linear this disc coincides with the line $l = (0, \ldots, 0, \zeta)$, $\zeta \in \mathbb{C}$; pushing forward J, we still obtain an almost complex structure coinciding with the standard one at the origin. Thus, we may assume that l is J-holomorphic in our coordinates. Similarly to the previous section, for every defining function r^j the value of the Levi form $L_0^J(r^j)(v)$ coincides with that of the Levi form $L_0^{J_{st}}(r^j)(v)$ with respect to J_{st} in the above coordinates.

For $\delta > 0$ consider the dilations $\Lambda_\delta : (z, w) \mapsto (\delta^{-1} z, \delta^{-1/2} w)$ and the induced structure $J_\delta := (\Lambda_\delta)_*(J)$. As we shall see, if the CR dimension of E is > 1, then the structures J_δ do not converge to J_{st} in general. Consider the Taylor expansion of the matrix function J:

$$J(Z) = J_{st} + L(Z) + \mathrm{O}\big(|Z|^2\big),$$

where $L(Z)$ is the linear part. We observe that L is an endomorphism of $\mathbb{R}^{2n}$ anti-linear with respect to the standard complex structure. We regard $L(Z)$ as a complex $n \times n$-matrix with entries $L_{qj}(Z)$ that are $\mathbb{R}$-linear ($\mathbb{C}$-valued) functions of Z. The following result can be proved by direct computation.

Lemma 4.4. *On has $J_\delta \to J_0$ as $\delta \to 0$, where $J_0 = J_{st} + L_0(w)$ and the matrix $L_0(w)$ (in the complex notation) has entries L^0_{qj} described as follows: $L^0_{qj} = 0$, for $q = 1, \dots, n$, $j = 1, \dots, m$ and for $q = n - m + 1, \dots, n$, $j = 1, \dots, n$; for $q = 1, \dots, m$, $j = n - m + 1, \dots, n$ one has $L^0_{qj}(w) = L_{qj}(0, w)$.*

Moreover, the above condition of the J-holomorphy of the line l implies that $L_{q,n-m}$ does not depend on w_{n-m}, that is, $L^0_{q,n-m} = L^0_{q,n-m}(0, w_1, \dots, w_{n-m-1})$.

Consider the manifolds $E_\delta := \Lambda_\delta(E)$ defined by the equations

$$r^J_\delta(z, w) := \delta^{-1} r^j(\delta z, \delta^{1/2} w) = 0, \quad j = 1, \dots, m.$$

Consider the Taylor expansion $r^j(z, w) = z_j + \bar z_j + 2\,\mathrm{Re}\, Q_j(z, w) + H_j(z, w) + \mathrm{O}(|Z|^2)$, where Q_j is the complex quadratic part and H_j the Hermitian part of the expansion. As $\delta \to 0$, we have $r^j \to r^j_0 := z_j + \bar z_j + 2\,\mathrm{Re}\, Q_j(0, w) + H_j(0, w)$. We point out that the biholomorphic (with respect to J_{st}) change of the variables $(z, w) \mapsto (z + Q(0, w), w)$ (where $Q = (Q_1, \dots, Q_m)$) does not change the line l, therefore we can execute it before the dilation. This allows us to assume that $Q_j(0, w) \equiv 0$. Thus, the functions r^j_δ converge to $z_j + \bar z_j + H_j(0, w)$ as $\delta \to 0$. In this sense we view the manifold $E_0 = \{z_j + \bar z_j + H_j(0, w) = 0,\ j = 1, \dots, m\}$ as the limit of E_δ as $\delta \to 0$.

Our next aim is the description of J_0-holomorphic Bishop discs (with values in a sufficiently small neighborhood U of the origin) with boundaries attached to E_0. Let $f : \mathbb{D} \to U$ be a smooth map. To simplify the notations, we will denote by f_ζ the partial derivative $\frac{\partial f}{\partial \zeta}$.

Recall that the J_0-holomorphy condition for f can be written in the following form:

$$f_{\bar\zeta} + Q(f)\overline{f_\zeta} = 0,$$

where $Q(Z)$ is the complex $n \times n$ matrix of an operator the composite of which with complex conjugation is equal to the endomorphism $-(J_{st} + J_\delta(Z))^{-1}(J_{st} - J_\delta(Z))$ (which is an anti-linear operator with respect to the standard structure J_{st}). If f has the form $f(\zeta) = (z(\zeta), w(\zeta))$, then after direct computations of the matrix Q we obtain the equations of the J_0-holomorphy of f:

$$(z_j)_{\bar\zeta} = -(i/2)\left(\sum_{q=1}^{n-m} L^0_{jq}(w)\overline{(w_j)_\zeta}\right), \quad j = 1, \dots, m, \tag{7}$$

and

$$(w_j)_{\bar\zeta} = 0, \quad j = 1, \dots, n - m.$$

This gives one a direct description of all J_0-holomorphic discs. Fix a function $w \in (\mathcal{O}(\mathbb{D}))^{n-m} \times (C^k(\bar{\mathbb{D}}))^{n-m}$ for some fixed non-integer $k > 0$. Then integration of the above system (7) shows that

$$z_j(\zeta) = -T_{\mathrm{CG}}\left((i/2)\sum_{q=1}^{n-m} L^0_{jq}(w)\overline{(w_q)_\zeta}\right)(\zeta) + \phi_j(\zeta), \quad j = 1, \dots, m,$$

where, as before, T_{CG} is the Cauchy–Green transform and ϕ_j is a holomorphic function of the class $C^k(\bar{\mathbb{D}})$.

For $w \in (\mathcal{O}(\mathbb{D}))^{n-m} \times (C^k(\bar{\mathbb{D}}))^{n-m}$ let $\Psi_j(w)(\bullet)$ be the function

$$-T_{\mathrm{CG}}\left((i/2)\sum_{q=1}^{n-m} L^0_{jq}(w)\overline{(w_q)_\zeta}\right).$$

If w is fixed, then the boundary condition $(z, w)(b\mathbb{D}) \subset E_0$ holds if and only if

$$z_j(\zeta) = \big(\Psi_j(w) - I_S\big(\operatorname{Re}\Psi_j(w)\big)\big)(\zeta) - (1/2)I_S\big(H_j(0, w)\big)(\zeta) + iy_j, \; j = 1, \dots, m,$$

where $y_j \in \mathbb{R}$ and I_S is the Schwarz integral in the unit disc:

$$I_S(h)(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} h(\mathrm{e}^{i\tau})\frac{\mathrm{e}^{i\tau} + \zeta}{\mathrm{e}^{i\tau} - \zeta}\, d\tau. \tag{8}$$

This gives us a complete description of Bishop discs attached to E_0. In particular, we have the following result.

Lemma 4.5. *A map $(z, w) : \zeta \mapsto (z(\zeta), w(\zeta))$ is a J_0-holomorphic Bishop disc for E_0 if and only if $(z - \Psi_j(w) + I_S(\operatorname{Re}\Psi_j(w)), w)$ is a J_{st}-holomorphic Bishop disc for E_0.*

Similarly to the previous subsection, consider the map $w : \mathbb{D} \to \mathbb{C}^{n-m}$ of the following form:

$$w_1 = c_1,$$
$$\dots$$
$$w_{n-m-1} = c_{n-m-1},$$
$$w_{n-m}(\zeta) = c_{n-m} + \frac{t(\lambda + \zeta)}{1 + \lambda},$$

where the c_j are complex constants, $t > 0$ and $\lambda \in [0, 1]$. Then $(w_k)_\zeta = 0$ for $k = 1, \dots, n - m - 1$. On the other hand, by our construction $L^0_{jn-m}(0, \dots, 0, \zeta) \equiv 0$. Recall here that the L^0_{jk} participate in the linear part $L(Z)$ of the Taylor expansion of J at the origin and are $\mathbb{R}$-linear in w. Hence the $\mathbb{R}$-linear function L^0_{jn-m} is independent of w_{n-m} and $\sum_{k=1}^{n-m} L^0_{jk}(w)\overline{(w_k)_\zeta} = L^0_{jn-m}(c_1, \dots, c_{n-m-1})t/(1 + \lambda)$ is constant with respect to the variable ζ.

Let $a_j(c_1, \dots, c_{n-m-1}) = -(i/2)L^0_{jn-m}(c_1, \dots, c_{n-m-1})t/(1 + \lambda)$. Then it follows from Eqs. (7) that $z_j(\zeta) = a_j\bar{\zeta} + \phi_j(\zeta)$ with holomorphic ϕ_j. We set $\phi_j = -\bar{a}_j\zeta +$

Φ_j. Then the inclusion $(z, w)(b\mathbb{D}) \subset E_0$ is equivalent to the relation $\operatorname{Re} \Phi_j(\zeta) = -(1/2)H_j(0, w(\zeta))$ for $\zeta \in b\mathbb{D}$ meaning that (Φ, w) is a Bishop disc for E_0 with respect to J_{st}. In view of the condition $\zeta\bar{\zeta} = 1$, the right-hand side represents a real polynomial of degree 1 in ζ, so that the Φ_j are complex polynomials of degree at most 1 and can easily be explicitly written; since the Hermitian quadratic form $H_m(0, w)$ does not vanish on the line $l = (0, \dots, 0, \zeta)$, it contains the term with negative coefficient, and without loss of generality we can assume that this coefficient is -1, other terms of the form $H_m(0, w)$ are independent of w_{n-m}, and the other forms $H_j(0, w)$ contain no term $|w_{n-m}|^2$. Then we obtain a formula for z_{n-m} similar to the one for z_{n-1} in the previous section and explicit expressions for the z-component of a J_0-holomorphic Bishop disc $(z, w)(c_1, \dots, c_{n-m}, t, \lambda, y_1, \dots, y_m)(\bullet)$ determined by the parameters c_j, t, λ, y_k:

$$\begin{aligned}
z_j(\zeta) &= iy_j + a_j(c)\bar{\zeta} - \bar{a}_j(c)\zeta - \frac{1}{2}H_j\left(0, c_1, \dots, c_{n-m-1}, c_{n-m} + \frac{t\lambda}{1+\lambda}\right) \\
&\qquad - \frac{t}{1+\lambda}\zeta l_j(c), \quad j = 1, \dots, m-1, \\
z_{n-m}(\zeta) &= \frac{1}{2}\left(c_{n-m}\bar{c}_{n-m} + \frac{t^2}{(1+\lambda)^2}(\lambda^2+1)\right) + \frac{t\lambda}{1+\lambda}\bar{c}_{n-m} + iy_{n-1} \\
&\qquad + \left(\frac{t\bar{c}_{n-m}}{1+\lambda} + \frac{t^2\lambda}{(1+\lambda)^2}\right)\zeta + a_m(c)\bar{\zeta} - \bar{a}_m(c)\zeta \\
&\qquad - \frac{1}{2}H_{n-m}(0, c_1, \dots, c_{n-m-1}, 0),
\end{aligned}$$

where the $a_j(c)$ are defined above and the l_j are homogeneous linear forms of $c_1, \dots, c_{n-m-1}$. As pointed out already, for $c_j = 0$, $j = 1, \dots, n-m+1$, these are just J_{st}-holomorphic Bishop discs.

Finally, it is easy to see (by computing the rank of the corresponding map; cf. the previous subsection) that the constructed family of J_0-holomorphic Bishop discs sweeps out a manifold with boundary E_0. So we may use the implicit function theorem to construct a perturbed family of J_δ-holomorphic Bishop discs sweeping out a manifold with boundary E_δ.

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