



Global singularization and the failure of SCH

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ARTICLE INFO

Article history:

Received 9 March 2009
 Received in revised form 3 November 2009
 Accepted 3 November 2009
 Available online 30 November 2009
 Communicated by A. Kechris

MSC:
 03E35
 03E55

Keywords:

Easton's theorem
 Prikrý-type forcings
 Hypermeasurable and strong cardinals
 Lifting

ABSTRACT

We say that κ is μ -hypermeasurable (or μ -strong) for a cardinal $\mu \geq \kappa^+$ if there is an embedding $j : V \rightarrow M$ with critical point κ such that $H(\mu)^V$ is included in M and $j(\kappa) > \mu$. Such a j is called a *witnessing embedding*.

Building on the results in [7], we will show that if V satisfies GCH and F is an Easton function from the regular cardinals into cardinals satisfying some mild restrictions, then there exists a cardinal-preserving forcing extension V^* where F is realised on all V -regular cardinals and moreover, all $F(\kappa)$ -hypermeasurable cardinals κ , where $F(\kappa) > \kappa^+$, with a witnessing embedding j such that either $j(F)(\kappa) = \kappa^+$ or $j(F)(\kappa) \geq F(\kappa)$, are turned into singular strong limit cardinals with cofinality ω . This provides some partial information about the possible structure of a continuum function with respect to singular cardinals with countable cofinality.

As a corollary, this shows that the continuum function on a singular strong limit cardinal κ of cofinality ω is virtually independent of the behaviour of the continuum function below κ , at least for continuum functions which are simple in that $2^\alpha \in \{\alpha^+, \alpha^{++}\}$ for every cardinal α below κ (in this case every κ^{++} -hypermeasurable cardinal in the ground model is witnessed by a j with either $j(F)(\kappa) \geq F(\kappa)$ or $j(F)(\kappa) = \kappa^+$).

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1. Introduction

In [2], Easton showed that if ZFC is consistent, so is the fact that the continuum function $\alpha \mapsto 2^\alpha$ on the regular cardinals is governed only by the two following conditions: $\alpha < \beta \rightarrow 2^\alpha \leq 2^\beta$, and $\text{cf}(2^\alpha) > \alpha$, where α, β are regular cardinals. That is if F is a class function $F : \text{Reg} \rightarrow \text{Card}$ satisfying

- $$\begin{aligned} (1) \quad & \alpha < \beta \rightarrow F(\alpha) \leq F(\beta), \quad \text{and} \\ (2) \quad & \text{cf}(F(\alpha)) > \alpha, \end{aligned} \tag{1.1}$$

then assuming GCH in the ground model, F is the continuum function in some cofinality-preserving forcing extension. If F satisfies (1.1), we call F an *Easton function*. We say that F is *realised* in the generic extension in question.

We will rephrase the above result using a notation which will allow for certain generalizations. Formally, let us work in Gödel–Bernays set theory with choice (GBC), which is more suitable when we deal with proper classes.¹ Easton showed the following: If

- $$\begin{aligned} (V, F, E) \models & \text{GCH} \wedge \\ & F \text{ is an Easton function} \wedge \\ & E \text{ is the class of regular cardinals,} \end{aligned} \tag{1.2}$$

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¹ If we view a class A as a collection of elements defined by some formula, as is customary in ZFC, then its interpretation can change with the universe in which we currently work: A^V may be different from A^W for $V \subseteq W$. In GB, A is rigid in the sense that it denotes the same collection of elements in V and W .

then there exists a generic extension $W \supseteq V$ such that

$$(W, F, E) \models E \text{ is the class of regular cardinals } \wedge \\ F \text{ is the continuum function on elements in } E. \quad (1.3)$$

(Note that the fact that W satisfies “ E is the class of regular cardinals” is an equivalent way of saying that the generic extension in question preserves regular cardinals, and so also cofinalities, and hence all cardinals.)

This shows that if we are interested in the axioms of ZFC alone, nothing more can be proved about the continuum function than what is present in the definition of an Easton function. One is tempted to extend this result above the axioms of ZFC, and include an additional property φ of cardinals. Typically, φ will concern some large cardinals. Large cardinals have in general strong reflection properties in that the value of 2^κ for a large cardinal κ depends in some way on the values of 2^α for $\alpha < \kappa$. This implies that we have to formulate the generalization with some care: If

$$(V, F, E, P) \models GCH \wedge \\ F \text{ is an Easton function satisfying } \varphi_0 \wedge \\ E \text{ is the class of regular cardinals } \wedge \\ P \subseteq E \text{ is a collection of cardinals satisfying } \varphi_1, \quad (1.4)$$

then there exists a generic extension $W \supseteq V$ such that

$$(W, F, E, P) \models E \text{ is the class of regular cardinals } \wedge \\ F \text{ is the continuum function on elements in } E \wedge \\ \text{All elements in } P \text{ satisfy } \varphi. \quad (1.5)$$

We then interpret such a result as the fact that the existence of cardinals with the property φ is compatible with every Easton function satisfying φ_0 , providing that we believe in the consistency of cardinals with the property φ_1 . Perhaps the best-known example of the situation where φ_1 is substantially stronger than φ is $2^\kappa = \kappa^{++}$ while κ is measurable; this requires consistency strength of a measurable cardinal κ such that $o(\kappa) = \kappa^{++}$ (see [12]).

To give some specific examples of such properties φ we will need the concept of a μ -hypermeasurable cardinal.

Definition 1.1. A cardinal κ is λ -hypermeasurable (or λ -strong), where λ is a cardinal number greater than κ^+ , if there is an elementary embedding j with a critical point κ from V into a transitive class M such that $\lambda < j(\kappa)$ and $H(\lambda)^V \subseteq M$. We call j in the above definition a *witnessing embedding*. If κ is λ -hypermeasurable for every λ , then κ is called *strong*.

Note that this definition is slightly different from the definition of an α -strong cardinal as in [19] or [21]. We use the more robust H -hierarchy rather than the V -hierarchy to gauge the strength of an embedding. For more information about this convention, see [7].

We say that an Easton function F satisfies the property ϕ if every $F(\kappa)$ -hypermeasurable cardinal κ is a closure point of F (i.e., for every $\mu < \kappa$, $F(\mu) < \kappa$) and for every $F(\kappa)$ -hypermeasurable cardinal κ there exists a witnessing embedding $j: V \rightarrow M$ such that $j(F)(\kappa) \geq F(\kappa)$.² We have shown the following in [7]: If

$$(V, F, E, P) \models GCH \wedge \\ F \text{ is an Easton function satisfying } \phi \wedge \\ E \text{ is the class of regular cardinals } \wedge \\ P \text{ is a collection of } F(\kappa)\text{-hypermeasurable cardinals,} \quad (1.6)$$

then there exists a generic extension W such that

$$(W, F, E, P) \models E \text{ is the class of regular cardinals } \wedge \\ F \text{ is the continuum function on } E \wedge \\ \text{All elements in } P \text{ are measurable cardinals.} \quad (1.7)$$

This shows that measurable cardinals can only restrict the continuum function by conditions occurring in ϕ , providing we believe in the existence of $F(\kappa)$ -hypermeasurable cardinals. This is almost optimal since it is provable that measurable cardinals have strong reflection properties (for instance a measurable cardinal cannot be the least cardinal where GCH fails; that is, if F satisfies GCH below a measurable cardinal κ then $j(F)(\kappa) = \kappa^+$ for every embedding j). Also the assumption on the consistency of $F(\kappa)$ -strong cardinals is almost optimal, see [14], and the comments at the end of this article.

Another example from [7] is the following: We say that an Easton function F is locally definable (this definition comes from [23]) iff there is a sentence ψ and a formula $\varphi(x, y)$ with two free variables such that ψ is true in V and for all cardinals γ , if $H(\gamma) \models \psi$, then $F[\gamma] \subseteq \gamma$ and

$$\forall \alpha, \beta \in \gamma (F(\alpha) = \beta \Leftrightarrow H(\gamma) \models \varphi(\alpha, \beta)). \quad (1.8)$$

² We identify $j(F)$ with $\bigcup_{\kappa \in \text{Card}} j(F \upharpoonright \kappa)$.

The following holds: If

$$(V, F, E, P) \models GCH \wedge$$

$$F \text{ is a locally definable Easton function } \wedge$$

$$E \text{ is the class of regular cardinals } \wedge$$

$$P \text{ is a collection of strong cardinals,} \tag{1.9}$$

then there exists a generic extension W such that

$$(W, F, E, P) \models E \text{ is the class of regular cardinals } \wedge$$

$$F \text{ is the continuum function on } E \wedge$$

$$\text{All elements in } P \text{ are strong cardinals.} \tag{1.10}$$

The present article extends this approach to include singular (strong limit) cardinals. The situation regarding singular cardinals and possible values of the continuum function is much more subtle than in the case of regular cardinals, and still not properly understood. Many deep results were shown which realise some predetermined Easton-type functions on all singular and regular cardinals. These techniques often involve a lot of collapsing of cardinals and may concentrate only on a segment of cardinals,³ and thus do not fit into the context of this article. However, by these techniques it is possible to show that it is consistent (from some hypermeasurable-type assumptions) that GCH can fail everywhere [3], GCH can hold at successors, but fail at limits [1], or that the continuum function on all cardinals can satisfy $2^\alpha = \alpha^{+n}$ for any fixed $n < \omega$ [24].

Our approach in this paper will be an intermediate one: we will not attempt to realise F on all cardinals, but we will realise F on some singular cardinals, while preserving all cardinals. Even with this modest approach it is possible to obtain new information about the behaviour of the continuum function on (some) singular cardinals. Note that we will not collapse cardinals: this means that our singular cardinals failing SCH will be former hypermeasurable cardinals, and thus high in the cumulative hierarchy (in particular, there are no limiting results provable for these cardinals, such as the Shelah’s bound $2^{\aleph_\omega} < \aleph_{\omega_4}$ for \aleph_ω strong limit).

It is long known that there are some natural connections between measurable cardinals failing GCH and singular cardinals failing SCH. For instance, using the original Prikry forcing from [26], it is easy to singularize a measurable cardinal failing GCH, thus obtaining a singular strong limit cardinal failing SCH. In fact, this connection is much deeper: by work of Mitchell and Gitik, we know that the consistency strength of the failure of SCH and the failure of GCH at a measurable is the same. This also extends to more general situations, when 2^κ is very large. See [17,12,13].

However, there is one basic difference between measurable cardinals failing GCH and singular cardinals failing SCH. While measurable cardinals have strong reflection properties as regards the continuum function below these cardinals, singular cardinals of cofinality ω probably do not have any such reflection properties (it is for instance consistent that GCH holds below κ , and κ fails SCH; this can happen already at \aleph_ω , see for instance [15]). It is important to emphasize that we now refer to cardinals of cofinality ω . Once we consider singular cardinals of uncountable cofinalities, we again witness reflection properties: a well-known theorem of Silver claims that if SCH fails at κ of uncountable cofinality then it already fails on a stationary set below κ (in fact on a closed unbounded set, see [27,1] for more details).

Let F be an Easton function and let $CL(F)$ denote the closed unbounded class of closure points of F : $CL(F) = \{\alpha \mid (\forall \beta < \alpha) F(\beta) < \alpha\}$. Given the presumed non-existence of reflection properties for singular cardinals of cofinality ω , the following **strong hypothesis** could, at least at the first glance, hold: If

$$(V, F, E, P) \models GCH \wedge$$

$$F \text{ is an Easton function with } P \subseteq CL(F) \wedge$$

$$E \text{ is the class of regular cardinals } \wedge$$

$$P \text{ is a collection of } F(\kappa)\text{-hypermeasurable cardinals with } \kappa^+ < F(\kappa), \tag{1.11}$$

then there exists a cardinal-preserving generic extension W such that

$$(W, F, E, P) \models F \text{ is the continuum function on } E \wedge$$

$$\text{All elements in } P \text{ are singular strong limit cardinals of cofinality } \omega. \tag{1.12}$$

Note. In applications we will study, we can assume a stronger property for W , i.e., that the class $E \setminus P$ is the class of regular cardinals in W . In practice this is often automatic for cardinal-preserving forcings because the hard work subsists in changing a regular cardinal into a singular one, while the preservation of regularity of cardinals not addressed in the forcing usually follows from the same argument as the preservation of cardinals (e.g., a chain condition). However, the status of the cardinals in $E \setminus P$ is not the main interest of this paper and so we will not explicitly refer to this issue in the rest of the paper.

To prove, or disprove, the above strong hypothesis seems too hard for current techniques. By combining the results from [7] with the Easton-supported iteration of a combination of the simple Prikry and extender-based Prikry forcing notions, we show in this paper the following weaker results.

³ The final model is then of the form V_κ for some inaccessible κ .

The first result follows straightforwardly from [7]. If F satisfies ϕ as in (1.6), then the following holds (see [Theorem 3.8](#) in this paper): If

$$\begin{aligned} (V, F, E, P) \models & GCH \wedge \\ & F \text{ is an Easton function satisfying } \phi \wedge \\ & E \text{ is the class of regular cardinals } \wedge \\ & P \text{ is a collection of } F(\kappa)\text{-hypermeasurable cardinals,} \end{aligned} \quad (1.13)$$

then there exists a cardinal-preserving generic extension W such that

$$\begin{aligned} (W, F, E, P) \models & F \text{ is the continuum function on } E \wedge \\ & \text{All elements in } P \text{ are singular strong limit cardinals of cofinality } \omega. \end{aligned} \quad (1.14)$$

The second result is the main interest of this paper. Let us say that F satisfies the property Ψ if the class of Mahlo cardinals is included in the class of closure points $CL(F)$, F is trivial at the successor of Mahlo cardinals (that is if μ is a Mahlo cardinal, then $F(\mu^+) = \max(F(\mu), \mu^{++})$)⁴, and moreover: for every $F(\kappa)$ -hypermeasurable cardinal κ , where $\kappa^+ < F(\kappa)$, there exists a witnessing embedding j such that either $j(F)(\kappa) \geq F(\kappa)$ or $j(F)(\kappa) = \kappa^+$. Then the following holds (see [Theorem 4.13](#)): If

$$\begin{aligned} (V, F, E, P) \models & GCH \wedge \\ & F \text{ is an Easton function satisfying } \Psi \wedge \\ & E \text{ is the class of regular cardinals } \wedge \\ & P \text{ is a collection of } F(\kappa)\text{-hypermeasurable cardinals with } \kappa^+ < F(\kappa), \end{aligned} \quad (1.15)$$

then there exists a cardinal-preserving generic extension W such that

$$\begin{aligned} (W, F, E, P) \models & F \text{ is the continuum function on } E \wedge \\ & \text{All elements in } P \text{ are singular strong limit cardinals of cofinality } \omega. \end{aligned} \quad (1.16)$$

Noticing that for an Easton function F which for every regular α satisfies $F(\alpha) \in \{\alpha^+, \alpha^{++}\}$ we can always find j such that either $j(F)(\kappa) \geq F(\kappa)$ or $j(F)(\kappa) = \kappa^+$, we obtain the following corollary (see [Corollary 4.25](#)): Let us denote by \mathcal{E} the following property of F : for every α , $F(\alpha) \in \{\alpha^+, \alpha^{++}\}$, and F is trivial at the successors of Mahlo cardinals (that is for every Mahlo cardinal μ , $F(\mu^+) = \max(F(\mu), \mu^{++})$). If

$$\begin{aligned} (V, F, E, P) \models & GCH \wedge \\ & F \text{ is an Easton function satisfying } \mathcal{E} \wedge \\ & E \text{ is the class of regular cardinals } \wedge \\ & P \text{ is a collection of } F(\kappa)\text{-hypermeasurable cardinals with } F(\kappa) = \kappa^{++}, \end{aligned} \quad (1.17)$$

then there exists a cardinal-preserving generic extension W such that

$$\begin{aligned} (W, F, E, P) \models & F \text{ is the continuum function on } E \wedge \\ & \text{All elements in } P \text{ are singular strong limit cardinals of cofinality } \omega. \end{aligned} \quad (1.18)$$

This shows that singular strong limit cardinals of cofinality ω have no global reflection properties formulated in terms of failure or truth of GCH below these cardinals.

Remark 1.2. We will make the following additional assumption about the Easton functions F considered in this paper: if κ is $F(\kappa)$ -hypermeasurable, then we can find for every witnessing embedding $j : V \rightarrow M$ a function $f_{F(\kappa)} : \kappa \rightarrow \kappa$ in V such that $j(f_{F(\kappa)})(\kappa) = F(\kappa)$. This is trivially true for “naturally” defined F ’s, but in the general case such $f_{F(\kappa)}$ ’s may not exist, and must be forced if we wish to have them (see [12]; these functions are relevant in the context of the extender-based Prikry forcings, see [10]). To avoid additional arguments, we will simply assume that we already have these functions $f_{F(\kappa)}$ in the ground model.

2. Extenders

In this section, we will review some basic facts about extenders.

Extenders and extender embeddings are described in detail in [21]. We will use a slightly different representation, as presented for instance in [10], *Extender-Based Prikry Forcing With a Single Extender*.

⁴ This technical condition is probably erasable, see Section 5 for more comments.

Definition 2.1. Let $j : V \rightarrow M$ be a witnessing embedding for λ -hypermeasurability of κ (see Definition 1.1), where λ is a cardinal greater than κ . An elementary embedding $j_E : V \rightarrow M_E$ is called the *extender embedding derived from j* if M_E is the transitive collapse of the class $\{j(f)(\alpha) \mid f : \kappa \rightarrow V, \alpha < \lambda\} \subseteq M$, and in particular $M_E = \{j_E(f)(\alpha) \mid f : \kappa \rightarrow V, \alpha < \lambda\}$.

M_E is identified with a direct limit of a directed system of ultrapowers $\langle M_\alpha \mid \alpha < \lambda \rangle$ where each measure E_α on κ is defined by $X \in E_\alpha \leftrightarrow \alpha \in j(X)$ and $M_\alpha = \text{Ult}(V, E_\alpha)$. The partial order on λ which determines the directed system is defined for $\alpha, \beta < \lambda$ by

$$\alpha \leq_E \beta \leftrightarrow \alpha \leq \beta \quad \text{and for some } f \in {}^\kappa \kappa, j(f)(\beta) = \alpha. \tag{2.19}$$

Clearly, $\alpha \leq_E \beta$ implies that there is a projection between E_α and E_β and subsequently an elementary embedding between M_α and M_β . Under GCH, one can show that \leq_E is μ -directed closed (i.e., for every subset X of λ of size $< \mu$ there is $\gamma \in \lambda$ such that γ is \leq_E -above every element in X), where

$$\mu = \min(\text{cf}(\lambda), \kappa^{++}). \tag{2.20}$$

Note that \leq_E is κ^{++} -directed closed whenever the cofinality of λ is at least κ^{++} . This will be used in the definition of the extender-based Prikry forcing later in the text.

It can be further shown (under GCH) that M_E contains $H(\lambda)$ of M , and that M_E is closed under κ -sequences in V whenever the cofinality of λ is at least κ^+ . It follows that under GCH we can witness λ -hypermeasurability by an extender embedding $j_E : V \rightarrow M_E$, where M_E is closed under κ -sequences in V if the cofinality of λ is at least κ^+ .

3. Iteration of the simple Prikry forcing

Assume GCH in the ground model throughout. Let F be an Easton function. In [7], we have defined a class of so called F -good cardinals as follows: κ is F -good if κ is closed under F (i.e., $\lambda < \kappa$ implies $F(\lambda) < \kappa$) and there is an embedding j witnessing the $F(\kappa)$ -hypermeasurability of κ such that $j(F)(\kappa) \geq F(\kappa)$. We have defined a reverse Easton forcing iteration \mathbb{P}^F and shown that the final model $V^{\mathbb{P}^F}$ realises the Easton function F , preserves all cofinalities and also preserves measurability of all F -good cardinals. It is natural to ask if it is possible to globally change the cofinality of all measurable cardinals in $V^{\mathbb{P}^F}$ while preserving all cardinals. Thus, if κ is measurable in $V^{\mathbb{P}^F}$ and GCH fails at κ , SCH will fail at κ if it remains a strong limit cardinal with cofinality ω . We show in this section that this is indeed possible, by iterating a forcing developed by Prikry in [26], see Definition 3.1 here, along (some) measurable cardinals. In fact, we shall show two ways of doing it: (i) An application of the iteration with full support developed by Magidor ([22]) and (ii) an application of the Easton-supported iteration introduced by Gitik in [11] (see also [10] for the presentation of this iteration). The technique in (ii) will become essential in Section 4 where the extender-based Prikry forcing is included.

However, the use of the forcing as in Definition 3.1 implies that the cardinal κ where we want to fail SCH needs to be first a measurable cardinal failing GCH. By reflection properties of measurable cardinals, this implies failure of GCH on unboundedly many cardinals below κ . This limits unnecessarily the eligible Easton functions F if we aim at obtaining cardinals failing SCH and not care to have them measurable first. There exists a more complicated Prikry-style forcing developed by Magidor and Gitik in [9] which achieves this task: it cofinalizes a sufficiently large κ to a cofinality ω and simultaneously blows up its powerset. We study the iteration of this type of forcing in Section 4 obtaining some original results in this area.

3.1. Preliminaries

In this subsection, we briefly review known facts on iteration of Prikry-style forcing notions, based on [10].

We will call the forcing in Definition 3.1 “simple Prikry forcing” and denote it as $\text{Prk}(\kappa)$.

Definition 3.1. A condition in $\text{Prk}(\kappa)$ is of the form (s, A) where s is a finite increasing sequence in κ and A is a subset of κ which lies in some fixed normal κ -complete ultrafilter U on κ . We assume that $\max(s) < \min(A)$. We say that (s, A) is stronger than (t, B) , $(s, A) \leq (t, B)$, if s end-extends t , $A \subseteq B$ and $s \setminus t \subseteq B$. We say that (s, A) directly extends (t, B) , $(s, A) \leq^* (t, B)$, if (s, A) extends (t, B) and moreover $s = t$.

In the terminology of [10], $\text{Prk}(\kappa)$ is the canonical example of a *Prikry-type forcing notion*, that is P is of the form (P, \leq, \leq^*) , where $\leq^* \subseteq \leq$, and \leq^* is called a direct extension and \leq an extension. The following Prikry property holds: for every $p \in P$ and a sentence σ with fixed parameters in the language of P , there is $q \leq^* p$ deciding σ . The ordering \leq^* is typically more closed than \leq , which is used to show that cardinals below κ are not collapsed.

All antichains in $\text{Prk}(\kappa)$ have size at most $\kappa^{<\omega} = \kappa$, and hence $\text{Prk}(\kappa)$ is κ^+ -cc (this does not require GCH). The direct extension relation \leq^* is κ -closed which implies that $\text{Prk}(\kappa)$ does not add new bounded subsets of κ . It follows that $\text{Prk}(\kappa)$ preserves all cardinals. Note that every two direct extensions of a given condition are compatible.

We will now describe how to iterate the forcing $\text{Prk}(\kappa)$. Essentially, there are two basic options: the full support, and the Easton support. Both definitions are taken literally from [10]. We first review the full support iteration:

Definition 3.2 (*Full Support Iteration*). An iteration with full support for a class X of large cardinals (a parameter of the construction) $\mathbb{R}_X^{\text{full}} = \mathbb{R} = \langle (\mathbb{R}_\alpha, \dot{R}_\alpha) \mid \alpha \in \text{On} \rangle$ is defined by recursion along $\alpha < \text{On}$. We will suppress the superscript notation “full”, and (more often) subscript X in $\mathbb{R}_X^{\text{full}}$ if there is no risk of confusion.

For every $\alpha < \text{On}$ let \mathbb{R}_α be the set of all elements p of the form $\langle \dot{p}_\gamma \mid \gamma < \alpha \rangle$, where for every $\gamma < \alpha$,

$$p \restriction \gamma = \langle \dot{p}_\beta \mid \beta < \gamma \rangle \in \mathbb{R}_\gamma, \tag{3.21}$$

and $p \restriction \gamma \Vdash \dot{p}_\gamma$ is a condition in \dot{R}_γ , where \dot{R}_γ is $\text{Prk}(\gamma)$ if $\gamma \in X$ and \mathbb{R}_γ forces that κ is measurable, or a trivial forcing otherwise.

Let $p = \langle \dot{p}_\gamma \mid \gamma < \alpha \rangle$ and $q = \langle \dot{q}_\gamma \mid \gamma < \alpha \rangle$ be elements of \mathbb{R}_α . Then p is stronger than q , $p \leq q$, iff

- (1) For every $\gamma < \alpha$,

$$p \restriction \gamma \Vdash \dot{p}_\gamma \leq \dot{q}_\gamma \quad \text{in } \dot{R}_\gamma; \tag{3.22}$$

- (2) There exists a finite subset $b \subseteq \alpha$ so that for every $\gamma \in \alpha \setminus b$,

$$p \restriction \gamma \Vdash \dot{p}_\gamma \leq^* \dot{q}_\gamma \quad \text{in } \dot{R}_\gamma. \tag{3.23}$$

If the set in item (2) is empty, then we call p a direct extension of q and denote it as $p \leq^* q$.

Note that even if κ is a Mahlo cardinal, the forcing $\mathbb{R}_\kappa^{\text{full}}$ fails to be κ -cc. However, in certain applications (see [11]), it is useful to have κ -cc at stage κ of an iteration. We may achieve this by requiring that the conditions have the Easton support.

Definition 3.3 (*Easton Support Iteration*). Let again X be a class of large cardinals and a parameter of the iteration. Then the iteration $\mathbb{R}_X^{\text{Easton}} = \mathbb{R} = \langle (\mathbb{R}_\alpha, \dot{R}_\alpha) \mid \alpha \in \text{On} \rangle$ is defined by recursion along $\alpha < \text{On}$. We will suppress the superscript notation “Easton” and (more often) X in $\mathbb{R}_X^{\text{Easton}}$ if there is no risk of confusion.

For every $\alpha < \text{On}$ let \mathbb{R}_α be the set of all elements p of the form $\langle \dot{p}_\gamma \mid \gamma \in g \rangle$, where

- (1) $g \subseteq \alpha$;
- (2) g has the Easton support, i.e., for every inaccessible $\beta \leq \alpha$, $\beta > |g \cap \beta|$, provided that for every $\gamma < \beta$, $|\mathbb{R}_\gamma| < \beta$;
- (3) For every $\gamma \in g$,

$$p \restriction \gamma = \langle \dot{p}_\beta \mid \beta \in g \cap \gamma \rangle \in \mathbb{R}_\gamma, \tag{3.24}$$

and $p \restriction \gamma \Vdash \dot{p}_\gamma$ is a condition in \dot{R}_γ , where \dot{R}_γ is either $\text{Prk}(\gamma)$ if $\gamma \in X$ and \mathbb{R}_γ forces that κ is measurable, or a trivial forcing.

Let $p = \langle \dot{p}_\gamma \mid \gamma \in g \rangle$ and $q = \langle \dot{q}_\gamma \mid \gamma \in f \rangle$ be elements of \mathbb{R} . Then p is stronger than q , $p \leq q$, iff

- (1) $g \supseteq f$;
- (2) For every $\gamma \in f$,

$$p \restriction \gamma \Vdash \dot{p}_\gamma \leq \dot{q}_\gamma \quad \text{in } \dot{R}_\gamma; \tag{3.25}$$

- (3) There exists a finite subset $b \subseteq f$ so that for every $\gamma \in f \setminus b$,

$$p \restriction \gamma \Vdash \dot{p}_\gamma \leq^* \dot{q}_\gamma \quad \text{in } \dot{R}_\gamma. \tag{3.26}$$

If the set in item (3) is empty, then we call p a direct extension of q and denote it as $p \leq^* q$.

Note. This definition is a generalization of the usual notion of an iteration, as in [19]. It is formulated in this way to distinguish for a condition p between coordinates in the iteration which are not even in the support of p , and which are in the support of p , but are trivial there (distinction which does not exist in the usual definition of an iteration). This distinction is important in the context of the direct extension.

If for all $\gamma \in X$ the cardinal γ remains sufficiently large (measurable in this context) in $V^{\mathbb{R}_\gamma}$, then by results in [10], both iterations $\mathbb{R}_X^{\text{full}}$ and $\mathbb{R}_X^{\text{Easton}}$ are themselves Prikry type, i.e., if p is a condition in either of the forcings and σ is a sentence then there is a direct extension $q \leq^* p$ deciding σ .

Lemma 3.4. *Let X be a class of large cardinals and assume that the forcings $\mathbb{R}_X^{\text{full}}$ and $\mathbb{R}_X^{\text{Easton}}$ preserve measurability of $\gamma \in X$ at stage γ of iteration. Then both iterations preserve (under some mild cardinal arithmetic assumptions in the case of $\mathbb{R}_X^{\text{Easton}}$) all cardinals, and also all axioms of ZFC:*

- (1) At each cardinal κ , $\mathbb{R}_\kappa^{\text{full}} = \mathbb{R}$ factors into $\mathbb{R}_{\kappa+1} * \mathbb{R} \restriction \mathbb{R}_{\kappa+1}$ such that $\mathbb{R}_{\kappa+1}$ is κ^+ -cc and $\mathbb{R} \restriction \mathbb{R}_{\kappa+1}$ does not add new subsets of κ^+ . In particular, \mathbb{R} preserves all axioms of ZFC and all cardinals.
- (2) Assuming SCH, at each cardinal κ , $\mathbb{R}_\kappa^{\text{Easton}} = \mathbb{R}$ factors into $\mathbb{R}_{\kappa+1} * \mathbb{R} \restriction \mathbb{R}_{\kappa+1}$ such that $\mathbb{R}_{\kappa+1}$ preserves cardinals $\lambda \geq \kappa^+$ and $\mathbb{R} \restriction \mathbb{R}_{\kappa+1}$ does not add new subsets of κ^+ . In particular \mathbb{R} preserves all axioms of ZFC and all cardinals.

Proof. Ad (1). Let us denote $\mathbb{R} = \mathbb{R}_X^{\text{full}}$, and let κ be a cardinal. The interesting case is when κ is a limit of non-trivial stages of the iteration \mathbb{R} , i.e., if there is a $\lambda \leq \kappa$ and an increasing sequence of cardinals $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ such that $\kappa = \sup(\langle \kappa_\alpha \mid \alpha < \lambda \rangle)$ and each \dot{R}_{κ_α} is a name for the simple Prikry forcing. Since we are dealing with a full support iteration, we do not need to distinguish the cases when κ is regular, or singular. \mathbb{R}_κ is κ^+ -cc by the following argument: if $p \in \mathbb{R}_\kappa$ then there exists a finite subset $b \subseteq \kappa$ where the first coordinate of the condition in the Prikry forcing is non-trivial (at coordinates outside b there are only direct extensions of the empty condition, and these are compatible as we are dealing with the simple Prikry forcing), i.e., there is a finite sequence of names $\langle \dot{s}_\alpha \mid \alpha \in b \rangle$ with \dot{s}_α being a name for a non-empty finite sequence in κ_α . As there are only $\kappa^{<\omega} = \kappa$ many such sequences, it follows that there are at most κ many incompatible conditions, and hence \mathbb{R}_κ is κ^+ -cc. Since \dot{R}_κ is either trivial or the simple Prikry forcing, we also have that $\mathbb{R}_{\kappa+1}$ is κ^+ -cc. The fact that $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ does not add new subset of κ^+ follows from the fact that $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ satisfies the Prikry condition and the direct extension relation in $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ is κ^{++} -closed.

This is enough to argue that \mathbb{R} preserves all cardinals: assume that some κ^+ is collapsed to κ and factor \mathbb{R} into $\mathbb{R}_{\kappa+1}$ and $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$. Since $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ cannot collapse κ^+ , it must be $\mathbb{R}_{\kappa+1}$, but this is impossible as $\mathbb{R}_{\kappa+1}$ is κ^+ -cc. Preservation of axioms of ZFC follows by the fact that $\mathbb{R} \setminus \mathbb{R}_{\kappa+1}$ does not add new subsets of κ^+ for every κ (for more about preservation of axioms of ZFC by class forcings see [5] or [19]).

Ad (2). Let $\mathbb{R} = \mathbb{R}_X^{\text{Easton}}$, and let κ be the interesting case as above in (1). Unlike in (1) we cannot argue that every p in \mathbb{R}_κ is determined as regards compatibility by a finite sequence of names $\langle \dot{s}_\alpha \mid \alpha \in b \rangle$. We need to distinguish the cases when κ is regular and singular. Notice that in both cases, κ needs to be strong limit since it is the limit of a sequence $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$, $\lambda \leq \kappa$, of inaccessible cardinals.

Case 1: κ is regular. In this case κ is strong limit and regular, and hence inaccessible. It follows that $\kappa^{<\kappa} = \kappa$ and by Easton support of \mathbb{R}_κ , this is enough to conclude that \mathbb{R}_κ is κ^+ -cc. In fact, if κ is Mahlo, a standard argument shows that \mathbb{R}_κ is κ -cc. Since \dot{R}_κ is either trivial or the simple Prikry forcing, also $\mathbb{R}_{\kappa+1}$ is κ^+ -cc.

Case 2: κ is singular. In this case κ is a strong limit singular cardinal. Since \mathbb{R}_κ has size 2^κ , it is obviously $(2^\kappa)^+$ -cc. By SCH (this is the only place where we need an additional assumption), $2^\kappa = \kappa^+$ and so \mathbb{R}_κ is κ^{++} -cc. It follows that \mathbb{R}_κ preserves all cardinals $\lambda \geq \kappa^{++}$. By a standard argument based on the Prikry properties of \mathbb{R}_κ , we can also show that κ^+ is preserved; see [11] for details.⁵

Preservation of axioms of ZFC and of cardinals follows exactly as in (1).

(Lemma 3.4) □

Remark 3.5. Notice that $\mathbb{R}_X^{\text{full}} = \mathbb{R}^{\text{full}}$ has the following nice property: every two direct extensions p, q in \mathbb{R}^{full} of the empty condition $1_{\mathbb{R}^{\text{full}}}$ are compatible. This is very useful in showing that the initial segment $\mathbb{R}_\kappa^{\text{full}}$ of the iteration preserves measurability κ (see the first proof of [Theorem 3.8](#)). This contrasts with $\mathbb{R}_X^{\text{Easton}}$ which fails to have this property.

3.2. Global singularization—Simple Prikry forcing

We first review the statement of the theorem in [7]. If F is an Easton function, recall that we call a cardinal κ F -good if κ is closed under F , κ is $F(\kappa)$ -hypermeasurable, and this is witnessed by some j such that $j(F)(\kappa) \geq F(\kappa)$. We defined an iteration \mathbb{P}^F which crucially uses the Sacks forcing to add new subsets of α for inaccessible α (see [20] where the notion of the Sacks forcing for uncountable regular cardinals is introduced). We write $\text{Sacks}(\alpha, \beta)$ to denote the product of β -many copies of the Sacks forcing at α with support of size $\leq \alpha$. The fusion property of the Sacks forcing is useful in lifting extender-type embeddings, see [8,6] for more details. $\text{Add}(\alpha, \beta)$ denotes the usual Cohen forcing for adding β -many Cohen subsets of α .

Definition 3.6. Let F be an Easton function and $\langle i_\alpha \mid \alpha < \text{On} \rangle$ be an increasing enumeration of the closure points of F .

We will define an iteration $\mathbb{P}^F = \langle \langle \mathbb{P}_{i_\alpha} \mid \alpha < \text{On} \rangle, \langle \dot{Q}_{i_\alpha} \mid \alpha < \text{On} \rangle \rangle$ indexed by $\langle i_\alpha \mid \alpha < \text{On} \rangle$ such that:

- If i_α is not an inaccessible cardinal, then

$$\mathbb{P}_{i_{\alpha+1}} = \mathbb{P}_{i_\alpha} * \dot{Q}_{i_\alpha}, \tag{3.27}$$

where \dot{Q}_{i_α} is a name for $\prod_{i_\alpha < \lambda < i_{\alpha+1}} \text{Add}(\lambda, F(\lambda))$ (λ ranges over regular cardinals and the product has the Easton support).

- If i_α is an inaccessible cardinal, then

$$\mathbb{P}_{i_{\alpha+1}} = \mathbb{P}_{i_\alpha} * \dot{Q}_{i_\alpha}, \tag{3.28}$$

where \dot{Q}_{i_α} is a name for $\text{Sacks}(i_\alpha, F(i_\alpha)) \times \prod_{i_\alpha < \lambda < i_{\alpha+1}} \text{Add}(\lambda, F(\lambda))$ (λ ranges over regular cardinals and the product has the Easton support).

- If γ is a limit ordinal, then \mathbb{P}_{i_γ} is an inverse limit unless i_γ is a regular cardinal, in which case \mathbb{P}_{i_γ} is a direct limit (the usual Easton support).

⁵ In fact, it is known from the results in inner model theory that it is very hard to collapse successors of singular cardinals. Thus if we for instance assume that there is no inner model with a Woodin cardinal in our universe, κ^+ cannot be collapsed by a general inner model argument.

Theorem 3.7 ([7]). *Let GCH holds and let F be an Easton function. Then the forcing \mathbb{P}^F in Definition 3.6 realises F , preserves all cofinalities, and preserves measurability of all F -good cardinals.*

Now we can show:

Theorem 3.8. *Let F be an Easton function and \mathbb{P}^F the forcing notion from [7]. Let Δ denote the class of F -good cardinals. Assume that GCH holds in V . Then: There is a forcing iteration \mathbb{R} of the simple Prikry forcing such that in the generic extension by $\mathbb{P}^F * \mathbb{R}$ all cardinals are preserved, the function F is realised and if κ is in Δ , then its cofinality is changed to ω .*

These are the relevant properties of the generic extension $V[G]$ by \mathbb{P}^F :

- (1) $V[G]$ is a cofinality-preserving extension of V realising F .
- (2) $V[G]$ satisfies SCH.
- (3) All F -good cardinals of V , i.e., all $\kappa \in \Delta$, remain measurable in $V[G]$.
- (4) The measurability of $\kappa \in \Delta$ is witnessed in $V[G]$ by some extender embedding $j^* : V[G] \rightarrow M[j^*(G)]$, where j^* lifts some extender embedding $j : V \rightarrow M$ witnessing the F -goodness of κ in V .

We will give two proofs of the theorem. The author first constructed a proof given as Proof 2 using an iteration with the Easton support. The reason for the use of the Easton support will become apparent in Section 4, where the extender-based Prikry forcing is added into our iteration.

M. Magidor in personal communication suggested to the author that in the case of the simple Prikry forcing the iteration with full support, as in [22], gives an easier proof. We include this proof as Proof 1 because we think it is instructive to compare these two techniques.

Proof 1: Full support iteration

This is just an application of Magidor's technique in [22] to the generic extension $V^{\mathbb{P}^F}$. We still review the proof to make the argument self-contained.

Work in $V[G]$ and let $\mathbb{R}_X^{\text{full}} = \mathbb{R}$ be defined as in Definition 3.2, with X defined to contain all measurable cardinals in the ground model $V[G]$. By Lemma 3.4, \mathbb{R} preserves cardinals, and obviously does not change the continuum function in $V[G]$ – hence F is still realised in a generic extension by \mathbb{R} . It remains to verify that all elements of Δ will be cofinalized to a cofinality ω . In fact, we show that all measurable cardinals in $V[G]$ will be cofinalized.

Let us denote by \mathcal{M} the class of measurable cardinals in $V[G]$. Note that in general $\Delta \subseteq \mathcal{M}$, but $\Delta = \mathcal{M}$ may not be true. Clearly, it is enough to show

$$\text{For every } \alpha \in \mathcal{M}, \mathbb{R}_\alpha \Vdash \alpha \text{ is measurable.} \quad (3.29)$$

Note that if measurable cardinals in \mathcal{M} are bounded in α , then α is trivially measurable after forcing with \mathbb{R}_α (because the size of the non-trivial part of \mathbb{R}_α is $< \alpha$). So we will concentrate on the case when measurable cardinals are unbounded below α .

The proof uses the following property of the full support iteration \mathbb{R} of the simple Prikry forcing:

$$\text{For all } p, q \leq^* 1_{\mathbb{R}}, p, q \text{ are } \leq^* \text{-compatible.} \quad (3.30)$$

This property is essential for the definition (3.31).

Let κ in \mathcal{M} be fixed and let $j : V[G] \rightarrow N$ be any embedding witnessing measurability of κ in $V[G]$, where N is some transitive model. We shall show that \mathbb{R}_κ forces that κ is measurable. Let H_κ be a generic for \mathbb{R}_κ . Note that $j(\mathbb{R}_\kappa) = \mathbb{R}_\kappa$, and so in particular a $p \in \mathbb{R}_\kappa$ is an initial segment of a condition in $j(\mathbb{R}_\kappa)$ of N . Define a measure U on κ in $V[G][H_\kappa]$ as follows:

$$X \in U \text{ iff } \exists p \in H_\kappa, \exists p' \Vdash p' \leq^* 1_{j(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa} \text{ and } p \hat{\wedge} p' \Vdash \kappa \in j(\dot{X}), \quad (3.31)$$

where \dot{X} is a \mathbb{R}_κ -name for a subset of κ . We claim that U is a κ -complete uniform ultrafilter in $V[G][H_\kappa]$. In the paragraphs below a primed condition (e.g., p') will refer to elements of $j(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$, while a non-primed condition (e.g., p) will refer to elements of $H_\kappa \subseteq \mathbb{R}_\kappa$ (unless stated otherwise). Note that the \leq^* relation in $j(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$ is κ -closed.

We first state a simple fact:

Fact 3.9. *If σ is a sentence with fixed parameters in the forcing language of $j(\mathbb{R}_\kappa)$, then there are r, r' , such that $r \in H_\kappa$, $1_{\mathbb{R}_\kappa} \Vdash r' \leq^* 1_{j(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa}$ and $r \hat{\wedge} r'$ decides σ .*

Proof. The empty condition $1_{\mathbb{R}_\kappa}$ forces that for some $r' \leq^* 1_{j(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa}$, either $r' \Vdash \sigma$ or $r' \Vdash \neg \sigma$ (this is because the tail iteration $j(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$ satisfies the Prikry condition; we identify here a $j(\mathbb{R}_\kappa)$ -name for a parameter with a \mathbb{R}_κ -name for a $j(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$ -name as usual). This is a disjunction and as such must be decided by some element r in H_κ : either $r \hat{\wedge} r' \Vdash \sigma$ or $r \hat{\wedge} r' \Vdash \neg \sigma$. (Fact 3.9) \square

We finish the first proof of Theorem 3.8 by the following lemma:

Lemma 3.10. *U defined in (3.31) is a κ -complete uniform ultrafilter in $V[G][H_\kappa]$.*

Proof. U is correctly defined. Note that if \dot{X}_0 and \dot{X}_1 are two names and they interpret as the same subset of κ in $V[G][H_\kappa]$, i.e., $(\dot{X}_0)^{H_\kappa} = (\dot{X}_1)^{H_\kappa}$, then they are decided in the same way by conditions according to (3.31): Assume for contradiction that there are $r_0 \hat{\wedge} r'_0$ and $r_1 \hat{\wedge} r'_1$ such that $r_0 \hat{\wedge} r'_0 \Vdash \kappa \in j(\dot{X}_0)$ and $r_1 \hat{\wedge} r'_1 \Vdash \kappa \notin j(\dot{X}_1)$. Let $p \in H_\kappa$ force that $\dot{X}_0 = \dot{X}_1$; $j(p)$ thus forces $j(\dot{X}_0) = j(\dot{X}_1)$ and is of the form $p \hat{\wedge} p'$ where p' is forced by p to be a direct extension of 1 (because there is only finite number of coordinates with non-direct extensions in p and hence these coordinates are bounded in $j(p)$ below κ). It follows that all these three conditions $r_0 \hat{\wedge} r'_0$, $r_1 \hat{\wedge} r'_1$, $p \hat{\wedge} p'$ are compatible which is a contradiction.

U is a filter. The empty condition in $j(\mathbb{R}_\kappa)$ forces that $\kappa \in j(\kappa)$, and so $\kappa \in U$. Let X, Y be in U and let \dot{X}, \dot{Y} be their respective names. If $p \hat{\wedge} p'$ forces that κ is in $j(\dot{X})$ and $r \hat{\wedge} r'$ forces the same for $j(\dot{Y})$ then clearly the common lower bound forces that κ is in the intersection. If $X \subseteq Y$ are subsets of κ , then we fix some $p \in H_\kappa$ which forces $\dot{X} \subseteq \dot{Y}$ and argue as above that if $r_0 \hat{\wedge} r'_0$ forces $\kappa \in j(\dot{X})$ and $r_1 \hat{\wedge} r'_1$ decides $\kappa \in j(\dot{Y})$, then it must decide it positively, otherwise we would reach contradiction by compatibility of $p \hat{\wedge} p'$, $r_0 \hat{\wedge} r'_0$ and $r_1 \hat{\wedge} r'_1$.

U is uniform. It is clearly enough to notice that no $\alpha < \kappa$ is in U as a subset. This is obvious from the fact that $j(\alpha) = \alpha$. Note that combined with κ -completeness of U (see below), this shows that κ remains regular after forcing with \mathbb{R}_κ .

U is an ultrafilter. Let X be a subset of κ , and X^c its complement. Let $p \in H_\kappa$ force that \dot{X} is a complement of \dot{X}^c . Then $j(p) \Vdash \kappa \in j(\dot{X} \cup \dot{X}^c)$. By Fact 3.9, the are $r_0 \hat{\wedge} r'_0$ and $r_1 \hat{\wedge} r'_1$ deciding whether or not κ is in $j(\dot{X})$ or $j(\dot{X}^c)$, respectively. By the compatibility of $r_0 \hat{\wedge} r'_0$, $r_1 \hat{\wedge} r'_1$ and $j(p)$, it must be that exactly one of these conditions decides its relevant sentence positively, otherwise we could consider a common lower bound and derive a contradiction.

U is a κ -complete ultrafilter. Let $(X_\alpha \mid \alpha < \delta)$ be sets in U for some $\delta < \kappa$. By definition (3.31), there are $p_\alpha \hat{\wedge} p'_\alpha$, $\alpha < \delta$, forcing that κ is in $j(\dot{X}_\alpha)$. Let $r \hat{\wedge} r'$, $r \in H_\kappa$, $1_{\mathbb{R}_\kappa} \Vdash r' \leq^* 1_{j(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa}$ decide the sentence $\kappa \in \bigcap_{\alpha < \delta} j(\dot{X}_\alpha)$. We claim that $r \hat{\wedge} r'$ must decide the sentence positively. Assume otherwise. Let \bar{p} be forced by $1_{\mathbb{R}_\kappa}$ to be the greatest lower bound of p'_α 's (it exists because the \leq^* relation in $j(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$ is κ -closed) and choose a condition s' forced by $1_{\mathbb{R}_\kappa}$ to be \leq^* below r' and \bar{p} . Then also $r \hat{\wedge} s'$ decides $\kappa \in \bigcap_{\alpha < \delta} j(\dot{X}_\alpha)$ negatively. There must be some $r_0 \leq r$ in H_κ and $r_0 \Vdash s'_0 \leq s'$ and α such that $r_0 \hat{\wedge} s'_0$ forces $\kappa \notin j(\dot{X}_\alpha)$. However, this is a contradiction since $r_0 \hat{\wedge} s'_0$ is compatible with $p_\alpha \hat{\wedge} p'_\alpha$. (Lemma 3.10) \square

This ends the first proof of Theorem 3.8 (note that GCH or SCH was never used in the argument).

Proof 2: Easton support iteration

Now we will give an alternative proof of Theorem 3.8. To motivate this alternative (and harder) proof, we will anticipate a little. In Section 4, we will include another forcing into \mathbb{R} , the extender-based Prikry forcing $\text{Prk}_E(\kappa, \lambda)$ (see Definition 4.6). This forcing fails to satisfy the condition that every two direct extension of an empty condition in $\text{Prk}_E(\kappa, \lambda)$ are compatible, and hence the definition in (3.31) will no longer be correct. Another technique will be needed, along the lines in the definition (3.40). To make this definition workable, however, we will need Lemma 3.14, which requires the Easton support of the forcing \mathbb{R} .

Work in $V[G]$ and let $\mathbb{R}_X^{\text{Easton}} = \mathbb{R}$ be defined as in Definition 3.3, with $X = \Delta$. By Lemma 3.4, \mathbb{R} preserves cardinals, and obviously does not change the continuum function in $V[G]$ – hence F is still realised in a generic extension by \mathbb{R} . It remains to verify that all elements of Δ will be cofinalized to a cofinality ω . Clearly, as in the first proof, it is enough to show

$$\text{For every } \alpha \in \Delta, \quad \mathbb{R}_\alpha \Vdash \alpha \text{ is measurable.} \tag{3.32}$$

Note that unlike in Proof 1 the argument is now limited to elements in Δ ; it may not include all measurable cardinals in $V[G]$.

Let $\kappa \in \Delta$ be fixed. κ is a measurable cardinal in $V[G]$ and this is witnessed by an embedding $j^* : V[G] \rightarrow M[j^*(G)] =_{df} M^*$ which is a lift of an embedding $j : V \rightarrow M$ in V . Recall that the original j was an extender embedding, i.e.,

$$M = \{j(f)(\alpha) \mid f : \kappa \rightarrow V, \alpha < F(\kappa)\}. \tag{3.33}$$

The lifted j^* is also an extender embedding so that

$$M^* = \{j^*(f^*)(\alpha) \mid f^* : \kappa \rightarrow V[G], \alpha < F(\kappa)\}. \tag{3.34}$$

Note that each f^* is defined from some $f \in V$ with its range containing only \mathbb{P}^F -names by setting

$$f^*(\alpha) = (f(\alpha))^G, \quad \text{for each } \alpha \in \text{dom}(f). \tag{3.35}$$

Preservation of measurability of κ by \mathbb{R}_κ follows directly from [10] if $F(\kappa) = \kappa^+$ (or if the cofinality of $F(\kappa)$ is κ^+). We provide a general argument which works for arbitrary $F(\kappa)$ (assuming $\kappa \in \Delta$). Before we start the proof, recall that the Easton-supported iteration $\mathbb{R}^{\text{Easton}}$ fails to satisfy the property that all direct extensions of a given condition are compatible. Thus we cannot proceed as in (3.31).

In order to show that κ remains measurable in \mathbb{R}_κ we have to define a measure at κ . Following the argument in [10] we will find a family of conditions in $j^*(\mathbb{R}_\kappa)$ which will answer compatibly the questions

$$\text{“is } \kappa \text{ in } j^*(\dot{X})\text{”,} \tag{3.36}$$

where the \dot{X} 's are \mathbb{R}_κ -names for subsets of κ . If $F(\kappa) > \kappa^+$, then there are more than κ^+ -many such names \dot{X} and this prevents us from taking lower bounds when constructing the (to-be) compatible family of conditions (M^* is closed only under κ -sequences in $V[G]$). A standard way to circumvent this obstacle is to group the \leq^* -dense open sets (see Definition 3.11)

corresponding to the relevant questions into κ^+ -many segments such that each segment can be determined by a single condition (each segment will typically have size greater than κ^+).

The basic idea of the proof is to show that this grouping can be achieved by considering a family $\{f_\alpha \mid f_\alpha : \kappa \rightarrow H(\kappa)^V, \alpha < \kappa^+\}$ in V which determines a family $\{f_\alpha^* \mid f_\alpha^* : \kappa \rightarrow H(\kappa)^{V[G]}, \alpha < \kappa^+\}$ of functions in $V[G]$ which is universal in that the ranges of $j^*(f^*)$'s capture all \leq^* -dense opens sets in $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_{\kappa+1}$ (and in particular the \leq^* -dense open sets corresponding to the questions (3.36)). Thus we will “borrow” some degree of GCH at κ from the original V . Note that this requires that \mathbb{R}_κ has in some sense small chain condition (see the argument in Lemma 3.14), which is ensured by the Easton support.

- Definition 3.11.** (1) $D \subseteq \mathbb{R}_\kappa$ is \leq^* -dense open if D is open and for every $p \in \mathbb{R}_\kappa$ there is $d \in D$ and $d \leq^* p$.
 (2) We say that p and q are \leq^* -compatible (or *direct compatible*) if there is a direct extension below p and q . We say that p and q are \leq^* -incompatible, if there is no direct extension below p and q .
 (3) $A \subseteq \mathbb{R}_\kappa$ is a \leq^* -antichain if all elements of A are \leq^* -incompatible. A is a maximal \leq^* -antichain if $a \notin A$ implies that there is some $\bar{a} \in A$ such that a and \bar{a} are direct compatible.
 (4) We say that $\langle \mathbb{R}_\kappa, \leq^* \rangle$ is κ -cc if all \leq^* -antichains are smaller than κ .

Note that a \leq^* -antichain may not be an antichain in the usual \leq -relation. However, every antichain is also a \leq^* -antichain. But a maximal antichain may not be a maximal \leq^* -antichain.

As regards the \leq^* -chain condition, notice by way of example that $\langle \text{Prk}(\kappa), \leq^* \rangle$ is still κ^+ -cc as conditions with the same first coordinate are direct compatible.

The usual correspondence between dense sets and antichains still holds:

Fact 3.12. Assume $G \subseteq \mathbb{R}_\kappa$ hits all \leq^* -maximal antichains. Then it hits all \leq^* -dense open sets.

The reason for introducing \leq^* -antichains is that there are generally smaller than \leq^* -dense open sets in \mathbb{R}_κ .

Lemma 3.13. $\langle \mathbb{R}_\kappa, \leq^* \rangle$ is κ -cc.

Proof. Emulate the usual proof for the Easton-supported iteration (see for instance [19], Theorem 16.9 and 16.30). The basic setup of the argument is that using the Fodor’s theorem one can find for every κ -sequence $\langle p_\xi \mid \xi < \kappa \rangle$ of conditions in \mathbb{R}_κ a stationary subset S of κ such that

- (1) For every $\xi \in S$ it holds that $\text{supp}(p_{\xi'}) \subseteq \xi$ for every $\xi' < \xi$, and
- (2) There is some γ such that for all $\xi \in S$, $\text{supp}(p_\xi) \cap \xi \subseteq \gamma$.

Now consider the sequence $\langle p_\xi \upharpoonright \gamma \mid \xi \in S \rangle$. Since \mathbb{R}_γ has size less than κ , $\langle \mathbb{R}_\gamma, \leq^* \rangle$ is certainly κ -cc. It follows there are η, ξ such that $\gamma < \eta < \xi$ with $p_\eta \upharpoonright \gamma$ and $p_\xi \upharpoonright \gamma$ being direct compatible (in fact $p_\eta \upharpoonright \gamma$ and $p_\xi \upharpoonright \gamma$ can be taken to be identical). By the properties of S , the supports of p_ξ and p_η are disjoint outside γ , and consequently p_ξ and p_η are direct compatible.

(Lemma 3.13) \square

Let H_κ be a generic filter for \mathbb{R}_κ . It is also a generic filter for $j^*(\mathbb{R})_\kappa$ over M^* . Let us assume that $j^*(\mathbb{R}_\kappa)$ is non-trivial at κ , that is $\kappa \in j^*(\Delta)$, which means that \dot{R}_κ of $j^*(\mathbb{R}_\kappa)$ is $\text{Prk}(\kappa)$ (we in general cannot eliminate the possibility that $\kappa \in j^*(\Delta)$ because our j^* comes from some fixed hypermeasurable embedding j). If $\kappa \notin j^*(\Delta)$, then the argument proceeds identically (and is easier in that the forcing at κ is trivial).

As $j(F)(\kappa) \geq F(\kappa)$, the least measurable cardinal above κ is greater than $F(\kappa)$ and hence $\text{Prk}(\kappa)$ forces over $M^*[H_\kappa]$ that $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_{\kappa+1}$ is $(F(\kappa)^+)$ - \leq^* -closed.

Lemma 3.14. Let σ in M^* be a $j^*(\mathbb{R})_{\kappa+1}$ -name (where $j^*(\mathbb{R})_{\kappa+1} = \mathbb{R}_\kappa * \text{Prk}(\kappa)$ in M^*) for a maximal \leq^* -antichain in $j^*(\mathbb{R}_\kappa) \setminus j^*(\mathbb{R})_{\kappa+1}$. We claim that there is a name $\bar{\sigma}$ such that $j^*(\mathbb{R})_{\kappa+1}$ forces $\bar{\sigma} = \sigma$, and moreover for some f in V and $\alpha < F(\kappa)$, $f : \kappa \rightarrow H(\kappa)^V$, we have that $j^*(f^*)(\alpha) = \bar{\sigma}$ (see (3.34) and (3.35) for the meaning of f^*). In particular there are only $(\kappa^\kappa)^V = \kappa^+$ functions f^* which enumerate (names for) maximal \leq^* -antichains in $j^*(\mathbb{R}_\kappa) \setminus j^*(\mathbb{R})_{\kappa+1}$.

Proof. We first argue that we can choose for σ a nice name $\bar{\sigma}$ which is an element of $H(j(\kappa))$ in M^* : By Lemma 3.13 applied to $j^*(\mathbb{R}_\kappa)$ we know that $j^*(\mathbb{R})_{\kappa+1}$ forces that σ is an antichain of size less than $j(\kappa)$, i.e., that it is an element of $H(j(\kappa))$ in a generic extension of M^* by $j^*(\mathbb{R})_{\kappa+1}$. W.l.o.g we can identify elements of $H(j(\kappa))$ in a generic extension of M^* with bounded subsets of $j(\kappa)$. Hence we know that $j^*(\mathbb{R})_{\kappa+1}$ forces that σ is a bounded subset of $j(\kappa)$. Moreover since $j^*(\mathbb{R})_{\kappa+1}$ is κ^+ -cc in M^* , it forces a bound on σ ; let $\alpha_\sigma < j(\kappa)$ be this bound:

$$M^* \models j^*(\mathbb{R})_{\kappa+1} \Vdash \sigma \subseteq \alpha_\sigma < j(\kappa). \tag{3.37}$$

Hence there is a nice $j^*(\mathbb{R})_{\kappa+1}$ -name for σ , to be denoted as $\bar{\sigma}$, which is an element of $H(j(\kappa))$ of M^* . We again identify $\bar{\sigma}$ with some bounded subset of $j(\kappa)$ in M^* .

Going back to the original V , notice that because $\bar{\sigma}$ is a bounded subset of $j(\kappa)$, it must have been added by the iteration $j(\mathbb{P}^F)_{j(\kappa)}$ over M . By $j(\kappa)$ -cc of the forcing $j(\mathbb{P}^F)_{j(\kappa)}$ in M , we can choose a nice $j(\mathbb{P}^F)_{j(\kappa)}$ -name $\bar{\bar{\sigma}}$ for $\bar{\sigma}$ which itself can be identified with a bounded subset of $j(\kappa)$, this time in M .

As a bounded subset of $j(\kappa)$ in M , $\bar{\bar{\sigma}}$ is an element of $H(j(\kappa))$ of M . It follows we can write $\bar{\bar{\sigma}}$ as $j(f)(\alpha)$ for some $f : \kappa \rightarrow H(\kappa), f \in V$, and $\alpha < F(\kappa)$. By defining $f^*(\gamma) = (f(\gamma))^G$ for every $\gamma < \kappa$ in the domain of f , we obtain

$$j^*(f^*)(\alpha) = (j(f)(\alpha))^{j(G)} = (\bar{\bar{\sigma}})^{j(G)} = \bar{\sigma}, \tag{3.38}$$

as desired.

(Lemma 3.14) \square

Work in $V[G][H_\kappa]$, where H_κ is a generic for \mathbb{R}_κ . To finish the proof of [Theorem 3.8](#), define the following construction: let $\langle f_\alpha^* \mid \alpha < \kappa^+ \rangle$ be some enumeration of the relevant f^* 's as identified in [Lemma 3.14](#). For each α , the family of names for \leq^* -antichains in the forcing $j^*(\mathbb{R}_\kappa) \setminus j^*(\mathbb{R})_{\kappa+1}$ in $M^*[H_\kappa]$ determined by $j^*(f_\alpha^*)$, i.e. $\{(j^*(f_\alpha^*)(\gamma))^{H_\kappa} \mid \gamma < F(\kappa)\} =_{df} \{A_{\alpha_\gamma} \mid \gamma < F(\kappa)\}$, exists in $M^*[H_\kappa]$ and has size less or equal $F(\kappa)$. We can assume that the empty condition in $\text{Prk}(\kappa)$, $1_{\text{Prk}(\kappa)}$, forces that each A_{α_γ} is a maximal \leq^* -antichain.

By induction construct for each family $\{A_{\alpha_\gamma} \mid \gamma < F(\kappa)\}$ a sequence of conditions $\langle q_{\alpha_\gamma} \in j^*(\mathbb{R}_\kappa) \setminus j^*(\mathbb{R})_{\kappa+1} \mid \gamma < F(\kappa) \rangle$ such that q_{α_γ} 's are forced by $1_{\text{Prk}(\kappa)}$ to form a \leq^* -decreasing chain in $j^*(\mathbb{R}_\kappa) \setminus j^*(\mathbb{R})_{\kappa+1}$. Choose each q_{α_γ} so that it is forced by $1_{\text{Prk}(\kappa)}$ to be a direct extension of some element in the maximal \leq^* -antichain A_{α_γ} (this can be arranged as each A_{α_γ} is (forced to be) a maximal \leq^* -antichain).

Let \bar{q}_α be the limit of the q_{α_γ} 's. Arrange the construction so that $1_{\text{Prk}(\kappa)}$ forces that for $\alpha < \kappa^+$, \bar{q}_α 's form a \leq^* -decreasing chain. Set

$$Q = \{q \in j^*(\mathbb{R}_\kappa) \setminus j^*(\mathbb{R})_{\kappa+1} \mid \exists \alpha < \kappa^+, 1_{\text{Prk}(\kappa)} \Vdash \bar{q}_\alpha \leq^* q\}. \tag{3.39}$$

The conditions in Q (compatibly) meet all maximal \leq^* -antichains in $j^*(\mathbb{R}) \setminus j^*(\mathbb{R})_{\kappa+1}$, and by [Fact 3.12](#) they meet all \leq^* -dense open sets in the same forcing.

Define a measure U as follows, where X is a subset of κ in $V[G][H_\kappa]$:

$$X \in U \text{ iff } \exists r \in H_\kappa, \exists p \Vdash r \leq^* 1_{\text{Prk}(\kappa)}, \text{ and } \exists q \in Q \text{ such that } r \hat{\wedge} p \hat{\wedge} q \Vdash \kappa \in j^*(\dot{X}). \tag{3.40}$$

The argument that U is a measure is analogous to the argument in the first proof for [\(3.31\)](#).

This ends the alternative proof of [Theorem 3.8](#).

Remark 3.15. One can argue (see [\[10\]](#)) that the models obtained in Proof 1 and Proof 2 are different. We do not know so far whether one can find an interesting statement related to our topic which distinguishes these two models.

4. Iteration of the extender-based Prikrý forcing

We will now include the extender-based Prikrý forcing in our iteration in order to remove (some of) the restrictions put on F by the techniques in the previous Section. However, this method – though much more powerful – will still not be completely general since the inclusion of the extender-based Prikrý forcing will bring some restrictions of its own.

The restrictions are caused by the two following reasons. Firstly, the extender-based Prikrý forcing at γ as developed in [\[10\]](#) does require for its correct definition some degree of GCH below γ . Secondly, when we iterate the extender-based Prikrý forcing below γ , we require that the forcing below γ is trivial (i.e., preserves GCH) on large enough a set.

These restrictions are inherently tied with the definition of the forcing: if γ should be large enough for the definition of the extender-based Prikrý forcing, it for instance cannot be the least measurable cardinal in the universe. This for instance implies that the iteration below γ cannot singularize all large cardinals below γ .

To avoid some restrictions of this kind to be put on F , one may ask if it is possible to first singularize the desired large cardinals over a model with GCH, and only then realise F on the remaining regular cardinals. However, we will show in [Section 4.1](#) that it may not be possible to do it (at least with obvious means).

4.1. Forcing after singularization tends to collapse cardinals

Assume a strong limit singular cardinal κ has cofinality ω in V^* and $\kappa^+ < 2^\kappa$ in V^* . Assume further that GCH holds below κ . Note that if the κ_i 's are cofinal in κ for $i < \omega$, then the size of $\prod_{i < \omega} \kappa_i$ is $\kappa^\omega = 2^\kappa$. This configuration for instance arises when V^* is a generic extension of V such that V satisfies GCH, κ is κ^{++} -hypermeasurable in V and we force with the extender-based Prikrý forcing which blows up the powerset of κ to κ^{++} and simultaneously cofinalizes κ .

If μ is a regular cardinal we write $\text{Add}(\mu, 1)$ for the Cohen forcing adding a new subset of μ . Conditions in $\text{Add}(\mu, 1)$ will be construed as defined on initial segments of μ (i.e., on ordinals less than μ) with range included in $\{0, 1\}$.

Definition 4.1. Under this notion, we say that p in $\text{Add}(\mu, 1)$, or more generally a generic for $\text{Add}(\mu, 1)$, codes $\delta < \mu$ at position $\delta' < \mu$ if p restricted to $[\delta', \delta' + \delta + 1)$ is a sequence of 1's followed by one 0, i.e., the 1's starting at δ' have order type δ and this segment is terminated by 0 to determine which ordinal is being coded.

Observation 4.2. Let $\langle \kappa_i \mid i < \omega \rangle$ be a sequence of regular cardinals cofinal in κ . Let $\mathbb{P} = \prod_{i < \omega}^{\text{FIN}} \text{Add}(\kappa_i, 1)$ be a product of Cohen forcings with finite support. Then κ is collapsed to ω in the generic extension $V^*\mathbb{P}$.

Proof. Let G be \mathbb{P} -generic over V^* , and g_i 's the generics for $\text{Add}(\kappa_i, 1)$'s. We define in $V[G]$ a function $h : \omega \rightarrow \kappa$ which is onto. Set $h(n)$ to be equal to an ordinal coded by g_n at position 0 according to [Definition 4.1](#), i.e., $h(n)$ is the ordinal corresponding to the order type of the initial segments of 1's in g_n .

We show that h is onto. To this end, let $\delta < \kappa$ be given. It is easy to see that D_δ is dense, where

$$D_\delta = \{p \in \mathbb{P} \mid \exists n < \omega, p(n) \text{ codes } \delta \text{ at } 0\}. \tag{4.41}$$

Now the observation follows. (Observation 4.2) \square

Let κ be as above. By Shelah's theorem, see for instance [\[19\]](#) Theorem 24.8, there exists an increasing sequence $\langle \lambda_n \mid n < \omega \rangle$ of regular cardinals with limit κ such that there is a sequence $\langle f_\xi \mid \xi < \kappa^+ \rangle$ of elements in $\prod_{n < \omega} \lambda_n$ such that $\langle f_\xi \mid \xi < \kappa^+ \rangle$ is $<_{\text{FIN}}$ -cofinal in $\prod_{n < \omega} \lambda_n$ modulo the ideal of finite sets FIN.

Observation 4.3. Let $\mathbb{P} = \prod_{i < \omega}^{\text{FULL}} \text{Add}(\lambda_i, 1)$ be the product of Cohen forcings with full support. Then \mathbb{P} collapses 2^κ to κ^+ .

Proof. Notice that \mathbb{P} has a big chain condition: it is $(2^\kappa)^+$ -cc, hence standard arguments to show that \mathbb{P} preserves cardinals above κ^+ fail.

Let G be generic for \mathbb{P} , with g_i for $i < \omega$ generics for $\text{Add}(\lambda_i, 1)$'s. Assume for simplicity first that $\langle f_\xi \mid \xi < \kappa^+ \rangle$ is cofinal on all coordinates, i.e., we do not allow the error modulo FIN. We define a function $h : \kappa^+ \rightarrow 2^\kappa = \prod_{i < \omega} \lambda_i$ which is onto as follows. For $\xi < \kappa^+$ let $h(\xi)$ be the sequence of ordinals $\langle \alpha_i \mid i < \omega \rangle$ in $\prod_{i < \omega} \lambda_i$ such that α_i is coded as in Definition 4.1 by g_i at the position $f_\xi(i)$ for each $i < \omega$.

We argue that h is onto. Let a sequence $s = \langle \beta_i \mid i < \omega \rangle$ in $\prod_{i < \omega} \lambda_i$ be given. By cofinality of $\langle f_\xi \mid \xi < \kappa^+ \rangle$, it is easy to see that the following set is dense:

$$D_s = \{p \in \mathbb{P} \mid \exists \xi < \kappa^+, p \text{ pointwise codes } s \text{ at places } f_\xi(i) \text{ for } i < \omega\}. \tag{4.42}$$

Now we rectify the argument to account for $<_{\text{FIN}}$ -cofinality. Define $h^* : \kappa^+ \rightarrow \prod_{i < \omega} \lambda_i$ as follows: let $h^*(\xi)$ be the family of all sequences $\langle \alpha_i \mid i < \omega \rangle$ such that $\langle \alpha_i \mid i < \omega \rangle$ is coded by g_i at position $f_\xi(i)$ for all but finitely many i 's. Note that the size of $h^*(\xi)$ is κ for every $\xi < \kappa^+$. The observation follows once we show that $\prod_{i < \omega} \lambda_i$ is covered by the union $\bigcup_{\xi < \kappa^+} h^*(\xi)$.

If p is a condition in \mathbb{P} let $\text{Code}_\xi(p)$ denote the family of sequences $\langle \alpha_i \mid i < \omega \rangle$ such that $p(i)$ codes α_i at position $f_\xi(i)$ for all but finitely many i 's. Let a sequence $s = \langle \beta_i \mid i < \omega \rangle$ in $\prod_{i < \omega} \lambda_i$ be given. By $<_{\text{FIN}}$ -cofinality of $\langle f_\xi \mid \xi < \kappa^+ \rangle$, it is easy to see that the following set is dense:

$$D_s = \{p \in \mathbb{P} \mid \exists \xi < \kappa^+, s \in \text{Code}_\xi(p)\}. \tag{4.43}$$

This proves the observation. (Observation 4.3) \square

A similar argument works for the iteration of Cohen forcings (and for iteration of other usual forcings, as well). It follows that it may be very hard to realise F below κ without collapsing cardinals if κ is a singular cardinal as above.

4.2. Preliminaries

We first review the definition of the extender-based Prikry forcing following [10], with some small corrections according to [4] in Definitions 4.6 and 4.7.

Let $\kappa < \lambda$ be cardinals, κ regular and λ of cofinality at least κ^{++} .

Definition 4.4. A commutative system of embeddings is called a *nice system* for (κ, λ) if the following conditions hold:

- (1) $\langle \lambda, \leq_E \rangle$ is a κ^{++} -directed partial order, i.e., if $\{\alpha_\xi \mid \xi < \kappa^+\}$ is a subset of λ then there is some $\bar{\alpha} < \lambda$ such that $\alpha_\xi \leq_E \bar{\alpha}$ for all $\xi < \kappa^+$.
- (2) $\langle U_\alpha \mid \alpha < \lambda \rangle$ is a Rudin–Keisler commutative sequence of κ -complete ultrafilters over κ with projections $\langle \pi_{\alpha\beta} \mid \beta \leq \alpha < \lambda, \alpha \geq_E \beta \rangle$.
- (3) For every $\alpha < \lambda$, $\pi_{\alpha\alpha}$ is the identity on a fixed set \bar{X} which belongs to every $U_\beta, \beta < \lambda$.
- (4) (Commutativity) For every $\alpha, \beta, \gamma < \lambda$ such that $\alpha \geq_E \beta \geq_E \gamma$ there is $Y \in U_\alpha$ so that for every $v \in Y$,

$$\pi_{\alpha\gamma}(v) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(v)). \tag{4.44}$$

- (5) For every $\alpha < \beta, \gamma < \lambda$ if $\gamma \geq_E \alpha, \beta$ then

$$\{v \in \kappa \mid \pi_{\gamma\alpha}(v) < \pi_{\gamma\beta}(v)\} \in U_\gamma. \tag{4.45}$$

- (6) U_κ is a normal ultrafilter.
- (7) $\kappa \leq_E \alpha$ when $\kappa \leq \alpha < \lambda$.
- (8) (Full commutativity at κ) For every $\alpha, \beta < \lambda$ and $v < \kappa$, if $\alpha \geq_E \beta$ then $\pi_{\alpha\kappa}(v) = \pi_{\beta\kappa}(\pi_{\alpha\beta}(v))$.
- (9) (Independence of the choice of projections to κ) For every α, β such that $\kappa \leq \alpha, \beta < \lambda, v < \kappa$,

$$\pi_{\alpha\kappa}(v) = \pi_{\beta\kappa}(v). \tag{4.46}$$

- (10) Each U_α is a P -point ultrafilter, i.e., for every $f \in {}^\kappa \kappa$, if f is not constant mod U_α , then there is $Y \in U_\alpha$ such that for every $v < \kappa, |Y \cap f^{-1}(\{v\})| < \kappa$.

The existence of a nice system for (κ, λ) follows in a straightforward way if κ is a λ -hypermeasurable cardinal and GCH holds. Implicit in [10] is the following weakening of the hypermeasurability assumption (and of GCH) which also implies the existence of a nice system for (κ, λ) .

Observation 4.5. Let κ be a regular cardinal and $\lambda > \kappa$ a cardinal with cofinality at least κ^{++} . Assume that $2^\kappa = \kappa^+$. Assume further that there exists an embedding $j : V \rightarrow M$ with a critical point κ such that

- (1) M is closed under κ -sequences in V .
- (2) $j(\kappa) > \lambda$.
- (3) $([\lambda]^{\leq \kappa^+})^V \subseteq M$.
- (4) $|[\lambda]^{\leq \kappa^+}| = \lambda$ in M (and hence also in V).
- (5) For some $f_\lambda : \kappa \rightarrow \kappa, j(f_\lambda)(\kappa) = \lambda$.

Then there exists a nice system for (κ, λ) .

Proof. Define for $\kappa \leq \alpha \leq \beta < \lambda$ that $\alpha \leq_E \beta$ iff $j(f)(\beta) = \alpha$ for some $f : \kappa \rightarrow \kappa$. The single interesting property which may fail to hold in this context (when we use a weaker embedding than a λ -hypermeasurable embedding, and GCH may not hold) is (1) in Definition 4.4, i.e., that $\langle \lambda, \leq_E \rangle$ is a κ^{++} -directed partial order. It is enough to verify that there exists in V an enumeration h such that $j(h)$ enumerates $[\lambda]^{\leq \kappa^+}$ in V and M in λ -many steps so that each subset of λ of size at most κ^+ occurs cofinally often in the enumeration. To this effect define h with a domain κ to satisfy (where $\mu_\alpha = |[f_\lambda(\alpha)]^{\leq \alpha^+}|$ in V): If α is a Mahlo cardinal, then h restricted to μ_α enumerates $[f_\lambda(\alpha)]^{\leq \alpha^+}$ so that each subset of $f_\lambda(\alpha)$ of size at most α^+ occurs cofinally often in the enumeration. Then $j(h)$ restricted to $(\mu_\kappa)^M = |[j(f_\lambda)(\kappa)]^{\leq \kappa^+}|^M = j(f_\lambda)(\kappa) = \lambda$ enumerates $[\lambda]^{\leq \kappa^+}$ both in V and M in λ -many steps and with cofinal repetitions.

See the construction of a nice system in [10] or [9] for the other properties. (Observation 4.5) □

Before we define the forcing notion, we first need some auxiliary definitions related to the nice system in Definition 4.4. Let us denote $\pi_{\alpha\kappa}(v)$ by v^0 , where $\kappa \leq \alpha < \lambda$ and $v < \kappa$ (this is independent of α). By 0 -increasing sequence of ordinals we mean a sequence $\langle v_1, \dots, v_n \rangle$ of ordinals below κ so that

$$v_1^0 < v_2^0 < \dots < v_n^0. \tag{4.47}$$

For every $\alpha < \lambda$ we shall always mean by writing $X \in U_\alpha$ that $X \subseteq \bar{X}$, in particular it will imply that for $v_1, v_2 \in X$ if $v_1^0 < v_2^0$ then the size of $\{\alpha \in X \mid \alpha = v_i^0\}$ is $< v_2^0$. The following weak version of normality holds since U_α is a P -point: if $X_i \in U_\alpha$ for $i < \kappa$ then also $X = \{v \mid \forall i < v^0, v \in X_i\} \in U_\alpha$.

Let $v < \kappa$ and $\langle v_1, \dots, v_n \rangle$ be a finite sequence of ordinals below κ . Then v is called *permitted* for $\langle v_1, \dots, v_n \rangle$ if $v^0 > \max(\{v_i^0 \mid 1 \leq i \leq n\})$.

Now we are ready to define the extender-based Prikry forcing notion.

Definition 4.6. The extender-based Prikry forcing $\text{Prk}_E(\kappa, \lambda)$ is defined as follows. The set of forcing conditions consists of all the elements p of the form $\{\langle \gamma, p^\gamma \rangle \mid \gamma \in g \setminus \{\max(g)\}\} \cup \{(\max(g), p^{\max(g)}, T)\}$, where

- (1) $g \subseteq \lambda$ is a set of cardinality $\leq \kappa$ which has a maximal element in \leq_E -ordering and $0 \in g$. Further let us denote g by $\text{supp}(p)$, $\max(g)$ by $\text{mc}(p)$, T by T^p and $p^{\max(g)}$ by p^{mc} (“mc” for maximal coordinate).
- (2) For $\gamma \in g$, p^γ is a finite 0 -increasing sequence.
- (3) T is a tree with a trunk p^{mc} consisting of 0 -increasing sequences. All the splittings in T are required to be on sets in $U_{\text{mc}(p)}$, i.e., for every $\eta \in T$, if $\eta \geq_T p^{\text{mc}}$ then the set

$$\text{Suc}_T(\eta) = \{v < \kappa \mid \eta \hat{\ } \langle v \rangle \in T\} \in U_{\text{mc}(p)}. \tag{4.48}$$

We also require that for $\eta_1 \geq_T \eta_0 \geq_T p^{\text{mc}}$,

$$T_{\eta_1} \text{ is a subtree of } T_{\eta_0}, \tag{4.49}$$

where T_η denotes the set of σ such that $\eta \hat{\ } \sigma$ belongs to T .

- (4) For every $\gamma \in g$, $\pi_{\text{mc}(p), \gamma}(\max(p^{\text{mc}}))$ is not permitted for p^γ .
- (5) For every $v \in \text{Suc}_T(p^{\text{mc}})$,

$$|\{\gamma \in g \mid v \text{ is permitted for } p^\gamma\}| \leq v^0. \tag{4.50}$$

- (6) $\pi_{\text{mc}(p), 0}$ project p^{mc} onto p^0 , in particular p^{mc} and p^0 are of the same length.

The ordering $\leq_{\text{Prk}_E(\kappa, \lambda)} = \leq$ is defined as follows:

Definition 4.7. We say that p extends q , $p \leq q$ if

- (1) $\text{supp}(p) \supseteq \text{supp}(q)$.
- (2) For every $\gamma \in \text{supp}(q)$, p^γ is an end-extension of q^γ .
- (3) $p^{\text{mc}(q)} \in T^q$.
- (4) For every $\gamma \in \text{supp}(q)$: $p^\gamma \setminus q^\gamma = \pi_{\text{mc}(q), \gamma}((p^{\text{mc}(q)} \setminus q^{\text{mc}(q)}) \upharpoonright \text{length}(p^{\text{mc}}) \setminus (i + 1))$, where $i \in \text{dom}(p^{\text{mc}(q)})$ is the largest such that $p^{\text{mc}(q)}(i)$ is not permitted for q^γ .
- (5) $\pi_{\text{mc}(p), \text{mc}(q)}$ maps T^p to a subtree of T^q .
- (6) For every $\gamma \in \text{supp}(q)$, for every $v \in \text{Suc}_{T^p}(p^{\text{mc}})$ if v is permitted for p^γ , then

$$\pi_{\text{mc}(p), \gamma}(v) = \pi_{\text{mc}(q), \gamma}(\pi_{\text{mc}(p), \text{mc}(q)}(v)). \tag{4.51}$$

The ordering $\leq_{\text{Prk}_E(\kappa, \lambda)}^* = \leq^*$ is defined as follows:

Definition 4.8. Let $p, q \in \text{Prk}_E(\kappa, \lambda)$. We say that p is a direct extension of q ($p \leq^* q$) if:

- (1) $p \leq q$.
- (2) For every $\gamma \in \text{supp}(q)$, $p^\gamma = q^\gamma$.

We state without a proof the following facts:

Fact 4.9. Assume the universe V satisfies the conditions set out in [Observation 4.5](#). Then the following hold of $\text{Prk}_E(\kappa, \lambda)$:

- (1) $\text{Prk}_E(\kappa, \lambda)$ is κ^{++} -cc.
- (2) \leq^* is κ -closed.
- (3) $(\text{Prk}_E(\kappa, \lambda), \leq, \leq^*)$ satisfies the Prikry property.
- (4) $\text{Prk}_E(\kappa, \lambda)$ preserves cardinals.
- (5) $\text{Prk}_E(\kappa, \lambda)$ does not add new bounded subsets of κ and the cofinality of κ is ω in the generic extension.
- (6) $\kappa^\omega = 2^\kappa = \lambda$ in the generic extension.

As regards (6), note that $\lambda \leq 2^\kappa$ is true by the way the forcing $\text{Prk}_E(\kappa, \lambda)$ is set up. The other direction, i.e., $2^\kappa \leq \lambda$ follows from the number of nice names for subsets of κ : by κ^{++} -cc of the forcing, this is λ^{κ^+} . Thus a direct way to ensure that $2^\kappa = \lambda$ is to have $\lambda^{\kappa^+} = \lambda$. In [Theorem 4.13](#) this is achieved by restricting F to be trivial at κ^+ (for κ a Mahlo cardinal). This restriction is also important to ensure that $\text{Prk}_E(\kappa, \lambda)$ behaves correctly over the relevant generic extensions (see [Theorem 4.13](#)).

4.3. Global singularization—Extender-based Prikry forcing

Definition 4.10. We say that an Easton function F is *mild* if it satisfies:

- (1) For all Mahlo cardinals α , α is closed under F , i.e., $\beta < \alpha \rightarrow F(\beta) < \alpha$.
- (2) For all Mahlo cardinals α , $F(\alpha^+) = \max(\alpha^{++}, F(\alpha))$.

Notice that (2) implies that if $F(\alpha) > \alpha^+$ for α a Mahlo cardinal, then the cofinality of $F(\alpha)$ is always greater or equal to α^{++} .

Remark 4.11. The condition that Mahlo cardinals are closure points of F is a technical one – it ensures that we have a control over the iteration points of the forcings \mathbb{P} and \mathbb{R} which we are going to use (see below). Roughly speaking, if $j : V \rightarrow M$ witnesses $F(\kappa)$ -hypermeasurability of κ , then we need to make sure that $j(\mathbb{P})$ contains no iteration points in the interval $(\kappa, F(\kappa)]$. This is automatic if $j(F)(\kappa) \geq F(\kappa)$, but it may fail when $j(F)(\kappa) = \kappa^+$. We resolve this problem by ignoring closure points of F which are not Mahlo cardinals (in fact Mahlo limits of Mahlo cardinals, see [Definition 4.15](#), and [Remark 4.16](#)). Also, κ is always a Mahlo cardinal in M (and a Mahlo limit of Mahlo cardinals), for $j : V \rightarrow M$ as above, and so it will be an iteration point of $j(\mathbb{P})$.

Let us define the following notation

$$\begin{aligned} \theta &= \{\kappa \mid F(\kappa) > \kappa^+ \wedge \kappa \text{ is } F(\kappa)\text{-hypermeasurable}\}, \\ \theta_E &= \{\kappa \in \theta \mid \text{there is a witnessing embedding } j : V \rightarrow M \text{ such that } j(F)(\kappa) = \kappa^+\}, \\ \theta_P &= \{\kappa \in \theta \setminus \theta_E \mid \text{there is a witnessing embedding } j_E : V \rightarrow M \text{ such that } j(F)(\kappa) \geq F(\kappa)\}. \end{aligned} \tag{4.52}$$

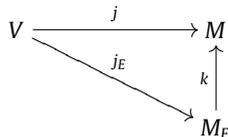
To motivate the notation, think of cardinals in θ_E as reserved for the extender-based Prikry forcing (“E”), and in θ_P as reserved for the simple Prikry forcing (“P”).

Note that we can w.l.o.g. use just extender embeddings as witnessing embeddings.

Fact 4.12. (GCH) If κ is $F(\kappa)$ -hypermeasurable, where F is an Easton function, and $j : V \rightarrow M$ is a witnessing embedding, then j can be factored through some $j_E : V \rightarrow M_E$ and $k : M_E \rightarrow M$ such that j_E is an extender embedding witnessing the $F(\kappa)$ -hypermeasurability of κ . Moreover,

- (1) If $j(F)(\kappa) \geq F(\kappa)$, then also $j_E(F)(\kappa) \geq F(\kappa)$.
- (2) If $j(F)(\kappa) = \kappa^+$, then also $j_E(F)(\kappa) = \kappa^+$.

Proof. Consider the following commutative triangle:



By the construction of the extender, it follows that k is the identity on $F(\kappa)$. The following holds: $k(j_E(F)(\kappa)) = k(j_E(F))(k(\kappa)) = k(j_E(F))(\kappa) = j(F)(\kappa)$.

Ad (1). If $\mu = j_E(F)(\kappa) < F(\kappa)$ were true, then k would be the identity at μ , implying that $j(F)(\kappa) = \mu$, which is a contradiction.

Ad (2). Similarly, if $j(F)(\kappa) = \kappa^+$, then $j_E(F)(\kappa)$ must be κ^+ so that $k(j_E(F)(\kappa))$ is κ^+ . (Fact 4.12) \square

We will prove the following theorem:

Theorem 4.13. Assume GCH and let F be a mild Easton function. Then there is a cardinal-preserving forcing extension V^* of V such that F is realised in V^* on all V -regular cardinals and moreover all elements of $\theta_E \cup \theta_P$ as in (4.52) are strong limit singular cardinals with cofinality ω (and hence fail SCH).

Remark 4.14. As a condition of non-triviality, we will assume that the elements in $\theta_p \cup \theta_E$ are unbounded in the cardinal numbers. If they are bounded the construction below can be straightforwardly localized.

The proof will occupy the rest of the paper. Our forcing will be a two-stage iteration $\mathbb{P} * \mathbb{R}$ where \mathbb{P} is cofinality-preserving and realises F on all regular cardinals except θ_E (and successors of cardinals in θ_E) and preserves sufficient largeness of all cardinals in $\theta_p \cup \theta_E$. The forcing \mathbb{R} will be an Easton-supported iteration of a combination of the simple Prikry forcing and the extender-based Prikry forcing which will complete the realisation of F and simultaneously cofinalise cardinals in $\theta_p \cup \theta_E$. Note that the Easton support of \mathbb{R} in combination with the simple Prikry forcing will require the technique in Proof 2 of 3.8.

We will define \mathbb{P} as follows:

Definition 4.15. Let F be a mild Easton function and $\langle i_\alpha \mid \alpha < \text{On} \rangle$ be an increasing enumeration of Mahlo limits of Mahlo cardinals in V .

We will define an iteration $\mathbb{P} = \langle \mathbb{P}_{i_\alpha} \mid \alpha < \text{On} \rangle, \langle \dot{Q}_{i_\alpha} \mid \alpha < \text{On} \rangle$ indexed by $\langle i_\alpha \mid \alpha < \text{On} \rangle$ such that:

- If i_α is an element of θ_E or

$$F(i_\alpha) = i_\alpha^+, \tag{4.53}$$

then

$$\mathbb{P}_{i_{\alpha+1}} = \mathbb{P}_{i_\alpha} * \dot{Q}_{i_\alpha}, \tag{4.54}$$

where \dot{Q}_{i_α} is a name for $\prod_{i_\alpha^+ < \lambda < i_{\alpha+1}} \text{Add}(\lambda, F(\lambda))$ (λ ranges over regular cardinals and the product has the Easton support). Note that the forcing is empty at i_α and i_α^+ .

- If i_α is not in θ_E and $F(i_\alpha) > i_\alpha^+$, then

$$\mathbb{P}_{i_{\alpha+1}} = \mathbb{P}_{i_\alpha} * \dot{Q}_{i_\alpha}, \tag{4.55}$$

where \dot{Q}_{i_α} is a name for $\text{Sacks}(i_\alpha, F(i_\alpha)) \times \prod_{i_\alpha^+ < \lambda < i_{\alpha+1}} \text{Add}(\lambda, F(\lambda))$ (λ ranges over regular cardinals and the product has the Easton support). Recall that $\text{Sacks}(i_\alpha, F(i_\alpha))$ denotes the product of the Sacks forcing at i_α with support of size $\leq i_\alpha$. Note that the forcing is empty at i_α^+ .

- If γ is a limit ordinal, then \mathbb{P}_{i_γ} is an inverse limit unless i_γ is a regular cardinal, in which case \mathbb{P}_{i_γ} is a direct limit (the usual Easton support).

Remark 4.16. The iteration defined above uses as iteration points Mahlo limits of Mahlo cardinals. If we defined the iteration points to be just Mahlo cardinals, it would cause some technical complications in the argument ($F(\kappa)$ might be an iteration point of $j(\mathbb{P})$, see proof of item (2) in Lemma 4.17 for more comments).

Lemma 4.17. Assuming GCH, \mathbb{P} preserves all cofinalities, realises F on all regular cardinals except elements of θ_E and successors of elements in θ_E and moreover:

- (1) If $\kappa \in \theta_p$, then κ remains measurable in $V^\mathbb{P}$.
- (2) If $\kappa \in \theta_E$, then there is a nice system for $(\kappa, F(\kappa))$ in $V^\mathbb{P}$.
- (3) SCH holds in $V^\mathbb{P}$.

Proof. Preservation of cofinalities and realisation of F follows from [7] (by definition of mildness, F is realised at the successor of a Mahlo limit κ of Mahlo cardinals if it is realised at κ).

Let G be \mathbb{P} -generic.

Ad (1).

Let $j : V \rightarrow M$ be an embedding witnessing that κ is in θ_p , in particular $j(F)(\kappa) \geq F(\kappa)$, which implies $j(F)(\kappa) > \kappa^+$. κ is a Mahlo limit of Mahlo cardinals both in V and M , and so by definition of \mathbb{P} , $j(\mathbb{P})$ at κ is $\text{Sacks}(\kappa, j(F)(\kappa))$ of $M^{j(\mathbb{P})^\kappa}$ unless κ is in $(\theta_E)^M$. The forcing at stage κ^+ is empty both on the V side and M side. Moreover, the next Mahlo limit of Mahlo cardinals in M is certainly bigger than $j(F)(\kappa)$ (by definition of the forcing, $j(F)(\kappa)$ is less than the next Mahlo cardinal in M above κ), and so the setup of (1) is thus identical to the situation in [7] once we show that κ is not in $(\theta_E)^M$.

Assume for contradiction that κ is in $(\theta_E)^M$, that is κ is $j(F)(\kappa)$ -hypermeasurable in M and there is a witnessing embedding $i : M \rightarrow N$ such that $i(j(F)(\kappa)) = \kappa^+$. We will argue that $i \circ j : V \rightarrow N$ witnesses that κ is in θ_E , thus reaching a contradiction. The composition $i \circ j$ is clearly an elementary embedding with critical point κ and $i(j(\kappa)) > F(\kappa)$. It remains to show that N contains $H(F(\kappa))^V$. But this follows from the fact that $j(F)(\kappa) \geq F(\kappa)$ and the assumption that κ is $j(F)(\kappa)$ -hypermeasurable in M : N contains $H(j(F)(\kappa))^M \supseteq H(F(\kappa))^M$ and hence also $H(F(\kappa))^V$.

The rest of the proof of (1) is identical to [7] (see also the proof of (2) for some details).

Ad (2).

We want to show that κ retains sufficient “largeness” for a reasonable definition of $\text{Prk}_E(\kappa, F(\kappa))$ in $V[G]$. We show that if we lift $j : V \rightarrow M$ witnessing $F(\kappa)$ -hypermeasurability to j^* using the argument [7] then j^* will satisfy all properties identified in Observation 4.5. This will imply that there exists a nice system for $(\kappa, F(\kappa))$ in the generic extension by G .

We clearly have that $2^\kappa = \kappa^+$ in $V[G]$ since \mathbb{P} is empty (trivial) at elements of θ_E (and κ is an iteration point of \mathbb{P}).

Assuming the knowledge of the lifting argument in [7], the following points ensure that j lifts to some j^* :

- \mathbb{P} is empty at κ because $\kappa \in \theta_E$, and $j(\mathbb{P})$ likewise is empty at κ because $j(F)(\kappa) = \kappa^+$. It follows that the lifting argument can avoid the otherwise hard stage of lifting the generic at κ .
- Both \mathbb{P} and $j(\mathbb{P})$ are also empty at κ^+ .
- The next iteration point of \mathbb{P} and $j(\mathbb{P})$ above κ is strictly greater than $F(\kappa)$. Note: In M , the next Mahlo limit of Mahlo cardinals after κ must be strictly greater than $F(\kappa)$ because no new Mahlo cardinals can appear in the interval $(\kappa, F(\kappa))$ in M . However, $F(\kappa)$ can become a Mahlo cardinal in M – but not a Mahlo limit of Mahlo cardinals, see [7], Observation 2.8.
- The work required to show that j lifts is thus limited to arguing that we can find a generic for the interval $[\kappa^{++}, F(\kappa)]$ on the M -side. Here we proceed exactly as in [7].

Let $j^* : V[G] \rightarrow M[j^*(G)]$ be the lifting of j .

It is clear that this j^* witnesses the first two conditions in [Observation 4.5](#) from the following list (identifying V with our $V[G]$ and M with $M[j^*(G)]$), j^* with j , and $F(\kappa)$ with λ):

- (1) $M[j^*(G)]$ is closed under κ -sequences in $V[G]$.
- (2) $j^*(\kappa) > F(\kappa)$.
- (3) $([F(\kappa)]^{\leq \kappa^+})^{V[G]} \subseteq M[j^*(G)]$.
- (4) $|[F(\kappa)]^{\leq \kappa^+}| = F(\kappa)$ in $M[j^*(G)]$ (and hence also in $V[G]$).
- (5) For some $f_{F(\kappa)} : \kappa \rightarrow \kappa$, $j^*(f_{F(\kappa)})(\kappa) = F(\kappa)$.

Condition (5) can by [Remark 1.2](#) be assumed for the original j , so it holds for our j^* as well.

It remains to verify conditions (3) and (4).

To verify (3), let X be a subset of $F(\kappa)$ of size at most κ^+ in $V[G]$. By the definition of F , the cofinality of $F(\kappa)$ is at least κ^{++} and so there is some $\alpha < F(\kappa)$ such that $X \subseteq \alpha$. Since \mathbb{P} is trivial at κ and κ^+ , X must be added by \mathbb{P}_κ . But now (3) follows since if \dot{X} is a nice \mathbb{P}_κ -name for $X \subseteq \alpha$, then $\dot{X} \in H(F(\kappa))$, and so $\dot{X} \in M$; as also $\mathbb{P}_\kappa = j(\mathbb{P})_\kappa$ and G_κ is M -generic, we obtain that $\dot{X}^{G_\kappa} = X \in M[j^*(G)]$.

Now also (4) follows easily: we argued in the proof of (3) that all subsets of $F(\kappa)$ of size at most κ^+ have nice names which are included in $H(F(\kappa))$; this is also true in M and since M satisfies GCH, $H(F(\kappa))$ has size $F(\kappa)$ in M , and so the number of nice $j(\mathbb{P})_\kappa = \mathbb{P}_\kappa$ -names is $F(\kappa)$. This implies (4) as desired.

Ad (3). SCH holds in $V^\mathbb{P}$ by standard arguments concerning Easton-supported iteration defined on regular cardinals over a model satisfying GCH. (Lemma 4.17) \square

We now define the iteration \mathbb{R} which will be an Easton-supported iteration defined on $\theta_p \cup \theta_E$ combining the simple Prikry forcing and the extender-based Prikry forcing. The definition is in the same form as in [Definition 3.3](#).

Definition 4.18. For every $\alpha < \text{On}$ let \mathbb{R}_α be a set of all elements p of the form $\langle \dot{p}_\gamma \mid \gamma \in g \rangle$, where

- (1) $g \subseteq \alpha$;
- (2) g has the Easton support, i.e., for every inaccessible $\beta \leq \alpha$, $\beta > |g \cap \beta|$, provided that for every $\gamma < \beta$, $|\mathbb{R}_\gamma| < \beta$;
- (3) for every $\gamma \in g$,

$$p \restriction \gamma = \langle \dot{p}_\beta \mid \beta \in g \cap \gamma \rangle \in \mathbb{R}_\gamma \tag{4.56}$$

and $p \restriction \gamma \Vdash \dot{p}_\gamma$ is a condition in \dot{R}_γ ,” where \dot{R}_γ is

- (a) $\text{Prk}(\gamma)$ if $\gamma \in \theta_p$ and \mathbb{R}_γ forces that γ is measurable, or
- (b) $\text{Prk}_E(\gamma, F(\gamma))$ if $\gamma \in \theta_E$ and \mathbb{R}_γ forces that the conditions in [Observation 4.5](#) hold, or
- (c) A trivial forcing otherwise.

Let $p = \langle \dot{p}_\gamma \mid \gamma \in g \rangle$ and $q = \langle \dot{q}_\gamma \mid \gamma \in f \rangle$ be elements of \mathbb{R} . Then p is stronger than q , $p \leq q$, iff

- (1) $g \supseteq f$;
 - (2) for every $\gamma \in f$,
- $$p \restriction \gamma \Vdash \dot{p}_\gamma \leq \dot{q}_\gamma \text{ in } \dot{R}_\gamma; \tag{4.57}$$

- (3) there exists a finite subset $b \subseteq f$ so that for every $\gamma \in f \setminus b$,

$$p \restriction \gamma \Vdash \dot{p}_\gamma \leq^* \dot{q}_\gamma \text{ in } \dot{R}_\gamma. \tag{4.58}$$

If the set in item (3) is empty, then we call p a direct extension of q and denote it as $p \leq^* q$.

By general results in [10], the forcing \mathbb{R} is itself a Prikry-type forcing.

Lemma 4.19. \mathbb{R} applied over $V^\mathbb{P}$ preserves all cardinals and axioms of ZFC.

Proof. The proof is essentially the same as in [Lemma 3.4\(2\)](#). (Lemma 4.19) \square

It remains to verify the hardest part, that is that the forcing \mathbb{R} is defined on all elements of $\theta_p \cup \theta_E$ as desired. We will deal separately with θ_p and θ_E .

Lemma 4.20. If $\kappa \in \theta_p$, then \mathbb{R}_κ forces over $V^\mathbb{P}$ that κ remains measurable.

Proof. If non-trivial stages of \mathbb{R}_κ are bounded below κ , then κ obviously remains measurable.

So assume that \mathbb{R}_κ is unbounded in κ . The proof follows the argument given as Proof 2 of Theorem 3.8. We will review Proof 2 and indicate how it applies to our present case.

Recall that we can fix an extender embedding $j^* : V[G] \rightarrow M[j^*(G)] = M^*$ with critical point κ which is a lift of some extender embedding $j : V \rightarrow M$ witnessing $F(\kappa)$ -hypermeasurability of κ in V . We also have that $j(F)(\kappa) \geq F(\kappa)$. See the beginning of Proof 2 on page 17 for more details.

Definition 3.11, Fact 3.12 and Lemma 3.13 apply as they stand since they only depend on general properties of Easton-supported Prikry-type iterations.

Let H_κ be a generic filter for \mathbb{R}_κ . H_κ is also a generic filter for $j^*(\mathbb{R})_\kappa$ over M^* . Notice that the next stage of iteration \dot{R}_κ after $j^*(\mathbb{R})_\kappa$ is either the Prikry forcing $\text{Prk}(\kappa)$ of $M^*[H_\kappa]$, or the empty forcing. This follows from Definition 4.18 of \mathbb{R} : a necessary condition for \dot{R}_γ to be $\text{Prk}(\gamma)$ is that GCH fails at γ ($\gamma \in \theta_p$ over $V[G]$ implies that $2^\gamma = F(\gamma) > \gamma^+$); similarly, a necessary condition for \dot{R}_γ to be $\text{Prk}_E(\gamma, F(\gamma))$ is that GCH holds at γ (see the conditions in Observation 4.5). However, $j(F)(\kappa) \geq F(\kappa) > \kappa^+$ implies that GCH fails in M^* at κ , hence $j^*(\mathbb{R})$ cannot be the extender-based Prikry forcing at κ . W.l.o.g. we will assume that \dot{R}_κ is the Prikry forcing $\text{Prk}(\kappa)$ of $M^*[H_\kappa]$.

As in Proof 2, we have that $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_{\kappa+1}$ is $F(\kappa)^+ \text{-}\leq^*$ -closed.

The key Lemma 3.14 also applies as it stands since it only depends on the general properties of the iteration \mathbb{R} . Finally, the rest of the argument is also valid in the present case, as can be easily checked. (Lemma 4.20) \square

We now turn to the elements of θ_E .

Lemma 4.21. *If $\kappa \in \theta_E$, then \mathbb{R}_κ forces over $V^{\mathbb{P}}$ that there is a nice system for $(\kappa, F(\kappa))$, and so $\text{Prk}_E(\kappa, F(\kappa))$ can be correctly defined.*

Proof. Let $j^* : V[G] \rightarrow M[j^*(G)]$ be an embedding as in Lemma 4.17, item (2).

If non-trivial stages of \mathbb{R}_κ are bounded below κ , then j^* lifts easily and there is a nice system for $(\kappa, F(\kappa))$.

So assume that \mathbb{R}_κ is unbounded in κ . Let H_κ be a \mathbb{R}_κ -generic over $V[G]$, where G is \mathbb{P} -generic. Set $\lambda = F(\kappa)$ and $M^* = M[j^*(G)]$; recall that the cofinality of λ is strictly greater than κ^+ .

We will verify that we can construct in $V[G * H_\kappa]$ an embedding $k : V[G * H_\kappa] \rightarrow N$ for some N such that all conditions in Observation 4.5 are satisfied. It will not in general be the case that k extends $j^* : V[G] \rightarrow M^*$. The technique used will be a generalization of the argument in [18].

In order to construct an embedding k we will define a sequence of measures $\langle E_\alpha^+ \mid \alpha < \lambda \rangle$ and show that the direct limit N of the extender sequence is well founded and the resulting $k : V[G * H_\kappa] \rightarrow N$ satisfies the required conditions.

We know that $2^\kappa = \kappa^+$ in $V[G]$, and by Lemma 4.17 we also know that $j^* : V[G] \rightarrow M^*$ is an extender embedding lifting the original embedding, and that the conditions (1)–(5) on page 31 hold. In particular $j^*(F)(\kappa) = \kappa^+$ and so the forcing $j^*(\mathbb{R})$ is empty at the interval $[\kappa, \lambda]$ (recall that λ is smaller than the first Mahlo limit of Mahlo cardinals above κ and so is certainly smaller than the first measurable cardinal above κ).

The iteration $j^*(\mathbb{R}_\kappa)$ factors into $j^*(\mathbb{R})_\kappa = \mathbb{R}_\kappa$ and the tail forcing $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$. It follows that the tail forcing $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$ is $\lambda^+ \text{-}\leq^*$ -closed in $M^*[H_\kappa]$ (i.e., the direct extension relation is $\lambda^+ \text{-}\leq^*$ -closed). Thus we can hit all \leq^* -dense open sets in $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$ in $M^*[H_\kappa]$ in κ^+ -many steps using the following standard construction: If f is a function $f : \kappa \rightarrow H(\kappa^+)$ with its range containing just \leq^* -dense open sets of \mathbb{R}_κ , then $j(f) \upharpoonright \lambda$ determines in $M^*[H_\kappa]$ a family $\{D_\alpha \mid \alpha < \lambda\}$ of λ -many \leq^* -dense open sets in $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$. By closure of the \leq^* -ordering, the intersection $\bigcap_{\alpha < \lambda} D_\alpha$ is \leq^* -dense. Since there are only κ^+ -many functions from $\kappa \rightarrow H(\kappa^+)$, by $2^\kappa = \kappa^+$ in $V[G]$, we can build in $V[G * H_\kappa]$ a \leq^* -decreasing sequence of conditions

$$\langle r_\alpha \mid \alpha < \kappa^+ \rangle \tag{4.59}$$

in $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$ hitting all \leq^* -dense open sets in $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$.

Let a sequence as in (4.59) be fixed.

Definition of an extender. For every $\alpha < \lambda$ define in $V[G * H_\kappa]$ a measure E_α^+ as follows: if \dot{X} is a \mathbb{R}_κ -name for a subset of κ , then

$$X \in E_\alpha^+ \text{ iff } \exists p \in H_\kappa, \exists \gamma \text{ s.t. } p \hat{\cap} r_\gamma \Vdash \alpha \in j^*(\dot{X}). \tag{4.60}$$

By compatibility of all r_γ 's it is routine to verify that each E_α^+ is a κ -complete measure and moreover each E_α^+ extends the original measure $E_\alpha^* \in V[G]$ obtained from j^* by setting for $X \in V[G]$: $X \in E_\alpha^*$ iff $\alpha \in j^*(X)$.

We now argue that \leq_E as in Definition 4.4 existing in $V[G]$ determines in $V[G * H_\kappa]$ a κ^{++} -directed system of commutative embeddings between the ultrapowers $N_\alpha = \text{Ult}(V[G * H_\kappa], E_\alpha^+)$ for $\alpha < \lambda$. For $\alpha \leq_E \beta$ let $\pi_{\alpha,\beta}$ denote the function from $\kappa \rightarrow \kappa$ witnessing in $V[G]$ $\alpha \leq_E \beta$, in particular $j^*(\pi_{\alpha,\beta})(\beta) = \alpha$. We now argue that

$$\bar{\pi}_{\alpha,\beta} : N_\alpha \rightarrow N_\beta \tag{4.61}$$

is elementary, where for an equivalence class $[f]_{N_\alpha}$ in N_α we define

$$\bar{\pi}_{\alpha,\beta}([f]_{N_\alpha}) = [f \circ \pi_{\alpha,\beta}]_{N_\beta}. \tag{4.62}$$

In order to verify the elementarity of $\bar{\pi}_{\alpha,\beta}$, first notice that for every condition r in $j^*(\mathbb{R}_\kappa)$ which forces that $j^*(\dot{f})$ is a function from $j^*(\kappa)$ to $j^*(\kappa)$ and every φ we have due to $1_{j^*(\mathbb{R}_\kappa)} \Vdash j^*(\pi_{\alpha,\beta})(\beta) = \alpha$, the following

$$r \Vdash \varphi(j^*(\dot{f})(\alpha)) \leftrightarrow r \Vdash \varphi(j^*(\dot{f}) \circ j^*(\pi_{\alpha,\beta})(\beta)) \leftrightarrow r \Vdash \varphi(j^*(\dot{f}) \circ \pi_{\alpha,\beta})(\beta). \tag{4.63}$$

Now the elementarity follows from the equivalence (4.64) which determines the satisfaction relation in the ultrapowers N_α for $\alpha < \lambda$:

$$\{\xi < \kappa \mid V[G * H_\kappa] \models \varphi(f(\xi), \dots)\} \in E_\alpha^+ \text{ iff } V[G * H_\kappa] \models \exists p \in H_\kappa, \exists \gamma < \kappa^+ \Vdash \varphi(j^*(\dot{f})(\alpha), \dots), \quad (4.64)$$

where \dot{f} is a \mathbb{R}_κ -name for f which can be taken to be forced by $1_{\mathbb{R}_\kappa}$ to be a function with domain κ . In order to prove (4.64), we define a concrete \mathbb{R}_κ -name for the set $\{\xi < \kappa \mid V[G * H_\kappa] \models \varphi(f(\xi), \dots)\}$ by setting $\dot{X} = \{\langle \alpha, p \rangle \mid \alpha < \kappa, p \Vdash \varphi(\dot{f}(\alpha), \dots)\}$. By elementarity of j^* , $j^*(\dot{X}) = \{\langle \alpha, p \rangle \mid \alpha < j^*(\kappa), p \Vdash \varphi(j^*(\dot{f})(\alpha), \dots)\}$. The equivalence (4.64) follows from (4.65): For all $q \in j^*(\mathbb{R}_\kappa)$:

$$q \Vdash \alpha \in j^*(\dot{X}) \text{ iff } q \Vdash \varphi(j^*(\dot{f})(\alpha), \dots). \quad (4.65)$$

Hence we can conclude that $\pi_{\alpha, \beta}$, where $\alpha \leq_E \beta$, determine a commutative system of embeddings (they are commutative in $V[G]$, and so are in $V[G * H_\kappa]$).

The κ^{++} -directed closure of \leq_E in $V[G * H_\kappa]$ follows by the following argument: Every subset $X \subseteq \lambda$ in $V[G * H_\kappa]$ of size κ^+ is included in some $X' \supseteq X$ in $V[G]$ such that X' has size κ^+ in $V[G]$. This is true by κ -cc of \mathbb{R}_κ . It follows that if α is a common upper bound of ordinals in X' , then α is a common upper bound of ordinals in X .

Since the commutative system is sufficiently closed (ω_1 -closure would suffice here), the direct limit N of the ultrapowers $\langle N_\alpha \mid \alpha < \lambda \rangle$ must be well founded (otherwise we could find an ill-founded epsilon chain in one of the measure ultrapowers). Let $k : V[G * H_\kappa] \rightarrow N$ be the corresponding embedding, and $N = \{k(f)(\alpha) \mid f \in V[G * H_\kappa], f : \kappa \rightarrow V[G * H_\kappa], \alpha < \lambda\}$. By (4.63) and (4.64), it follows that the embeddings j^* and k are connected in the following way:

$$N \models \varphi(k(f)(\alpha), k(g)(\beta), \dots) \text{ iff } \exists p \in H_\kappa, \exists \gamma \in \kappa^+, p \hat{\sim} r_\gamma \Vdash \varphi(j^*(\dot{f})(\alpha), j^*(\dot{g})(\beta), \dots), \quad (4.66)$$

where \dot{f} and \dot{g} are \mathbb{R}_κ -names for f and g , respectively.

We will verify that $k : V[G * H_\kappa] \rightarrow N$ satisfies conditions in **Observation 4.5**. It is easy to check that \mathbb{R}_κ forces that $2^\kappa = \kappa^+$, and so 2^κ is still equal to κ^+ in $V[G * H_\kappa]$. We now check the remaining items.

Ad (1) Closure of N under κ -sequences. This follows from the κ^{++} -directed closure of \leq_E .

Ad (2) $k(\kappa) > \lambda$. First recall that the measures in the sequence $\langle E_\alpha^+ \mid \alpha < \lambda \rangle$ extend the measures in the original sequence $\langle E_\alpha \mid \alpha < \lambda \rangle$ derived from $j : V \rightarrow M$. If $\{j(f_\xi)(\alpha_\xi) \mid f_\xi \in V \cap {}^\kappa V, \alpha_\xi < \lambda\}$ is the set of increasing ordinals such that $\{j(f_\xi)(\alpha_\xi) \mid f_\xi \in V \cap {}^\kappa V, \alpha_\xi < \lambda\} = j(\kappa)$ in M , then $\{k(f_\xi)(\alpha_\xi) \mid f_\xi \in V \cap {}^\kappa V, \alpha_\xi < \lambda\}$ is an increasing subset of ordinals in $k(\kappa)$. It follows that $\lambda < j(\kappa) \leq k(\kappa)$.

Ad (5). We will show that $k(f_\lambda)(\kappa)$ is the supremum of ordinals less than λ , that is $k(f_\lambda)(\kappa) = \lambda$. This will enable us to refer to λ in N . Recall that $j(f_\lambda)(\kappa) = j^*(f_\lambda)(\kappa) = \lambda$. Clearly, $1_{j^*(\mathbb{R}_\kappa)} \Vdash \alpha < j^*(f_\lambda)(\kappa)$ for every $\alpha < \lambda$, and hence $\lambda \leq k(f_\lambda)(\kappa)$.

For the other direction, it suffices to show that if for some $p \in H_\kappa$ and $\alpha < \kappa^+$ and $\beta < \lambda$

$$p \hat{\sim} r_\alpha \Vdash j^*(\dot{g})(\beta) < j^*(f_\lambda)(\kappa), \text{ which is equivalent to } N \models k(g)(\beta) < k(f_\lambda)(\kappa), \quad (4.67)$$

then for some $\gamma < \lambda$, $\bar{\alpha} < \kappa^+$ and $\bar{p} \in H_\kappa$ we have

$$\bar{p} \hat{\sim} r_{\bar{\alpha}} \Vdash j^*(\dot{g})(\beta) = \gamma, \text{ which is equivalent to } N \models \gamma = k(g)(\beta). \quad (4.68)$$

But this is implied by the λ^+ -weak closure of the forcing above κ :

$$p \Vdash (\exists \gamma) r_{\bar{\alpha}} \Vdash j^*(\dot{g})(\alpha) = \gamma, \quad (4.69)$$

and hence there is some $\bar{p} \Vdash p$ and γ such that (4.68).

Ad (4). This follows from **Ad (5)** since

$$1_{j^*(\mathbb{R}_\kappa)} \Vdash (j^*(f_\lambda)(\kappa))^{\kappa^+} = j^*(f_\lambda)(\kappa), \quad (4.70)$$

which implies

$$N \models (k(f_\lambda)(\kappa))^{\kappa^+} = k(f_\lambda)(\kappa). \quad (4.71)$$

Ad (3) N contains all subsets of λ of size $\leq \kappa^+$ in $V[G * H_\kappa]$. First note that all ordinals $\alpha < \lambda$ can be represented in N using the identity function $\text{id} : \kappa \rightarrow \kappa$ by

$$\alpha = k(\text{id})(\alpha). \quad (4.72)$$

Moreover, the representation is the same in M^* and N : for every $\alpha < \lambda$

$$\alpha = k(\text{id})(\alpha) = j^*(\text{id})(\alpha). \quad (4.73)$$

We will use this representation to argue first that $[\lambda]^{\leq \kappa^+}$ of $V[G]$ is in N .

N contains $[\lambda]^{\leq \kappa^+}$ of $V[G]$.

Let $h : \kappa \rightarrow H(\kappa)$ be a function which is defined in $V[G]$ as follows: whenever $\xi < \kappa$ is an inaccessible cardinal, let $\mu_\xi < \kappa$ be the size of $[f_\lambda(\xi)]^{\leq \xi^+}$ in $V[G]$. Define h so that for every inaccessible $\xi < \kappa$, $h \upharpoonright \mu_\xi$ enumerates $[f_\lambda(\xi)]^{\leq \xi^+}$ in $V[G]$. Recall that $j^*(f_\lambda)(\kappa) = \lambda$, and that in M^* the size of $[\lambda]^{\leq \kappa^+}$ equals λ . Since $[\lambda]^{\leq \kappa^+}$ of M^* has size λ in M^* , it follows that

$$1_{j^*(\mathbb{R}_\kappa)} \Vdash |([\lambda]^{\leq \kappa^+})^{M^*}| = \lambda. \quad (4.74)$$

This implies that

$$1_{j^*(\mathbb{R}_\kappa)} \Vdash j^*(h) \upharpoonright \lambda \text{ enumerates } [\lambda]^{\leq \kappa^+} \text{ of } M^*. \tag{4.75}$$

As $[\lambda]^{\leq \kappa^+}$ of M^* equals $[\lambda]^{\leq \kappa^+}$ of $V[G]$ this gives:

$$1_{j^*(\mathbb{R}_\kappa)} \Vdash j^*(h) \upharpoonright \lambda \text{ enumerates } [\lambda]^{\leq \kappa^+} \text{ of } V[G]. \tag{4.76}$$

Note that $[\lambda]^{\leq \kappa^+}$ of $V[G]$ thus equals to the set $\{j^*(h)(\alpha) \mid \alpha < \lambda\}$, and so it suffices to show that for every $\alpha < \lambda$,

$$k(h)(\alpha) = j^*(h)(\alpha). \tag{4.77}$$

If $j^*(g)(\beta)$, for $g \in V[G]$, is in $j^*(h)(\alpha)$ then because the extender sequence $\langle E_\alpha^+ \mid \alpha < \lambda \rangle$ extends the sequence $\langle E_\alpha^* \mid \alpha < \lambda \rangle$ derived from $j^* : V[G] \rightarrow M^*$ (and $\langle E_\alpha^* \mid \alpha < \lambda \rangle$ itself extends the original sequence $\langle E_\alpha \mid \alpha < \lambda \rangle$ derived from $j : V \rightarrow M$), this shows that $j^*(g)(\beta)$ is in $k(h)(\alpha)$. The converse direction is shown as follows. Let $k(g)(\beta)$ in $k(h)(\alpha)$ be given. Note that $k(g)(\beta)$, where $g \in V[G * H_\kappa]$, is a name for an ordinal $< \lambda$ (this is because $k(h)(\alpha)$ is a subset of λ by Ad(5)). By (4.66), $k(g)(\beta) \in k(h)(\alpha)$ is equivalent to

$$\exists p \in G, \exists \xi < \kappa^+ \text{ such that } p \hat{\wedge} r_\xi \Vdash j^*(\dot{g})(\beta) \in j^*(h)(\alpha). \tag{4.78}$$

By (4.75), it holds that

$$p \hat{\wedge} r_\xi \Vdash j^*(\dot{g})(\beta) < \lambda. \tag{4.79}$$

Since the direct extension relation \leq^* of the tail forcing $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$ is λ^+ -closed, one of the r_ξ 's $\xi < \kappa^+$ the value of $j^*(\dot{g})(\beta)$. Hence there $\bar{\xi} < \kappa^+$, $\eta < \lambda$ and $q \in H_\kappa$ such that

$$q \hat{\wedge} r_{\bar{\xi}} \Vdash \eta = j^*(\text{id})(\eta) = j^*(\dot{g})(\beta), \tag{4.80}$$

which by (4.66) implies that $k(g)(\beta) = k(\text{id})(\eta) = \eta$. However, as the identity function id is now in $V[G]$ and not $V[G * H_\kappa]$, we can also conclude from (4.78) that

$$j^*(\text{id})(\eta) \in j^*(h)(\alpha), \tag{4.81}$$

which shows (4.77).

We will now show that H_κ is in N . This will be enough to conclude that $[\lambda]^{\leq \kappa^+}$ of $V[G * H_\kappa]$ is in N .

N contains $[\lambda]^{\leq \kappa^+}$ of $V[G * H_\kappa]$.

First notice that all elements x of $[\lambda]^{\leq \kappa^+}$ in $V[G * H_\kappa]$ have a \mathbb{R}_κ -name \dot{x} which is in $[\lambda]^{\leq \kappa^+}$ of $V[G]$. Such \dot{x} is thus included in N . Existence of such \dot{x} is straightforward by the fact that it is an element of κ^+ ($\mathbb{R}_\kappa \times \lambda$) of $V[G]$, which can be identified with $[\lambda]^{\leq \kappa^+}$ of $V[G]$. Since $x = (\dot{x})^{H_\kappa}$, it suffice to show that H_κ is in N .

We will argue that there is some $g \in V[G * H_\kappa]$ such that $H_\kappa \subseteq k(g)(\kappa)$, where $k(g)(\kappa)$ is a filter (this implies the desired claim). Let g be defined by $g(\alpha) = H_\kappa \cap \mathbb{R}_\alpha$. Let $p \in H_\kappa$ be fixed. Clearly there is some α such that

$$p \hat{\wedge} r_\alpha \Vdash p \in j^*(\dot{g})(\kappa). \tag{4.82}$$

Since p is in $[\lambda]^{\leq \kappa^+}$, we can write by (4.77), $p = j^*(h)(\alpha) = k(h)(\alpha)$ for some $\alpha < \lambda$, and so

$$p \hat{\wedge} r_\alpha \Vdash j^*(h)(\alpha) \in j^*(\dot{g})(\kappa), \tag{4.83}$$

which by (4.66) proves the desired claim $p \in k(g)(\kappa)$. (Lemma 4.21) \square

Remark 4.22. One might alternatively argue that the system of measures $\langle E_\alpha^+ \mid \alpha < \lambda \rangle$ satisfies the properties of the nice system in Definition 4.4. For instance the property (6) follows by the following argument: E_κ^+ is normal because the conditions are of the form $p \hat{\wedge} r_\alpha$ where p comes from a generic and hence decide everything compatibly, and r_α 's compatibly capture all weakly-closed dense sets in $j^*(\mathbb{R}_\kappa) \setminus \mathbb{R}_\kappa$. Hence if $p \hat{\wedge} r_\alpha \Vdash j^*(\dot{g})(\kappa) < j^*(\text{id})(\kappa) = \kappa$, then we can decide the value of $j^*(\dot{g})(\kappa)$ to be some $\beta < \kappa$. It follows that $[\text{id}]_{E_\kappa^+}$ is κ . We have considered interesting, however, to show the stronger result that the we can obtain an embedding k which will witness the existence of a nice system.

Lemma 4.23. F is realised in $V^{\mathbb{P} * \mathbb{R}}$ on all V -regular cardinals, and all elements in $\theta_p \cup \theta_E$ are singular strong limit cardinals with cofinality ω .

Proof. By Lemmas 4.20 and 4.21, the forcing \mathbb{R} is defined on all elements in $\theta_p \cup \theta_E$, and hence all elements in $\theta_p \cup \theta_E$ are strong limit singular cardinals of cofinality ω . Recall that \mathbb{P} realises F on all regular cardinals except on elements in θ_E (and their successors). It remains to check that \mathbb{R} preserves the continuum function on cardinals where it is already correct, and realises F on elements in θ_E (and their successors). The forcing $\text{Prk}_E(\kappa, F(\kappa))$ on κ in θ_E (and the successor of κ) by its definition (and the properties of κ in $V[G * H_\kappa]$). The fact that \mathbb{R} does not change the continuum function on cardinals outside θ_E follows from the factoring property of \mathbb{R} : for every ordinal γ , \mathbb{R} is equal to $\mathbb{R}_\gamma * \mathbb{R} \setminus \mathbb{R}_\gamma$, where $\mathbb{R} \setminus \mathbb{R}_\gamma$ does not add new bounded subsets of γ . (Lemma 4.23) \square

This ends the proof of Theorem 4.13.

(Theorem 4.13) \square

We now turn to a special kind Easton function which we can show a stronger result.

Definition 4.24. We say that an Easton function F is *toggle-like* if for all regular cardinals α ,

$$F(\alpha) = \alpha^+ \text{ or } \alpha^{++} \quad (4.84)$$

and for all Mahlo cardinals κ

$$F(\kappa^+) = \kappa^{++} \quad (4.85)$$

Intuitively, F “toggles” GCH on and off, and is trivial at the successors of Mahlo cardinals.

For the purposes of this article, we say that κ has the reflection property if the value of 2^κ depends on the values of 2^α for $\alpha < \kappa$. The following Corollary shows that singular limit cardinals of cofinality ω have virtually no global reflection properties if we formulate these in terms of failure or truth of GCH.

Corollary 4.25. *Let Σ be any subclass of cardinals κ such that κ is κ^{++} -hypermeasurable. Assume further that F is toggle-like and $F(\kappa) = \kappa^{++}$ for every $\kappa \in \Sigma$. Then there is a cardinal-preserving extension where F is realised on all V -regular cardinals, and all elements in Σ are singular strong limit cardinals of cofinality ω . In particular SCH fails at all elements in Σ .*

Proof. This follows immediately from [Theorem 4.13](#) because every toggle-like F is mild in the sense of [Definition 4.10](#), and $\theta = \theta_p \cup \theta_E$ in [\(4.52\)](#) for a toggle-like F : if κ is $F(\kappa)$ -hypermeasurable, where $F(\kappa) = \kappa^{++}$, then there is j such that either $\kappa^{++} = j(F)(\kappa) \geq \kappa^{++}$ or $j(F)(\kappa) = \kappa^+$. (Corollary 4.25) \square

Remark 4.26. Notice that [Corollary 4.25](#) can easily be extended to toggle-like functions F which toggle GCH and the failure of GCH by a fixed $n \in \omega$ for $n > 1$: $F(\alpha) = \alpha^+$ or $F(\alpha) = \alpha^{+n}$. The problem occurs when a third value of $F(\alpha)$ is allowed, see below.

5. Possible improvements and open questions

- (1) In [9] M. Magidor and M. Gitik comment that the ordering \leq_E used in the definition of the extender-based Prikry forcing needs to be only κ^+ -directed in order to define $\text{Prk}_E(\kappa, F(\kappa))$ suggests in the definition of a mild function we do not need the restriction that F is trivial at successors of Mahlo cardinals (this restriction makes sure we capture all subsets of $F(\kappa)$ is size $\leq \kappa^+$). [Corollary 4.25](#) would then be completely general because the side condition on the behaviour of a toggle-like F on successors of Mahlo cardinals would be dropped. Some arguments using just the κ^+ -directed closure of the ordering \leq_E can in fact be found in works by M. Gitik and C. Merimovich, see for instance [16,25]. However, we have decided to use the easier formulation of $\text{Prk}_E(\kappa, \lambda)$ with κ^{++} -directed closure, not least because it is well-described in the available literature (see [10,9]).
- (2) Admittedly, there is a large gap between the almost complete result in the case of a toggle-like F in [Corollary 4.25](#) and in a general case in [Theorem 4.13](#). However, there are some inherent restrictions which limit the generalization of these results. The main problem is that the extender-based Prikry forcing does not satisfy the property that every two direct extensions of condition are compatible. This precludes the definition of a measure as in [\(3.31\)](#) or [\(3.40\)](#). We have overcome this problem by demanding that $j(F)$ is trivial at that stage, that is equal to κ^+ . Note that it is possible to avoid this restriction if we have $j(F)(\kappa) \geq F(\kappa)$ which allows us to first realise F and preserve measurability of κ , and then use the simple Prikry forcing $\text{Prk}(\kappa)$.

Even for an Easton function F which for every regular α takes its value in the set $\{\alpha^+, \alpha^{++}, \alpha^{+++}\}$ we encounter this problem in the following case (let us denote such an F as F_3): Assume there is κ such that κ is κ^{+3} -hypermeasurable, $F_3(\kappa) = \kappa^{+3}$, and every witnessing j satisfies $j(F_3)(\kappa) = \kappa^{++}$. Then κ is in θ , but not in $\theta_p \cup \theta_E$. Also, for every witnessing $j : V \rightarrow M$, κ is κ^{++} -hypermeasurable in M , and so the forcing is non-trivial here.

In some sense, the fact that it is hard to generalize the construction even to F_3 is not that surprising. It is known for instance that it is much harder to fail GCH everywhere with $2^\kappa = \kappa^{+3}$ than with $2^\kappa = \kappa^{+2}$ (personal discussion with J. Cummings and M. Magidor).

- (3) An obvious generalization of the present technique would be to allow for other cofinalities than just ω . This would require the use of Radin-type forcings, and one would need to face the issue of reflection for singular cardinals of uncountable cofinalities. In fact, this might nicely correspond with the restriction $j(F)(\kappa) \geq F(\kappa)$ placed on F in order to preserve measurability of κ .
- (4) We do not know whether the results in this paper would go through with the weaker hypothesis of $o(\kappa) = F(\kappa)$ instead of $F(\kappa)$ -hypermeasurability. It is known that the weaker hypothesis is the optimal one if we deal with one fixed cardinal κ , and do not force at successors of κ . However, the assumption of $F(\kappa)$ -hypermeasurability seems essential for the construction in [7] (which is used in this paper) where F is realised on all regular cardinals.

Acknowledgements

This paper is a revised and extended version of a part of my Ph.D. thesis. Great thanks are due to my advisor Sy D. Friedman who gave me guidance and a lot of help. I also wish to thank James Cummings, Moti Gitik and Menachem Magidor for the personal discussions. The work was supported by a postdoctoral grant of the Grant Agency of the Czech Republic 201/09/P115, and by a Short Visit Grant 2903 from the European Science Foundation: New frontiers of infinity—mathematical, philosophical and computational prospects.

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