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Asymptotic expansions for nonlocal diffusion equations in L^q-norms for $1 \leqslant q \leqslant 2$

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ABSTRACT

We study the asymptotic behavior for solutions to nonlocal diffusion models of the form $u_t(x,t) = J * u(x,t) - u(x,t) = \int_{\mathbb{R}^d} J(x-y)u(y,t)\,dy - u(x,t)$ in the whole $\mathbb{R}^d$ with an initial condition $u(x,0) = u_0(x)$. Under suitable hypotheses on J (involving its Fourier transform) and u_0, it is proved an expansion of the form

$$\left\| u(x,t) - \sum_{|\alpha| \leqslant k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int u_0(x) x^\alpha \, dx \right) \partial^\alpha K_t \right\|_{L^q(\mathbb{R}^d)} \leqslant C t^{-A},$$

where K_t is the regular part of the fundamental solution and the exponent A depends on J, q, k and the dimension d. Moreover, we can obtain bounds for the difference between the terms in this expansion and the corresponding ones for the expansion of $v_t(x,t) = -(-\Delta)^{\frac{s}{2}} v(x,t)$. Here we deal with the case $1 \leqslant q \leqslant 2$. The case $2 \leqslant q \leqslant \infty$ was treated previously, by other methods, in L.I. Ignat and J.D. Rossi (2008) [11].

1. Introduction

In this paper we study the asymptotic behavior as $t \to \infty$ of solutions to the nonlocal evolution problem

$$\begin{cases} u_t(x,t) = J * u - u(x,t), & t > 0,\ x \in \mathbb{R}^d, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $J * u$ is the usual convolution in the space variable given by

$$(J * u)(x,t) = \int_{\mathbb{R}^d} J(x-y)u(y,t)\,dy.$$

Here the kernel $J : \mathbb{R}^d \to \mathbb{R}$ is nonnegative and verifies $\int_{\mathbb{R}^d} J(x)\,dx = 1$.

For the heat equation, $v_t = \Delta v$, a precise asymptotic expansion in terms of the fundamental solution and its derivatives was found in [8]. In fact, if G_t denotes the fundamental solution of the heat equation, namely, $G_t(x) = (4\pi t)^{-d/2} e^{-|x|^2/(4t)}$, under adequate assumptions on the initial condition, we have

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$$\left\| u(x,t) - \sum_{|\alpha|\leqslant k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int_{\mathbb{R}^d} u_0(x) x^\alpha \right) \partial^\alpha G_t \right\|_{L^q(\mathbb{R}^d)} \leqslant C t^{-A} \tag{1.2}$$

with $A = (\frac{d}{2})(\frac{(k+1)}{d} + (1 - \frac{1}{q}))$. As pointed out by the authors in [8], the same asymptotic expansion can be done in a more general setting, dealing with the equation $u_t = -(-\Delta)^{\frac{s}{2}} u$, $s > 0$.

Our main objective here is to study if an expansion analogous to (1.2) holds for the nonlocal problem (1.1). In this paper, that can be viewed as a natural extension of [11], we deal with the case $1 \leqslant q \leqslant 2$. The cases $2 \leqslant q \leqslant \infty$ are derived from Hausdorff–Young's inequality and Plancherel's identity, see [11]. The cases analyzed here, $1 \leqslant q \leqslant 2$, are more tricky. They are reduced to L^2-estimates on the momenta of $\partial^\alpha K_t$ and therefore more restrictive assumptions on J have to be imposed.

Now we need to introduce some notation. We will denote by $f \sim g$ as $\xi \sim 0$ if $|f(\xi) - g(\xi)| = o(g(\xi))$ when $\xi \to 0$ and $f \lesssim g$ if there exists a constant c independent of the relevant quantities such that $f \leqslant cg$. We also use the standard notation $\widehat{J}$ for the Fourier transform of a function J that is given by

$$\widehat{J}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} J(x)\, dx.$$

Concerning the first term in the expansion (1.2), in [4] it is proved that if J verifies $\widehat{J}(\xi) - 1 \sim -|\xi|^s$ as $\xi \sim 0$, then the asymptotic behavior can be described as follows,

$$\lim_{t\to+\infty} \left\| t^{\frac{d}{s}} u\big(y t^{\frac{1}{s}}, t\big) - \left(\int_{\mathbb{R}^d} u_0 \right) G^s(y) \right\|_{L^\infty(\mathbb{R}^d)} = 0,$$

where $G^s(y)$ satisfies $\widehat{G^s}(\xi) = e^{-|\xi|^s}$. Also, it is proved in [4] that the fundamental solution $w(x,t)$ of problem (1.1) satisfies $w(x,t) = e^{-t}\delta_0(x) + K_t(x)$, where the function K_t (the regular part of the fundamental solution) is given by $\widehat{K_t}(\xi) = e^{-t}(e^{t\widehat{J}(\xi)} - 1)$.

Here we find a complete expansion for $u(x,t)$, a solution to (1.1), in terms of the derivatives of the regular part of the fundamental solution, K_t.

Theorem 1.1. *Assume that J satisfies*

$$\widehat{J}(\xi) - 1 \sim -|\xi|^s, \quad \xi \sim 0 \tag{1.3}$$

with $[s] > d/2$ and that for any $m \geqslant 0$ and α there exists $c(m,\alpha)$ such that

$$\left|\partial^\alpha \widehat{J}(\xi)\right| \leqslant \frac{c(m,\alpha)}{|\xi|^m}, \quad |\xi| \to \infty. \tag{1.4}$$

Then for any $1 \leqslant q \leqslant 2$, we have the asymptotic expansion

$$\left\| u(x,t) - \sum_{|\alpha|\leqslant k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int_{\mathbb{R}^d} u_0(x) x^\alpha \right) \partial^\alpha K_t \right\|_{L^q(\mathbb{R}^d)} \leqslant C t^{-A}, \tag{1.5}$$

for all $u_0 \in L^1(\mathbb{R}^d, 1 + |x|^{k+1})$. Here $A = \frac{(k+1)}{s} + \frac{d}{s}(1 - \frac{1}{q})$.

Here we denoted by $L^1(\mathbb{R}^d, 1 + |x|^{k+1})$ the space of functions

$$L^1\big(\mathbb{R}^d, 1 + |x|^{k+1}\big) = \left\{ f : \mathbb{R}^d \mapsto \mathbb{R}\colon \int_{\mathbb{R}^d} |f|(x)\big(1 + |x|^{k+1}\big)\, dx < \infty \right\}$$

endowed with the norm

$$\|f\|_{L^1(\mathbb{R}^d,\, |x|^{k+1})} = \int_{\mathbb{R}^d} |f|(x)\big(1 + |x|^{k+1}\big)\, dx.$$

Note that, when J has an expansion of the form $\widehat{J}(\xi) - 1 \sim -|\xi|^2$ as $\xi \sim 0$ (this happens for example if J is compactly supported), then the decay rate in $L^\infty(\mathbb{R}^d)$ of the solutions to the nonlocal problem (1.1) and the heat equation coincide (in both cases they decay as $t^{-\frac{d}{2}}$). Moreover, the first order term also coincide (in both cases it is a Gaussian).

Our next aim is to study if the higher order terms of the asymptotic expansion that we have found in Theorem 1.1 have some relation with the corresponding ones for the heat equation. Our next results say that the difference between them is of lower order.

Theorem 1.2. *Let J be as in Theorem (1.1) and assume in addition that there exists $r > 0$ such that*

$$\widehat{J}(\xi) - \left(1 - |\xi|^s\right) \sim |\xi|^{s+r}, \quad \xi \sim 0. \tag{1.6}$$

We also assume that all the derivatives of $\widehat{J}$ decay at infinity faster as any polinomial:

$$\left|\partial^\alpha \widehat{J}(\xi)\right| \leqslant \frac{c(m,\alpha)}{|\xi|^m}, \quad \xi \to \infty.$$

Then for any $1 \leqslant q \leqslant 2$ and any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$, there exists a positive constant $C = C(q,d,s,r)$ such that the following holds

$$\left\|\partial^\alpha K_t - \partial^\alpha G_t^s\right\|_{L^q(\mathbb{R}^d)} \leqslant Ct^{-\frac{d}{s}(1-\frac{1}{q})} t^{-\frac{|\alpha|+r}{s}}, \tag{1.7}$$

where G_t^s is defined by its Fourier transform $\widehat{G_t^s}(\xi) = \exp(-t|\xi|^s)$.

Note that these results do not imply that the asymptotic expansion $\sum_{|\alpha|\leqslant k} \frac{(-1)^{|\alpha|}}{\alpha!}(\int u_0(x)x^\alpha)\partial^\alpha K_t$ coincides with the expansion that holds for the equation $u_t = -(-\Delta)^{\frac{s}{2}}u$: $\sum_{|\alpha|\leqslant k} \frac{(-1)^{|\alpha|}}{\alpha!}(\int u_0(x)x^\alpha)\partial^\alpha G_t^s$ (see [8]). They only say that the corresponding terms agree up to a better order. When J is compactly supported or rapidly decaying at infinity, then $s = 2$ and we obtain an expansion analogous to the one that holds for the heat equation.

To end this introduction let us comment briefly on some of the available literature.

Equations like (1.1) and variations of it, have been recently widely used to model diffusion processes, for example, in biology, dislocations dynamics, etc. See, for example, [2,3,5–7,9,10,13,14]. As stated in [9], if $u(x,t)$ is thought of as the density of a single population at the point x at time t, and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $(J * u)(x,t) = \int_{\mathbb{R}^N} J(y-x)u(y,t)\,dy$ is the rate at which individuals are arriving to position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies Eq. (1.1). Eq. (1.1) is called *nonlocal diffusion equation* since the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$, but on all the values of u in a neighborhood of x through the convolution term $J * u$.

2. Proofs of the results

2.1. Preliminaries

First, let us obtain a representation of the solution using Fourier variables. A proof of existence and uniqueness of solutions using the Fourier transform (see [12]) is given in [4] (see also [11]). We repeat the main arguments here for the sake of completeness.

Theorem 2.1. *Let $u_0 \in L^1(\mathbb{R}^d)$ such that $\widehat{u_0} \in L^1(\mathbb{R}^d)$. There exists a unique solution $u \in C^0([0,\infty); L^1(\mathbb{R}^d))$ of (1.1), and it is given by*

$$\widehat{u}(\xi,t) = e^{(\widehat{J}(\xi)-1)t}\widehat{u_0}(\xi).$$

Proof. We have

$$u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}^d} J(x-y)u(y,t)\,dy - u(x,t).$$

Applying the Fourier transform we obtain $\widehat{u_t}(\xi,t) = \widehat{u}(\xi,t)(\widehat{J}(\xi)-1)$. Hence, $\widehat{u}(\xi,t) = e^{(\widehat{J}(\xi)-1)t}\widehat{u_0}(\xi)$. Since $\widehat{u_0} \in L^1(\mathbb{R}^d)$ and $e^{(\widehat{J}(\xi)-1)t}$ is continuous and bounded, the result follows by taking the inverse of the Fourier transform. □

Now we prove a lemma concerning the fundamental solution of (1.1).

Lemma 2.1. *Let $J \in \mathcal{S}(\mathbb{R}^d)$, the space of rapidly decreasing functions. The fundamental solution of (1.1), that is the solution of (1.1) with initial condition $u_0 = \delta_0$, can be decomposed as*

$$w(x,t) = e^{-t}\delta_0(x) + K_t(x), \tag{2.8}$$

where the function K_t is smooth and given by $\widehat{K_t}(\xi) = e^{-t}(e^{t\widehat{J}(\xi)} - 1)$. Moreover, if u is a solution of (1.1) it can be written as

$$u(x,t) = (w * u_0)(x,t) = \int_{\mathbb{R}^d} w(x-z,t)u_0(z)\,dz.$$

Proof. By the previous result we have $\widehat{w_t}(\xi,t) = \widehat{w}(\xi,t)(\widehat{J}(\xi)-1)$. Hence, as the initial datum verifies $\widehat{u_0} = \widehat{\delta_0} = 1$,

$$\widehat{w}(\xi,t) = e^{(\widehat{J}(\xi)-1)t} = e^{-t} + e^{-t}\big(e^{\widehat{J}(\xi)t} - 1\big).$$

The first part of the lemma follows applying the inverse Fourier transform in $\mathcal{S}(\mathbb{R}^d)$.

To finish the proof we just observe that $w * u_0$ is a solution of (1.1) (just use Fubini's theorem) with $(w * u_0)(x,0) = u_0(x)$. □

Remark 2.1. The above proof together with the fact that $\widehat{J}(\xi) \to 0$ (since $J \in L^1(\mathbb{R}^d)$) shows that if $\widehat{J} \in L^1(\mathbb{R}^d)$ then the same decomposition (2.8) holds and the result also applies.

To prove our result we need some estimates on the kernel K_t.

2.2. Estimates on K_t

In this subsection we obtain the long time behavior of the kernel K_t and its derivatives.

As we have mentioned in the introduction, in [11] the authors study the behavior of $L^q(\mathbb{R}^d)$-norms with $2 \leqslant q \leqslant \infty$. They use Hausdorff–Young's inequality in the case $q = \infty$ and Plancherel's identity for $q = 2$.

However the case $1 \leqslant q \leqslant 2$ is more tricky. In order to evaluate the $L^1(\mathbb{R}^d)$-norm of the kernel K_t we use the following inequality

$$\|f\|_{L^1(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{2n}} \big\||x|^n f\big\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2n}}, \tag{2.9}$$

which holds for $n > d/2$ and which is frequently attributed to Carlson (see for instance [1]). The use of the above inequality with $f = K_t$ imposes that $|x|^n \partial^\alpha K_t$ belongs to $L^2(\mathbb{R}^d)$. To guarantee that property and to obtain the decay rate for the $L^2(\mathbb{R}^d)$-norm of $|x|^n \partial^\alpha K_t$ we need to impose the hypotheses (1.3) and (1.4) in Theorem 1.1.

Lemma 2.2. *Assume that J verifies* (1.4) *and*

$$\widehat{J}(\xi) - 1 \sim -|\xi|^s, \quad \xi \sim 0$$

with $[s] > d/2$. Then for any index $\alpha = (\alpha_1, \ldots, \alpha_d)$,

$$\big\|\partial^\alpha K_t\big\|_{L^1(\mathbb{R}^d)} \lesssim t^{-\frac{|\alpha|}{s}}. \tag{2.10}$$

Moreover, for $1 < q < 2$ we have

$$\big\|\partial^\alpha K_t\big\|_{L^q(\mathbb{R}^d)} \lesssim t^{-\frac{d}{s}(1-\frac{1}{q})-\frac{|\alpha|}{s}}$$

for large t.

Remark 2.2. There is no restriction on s if J is such that

$$\big|\partial^\alpha \widehat{J}(\xi)\big| \lesssim \min\big\{|\xi|^{s-|\alpha|}, 1\big\}, \quad |\xi| \leqslant 1.$$

This happens if s is a positive integer and $\widehat{J}(\xi) = 1 - |\xi|^s$ in a neighborhood of the origin.

Remark 2.3. The case $\alpha = (0,\ldots,0)$ can be easily treated when J is nonnegative. As a consequence of the mass conservation (just integrate the equation and use Fubini's theorem, see [4]), $\int_{\mathbb{R}^d} w(x,t) = 1$, we obtain $\int_{\mathbb{R}^d} |K_t| \leqslant 1$ and therefore (2.10) follows with $\alpha = (0,\ldots,0)$.

Remark 2.4. The condition (1.4) imposed on J is satisfied, for example, for any smooth, compactly supported function J.

Proof of Lemma 2.2. The estimates for $1 < q < 2$ follow from the cases $q = 1$ and $q = 2$ by interpolation.

The case $q = 2$ was analyzed in [11], we refer to that paper for details but include here the main argument for the reader's convenience.

By Plancherel's identity we have

$$\big\|\partial^\alpha K_t\big\|_{L^2(\mathbb{R}^d)}^2 \leqslant e^{-2t} \int_{\mathbb{R}^d} \big|e^{t\widehat{J}(\xi)} - 1\big|^2 |\xi|^{2|\alpha|}\, d\xi.$$

Now, let us choose $R > 0$ such that

$$\big|\widehat{J}(\xi)\big| \leqslant 1 - \frac{|\xi|^s}{2} \quad \text{for all } |\xi| \leqslant R. \tag{2.11}$$

Putting out the exponentially small terms, it remains to estimate

$$\int_{|\xi|\leqslant R} \left|e^{t(\widehat{J}(\xi)-1)}\right|^2 |\xi|^{2|\alpha|}\, d\xi,$$

where R is given by (2.11). The behavior of $\widehat{J}$ near zero gives

$$\int_{|\xi|\leqslant R} \left|e^{t(\widehat{J}(\xi)-1)}\right|^2 |\xi|^{2|\alpha|}\, d\xi \lesssim \int_{|\xi|\leqslant R} e^{-t|\xi|^s}|\xi|^{2|\alpha|}\, d\xi \lesssim t^{-\frac{d}{s}-\frac{2|\alpha|}{s}}.$$

To deal with $q=1$, we use inequality (2.9) with $f=\partial^\alpha K_t$ and n such that $[s]\geqslant n>d/2$. We get

$$\left\|\partial^\alpha K_t\right\|_{L^1(\mathbb{R}^d)} \lesssim \left\|\partial^\alpha K_t\right\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{2n}} \left\||x|^n\partial^\alpha K_t\right\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2n}}.$$

The condition $n\leqslant [s]$ guarantees that $\partial^n_{\xi_j}\widehat{J}$ makes sense near $\xi=0$ and thus the derivatives $\partial^n_{\xi_j}\widehat{K_t}$, $j=1,\dots,d$, exist. Observe that the moment of order n of K_t imposes the existence of the partial derivatives $\partial^n_{\xi_j}\widehat{K_t}$, $j=1,\dots,d$.

We have, using the decay in L^2 that we have proved previously,

$$\left\|\partial^\alpha K_t\right\|_{L^1(\mathbb{R}^d)} \lesssim t^{-(\frac{d}{2s}+\frac{|\alpha|}{s})(1-\frac{d}{2n})} \left\||x|^n\partial^\alpha K_t\right\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2n}}.$$

Thus it is sufficient to prove that

$$\left\||x|^n\partial^\alpha K_t\right\|_{L^2(\mathbb{R}^d)} \lesssim t^{\frac{n}{s}-\frac{d}{2s}-\frac{|\alpha|}{s}}$$

for all sufficiently large t. Observe that by Plancherel's theorem

$$\int_{\mathbb{R}^d} |x|^{2n}\left|\partial^\alpha K_t(x)\right|^2 dx \leqslant c(n)\int_{\mathbb{R}^d}\left(x_1^{2n}+\cdots+x_d^{2n}\right)\left|\partial^\alpha K_t(x)\right|^2 dx = c(n)\sum_{j=1}^d \int_{\mathbb{R}^d}\left|\partial^n_{\xi_j}\left(\xi^\alpha\widehat{K_t}\right)\right|^2 d\xi$$

where $\xi^\alpha=\xi_1^{\alpha_1}\dots\xi_d^{\alpha_d}$. Therefore, it remains to prove that for any $j=1,\dots,d$, it holds

$$\int_{\mathbb{R}^d}\left|\partial^n_{\xi_j}\left(\xi^\alpha\widehat{K_t}\right)\right|^2 d\xi \lesssim t^{\frac{2n}{s}-\frac{d}{s}-\frac{2|\alpha|}{s}}, \quad \text{for } t \text{ large}. \tag{2.12}$$

We analyze the case $j=1$, the others follow by the same arguments. Leibnitz's rule gives

$$\partial^n_{\xi_1}\left(\xi^\alpha\widehat{K_t}\right)(\xi)=\xi_2^{\alpha_2}\dots\xi_d^{\alpha_d}\sum_{k=0}^n\binom{n}{k}\partial^k_{\xi_1}\left(\xi_1^{\alpha_1}\right)\partial^{n-k}_{\xi_1}(\widehat{K_t})(\xi)$$

and guarantees that

$$\begin{aligned}\left|\partial^n_{\xi_1}\left(\xi^\alpha\widehat{K_t}\right)(\xi)\right|^2 &\lesssim \xi_2^{2\alpha_2}\dots\xi_d^{2\alpha_d}\sum_{k=0}^n\left|\partial^k_{\xi_1}\left(\xi_1^{\alpha_1}\right)\right|^2\left|\partial^{n-k}_{\xi_1}\widehat{K_t}(\xi)\right|^2\\ &\lesssim \xi_2^{2\alpha_2}\dots\xi_d^{2\alpha_d}\sum_{k=0}^{\min\{n,\alpha_1\}}\xi_1^{2(\alpha_1-k)}\left|\partial^{n-k}_{\xi_1}\widehat{K_t}(\xi)\right|^2.\end{aligned}$$

The last inequality reduces (2.12) to the following one:

$$\int_{\mathbb{R}^d}\xi_1^{2(\alpha_1-k)}\xi_2^{2\alpha_2}\dots\xi_d^{2\alpha_d}\left|\partial^{n-k}_{\xi_1}\widehat{K_t}(\xi)\right|^2 d\xi \lesssim t^{\frac{2n}{s}-\frac{d}{s}-\frac{2|\alpha|}{s}}$$

for all $0\leqslant k\leqslant\min\{\alpha_1,n\}$. Using the elementary inequality (it follows from the convexity of the log function)

$$\xi_1^{2(\alpha_1-k)}\xi_2^{2\alpha_2}\dots\xi_d^{2\alpha_d}\lesssim\left(\xi_1^2+\cdots+\xi_d^2\right)^{\alpha_1-k+\alpha_2+\cdots+\alpha_d}=|\xi|^{2(|\alpha|-k)}$$

it remains to prove that for any r nonnegative and any m such that $n-\min\{\alpha_1,n\}\leqslant m\leqslant n$, the following inequality is valid,

$$I(r,m,t)=\int_{\mathbb{R}^d}|\xi|^{2r}\left|\partial^m_{\xi_1}\widehat{K_t}\right|^2 d\xi\lesssim t^{-\frac{d}{s}+\frac{2}{s}(m-r)}. \tag{2.13}$$

First we analyze the case $m=0$. In this case

$$I(r,0,t)=\int_{\mathbb{R}^d}|\xi|^{2r}\left|e^{t(\widehat{J}(\xi)-1)}-e^{-t}\right|^2 d\xi=e^{-2t}\int_{\mathbb{R}^d}|\xi|^{2r}\left|e^{t\widehat{J}(\xi)}-1\right|^2 d\xi.$$

Using that $|e^y-1|\leqslant 2|y|$ for $|y|$ small, say $|y|\leqslant c_0$, we obtain that

$$\left|e^{t\widehat{J}(\xi)}-1\right|\leqslant 2t\left|\widehat{J}(\xi)\right|\leqslant \frac{2t}{|\xi|^m}$$

for all $|\xi|\geqslant h(t)=(c_0t)^{\frac{1}{m}}$. Then

$$e^{-2t}\int_{|\xi|\geqslant h(t)}|\xi|^{2r}\left|e^{t\widehat{J}(\xi)}-1\right|^2 d\xi\lesssim t^2e^{-2t}\int_{|\xi|\geqslant h(t)}\frac{|\xi|^{2r}}{|\xi|^m}\,d\xi\leqslant te^{-t}c(m-2r)$$

provided that $2r<m-d$.

It remains to estimate

$$e^{-2t}\int_{|\xi|\leqslant h(t)}|\xi|^{2r}\left|e^{t\widehat{J}(\xi)}-1\right|^2 d\xi.$$

We observe that the term $e^{-2t}\int_{|\xi|\leqslant h(t)}|\xi|^{2r}\,d\xi$ is exponentially small, so we concentrate on

$$I(t)=e^{-2t}\int_{|\xi|\leqslant h(t)}\left|e^{t\widehat{J}(\xi)}\right|^2|\xi|^{2r}\,d\xi.$$

Now, let us choose $R>0$ such that

$$\left|\widehat{J}(\xi)\right|\leqslant 1-\frac{|\xi|^s}{2}\quad\text{for all } |\xi|\leqslant R. \tag{2.14}$$

Once R is fixed, there exists $\delta>0$ with

$$\left|\widehat{J}(\xi)\right|\leqslant 1-\delta\quad\text{for all } |\xi|\geqslant R. \tag{2.15}$$

Then

$$\begin{aligned}
\left|I(t)\right|&\leqslant e^{-2t}\int_{|\xi|\leqslant R}\left|e^{t\widehat{J}(\xi)}\right|^2|\xi|^{2r}\,d\xi+e^{-2t}\int_{R\leqslant|\xi|\leqslant h(t)}\left|e^{t\widehat{J}(\xi)}\right|^2|\xi|^{2r}\,d\xi\\
&\lesssim\int_{|\xi|\leqslant R}e^{2t(|\widehat{J}(\xi)|-1)}|\xi|^{2r}\,d\xi+e^{-2t\delta}\int_{R\leqslant|\xi|\leqslant h(t)}|\xi|^{2r}\,d\xi\\
&\lesssim\int_{|\xi|\leqslant R}e^{-t|\xi|^s}|\xi|^{2r}+\text{e.s.}\\
&=t^{-\frac{2r}{s}-\frac{d}{s}}\int_{|\eta|\leqslant Rt^{\frac{1}{s}}}e^{-|\eta|^s}|\eta|^{2r}+\text{e.s.}\\
&\lesssim t^{-\frac{2r}{s}-\frac{d}{s}}.
\end{aligned}$$

Observe that under hypothesis (1.4) no restriction on r is needed.

In what follows we analyze the case $m\geqslant 1$. First, we recall the following elementary identity

$$\partial^m_{\xi_1}\left(e^g\right)=e^g\sum_{i_1+2i_2+\dots+mi_m=m}a_{i_1,\dots,i_m}\left(\partial^1_{\xi_1}g\right)^{i_1}\left(\partial^2_{\xi_1}g\right)^{i_2}\dots\left(\partial^m_{\xi_1}g\right)^{i_m}$$

where $a_{i_1,\dots,i_m}$ are universal constants independent of g. Tacking into account that $\widehat{K_t}(\xi)=e^{t(\widehat{J}(\xi)-1)}-e^{-t}$, we obtain for any $m\geqslant 1$ that

$$\partial_{\xi_1}^m \widehat{K}_t(\xi) = e^{t(\widehat{J}(\xi)-1)} \sum_{i_1+2i_2+\cdots+mi_m=m} a_{i_1,\ldots,i_m} t^{i_1+\cdots+i_m} \prod_{j=1}^m \big[\partial_{\xi_1}^j \widehat{J}(\xi)\big]^{i_j}$$

and hence

$$\big|\partial_{\xi_1}^m \widehat{K}_t(\xi)\big|^2 \lesssim e^{2t|\widehat{J}(\xi)-1|} \sum_{i_1+2i_2+\cdots+mi_m=m} t^{2(i_1+\cdots+i_m)} \prod_{j=1}^m \big[\partial_{\xi_1}^j \widehat{J}(\xi)\big]^{2i_j}.$$

Using that all the partial derivatives of $\widehat{J}$ decay, as $|\xi| \to \infty$, faster than any polinomial in $|\xi|$, we obtain that

$$\int_{|\xi|>R} |\xi|^{2r} \big|\partial_{\xi_1}^m \widehat{K}_t(\xi)\big|^2 \, d\xi \lesssim e^{-\delta t} t^{2m}$$

where R and δ are chosen as in (2.14) and (2.15). Tacking into account that $n \leqslant [s]$ and that $|\widehat{J}(\xi) - 1 + |\xi|^s| \leqslant o(|\xi|^s)$ as $|\xi| \to 0$, we obtain

$$\big|\partial_{\xi_1}^j \widehat{J}(\xi)\big| \leqslant |\xi|^{s-j}, \quad j = 1, \ldots, n$$

for all $|\xi| \leqslant R$. Then for any $m \leqslant n$ and for all $|\xi| \leqslant R$ the following holds

$$\begin{aligned}\big|\partial_{\xi_1}^m \widehat{K}_t(\xi)\big|^2 &\lesssim e^{-t|\xi|^s} \sum_{i_1+2i_2+\cdots+mi_m=m} t^{2(i_1+\cdots+i_m)} \prod_{j=1}^m |\xi|^{2(s-j)i_j} \\ &\lesssim e^{-t|\xi|^s} \sum_{i_1+2i_2+\cdots+mi_m=m} t^{2(i_1+\cdots+i_m)} |\xi|^{\sum_{j=1}^m 2(s-j)i_j}.\end{aligned}$$

Using that for any $l \geqslant 0$,

$$\int_{\mathbb{R}^d} e^{-t|\xi|^s} |\xi|^l \, d\xi \lesssim t^{-\frac{d}{s}-\frac{l}{s}},$$

we obtain

$$\int_{|\xi|\leqslant R} |\xi|^{2r} \big|\partial_{\xi_1}^m K_t(\xi)\big|^2 d\xi \lesssim t^{-\frac{d}{s}} \sum_{i_1+2i_2+\cdots+mi_m=m} t^{2p(i_1,\ldots,i_d)-\frac{2r}{s}}$$

where

$$p(i_1, \ldots, i_m) = (i_1 + \cdots + i_m) - \frac{1}{s}\sum_{j=1}^m (s-j)i_j = \frac{1}{s}\sum_{j=1}^m j\, i_j = \frac{m}{s}.$$

This completes the proof. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Following [8] we obtain that the initial condition $u_0 \in L^1(\mathbb{R}^d, 1 + |x|^{k+1})$ has the following decomposition

$$u_0 = \sum_{|\alpha|\leqslant k} \frac{(-1)^{|\alpha|}}{\alpha!} \bigg(\int u_0 x^\alpha \, dx\bigg) D^\alpha \delta_0 + \sum_{|\alpha|=k+1} D^\alpha F_\alpha$$

where

$$\|F_\alpha\|_{L^1(\mathbb{R}^d)} \leqslant \|u_0\|_{L^1(\mathbb{R}^d,\, |x|^{k+1})}$$

for all multi-indexes α with $|\alpha| = k + 1$.

In view of (2.8) the solution u of (1.1) satisfies

$$u(x, t) = e^{-t} u_0(x) + (K_t * u_0)(x).$$

The first term being exponentially small it suffices to analyze the long time behavior of $K_t * u_0$. Using the above decomposition and Lemma 2.2, we get

$$\begin{aligned}\Big\|K_t * u_0 - \sum_{|\alpha|\leqslant k}\frac{(-1)^{|\alpha|}}{\alpha!}\Big(\int u_0(x)x^\alpha dx\Big)\partial^\alpha K_t\Big\|_{L^q(\mathbb{R}^d)} &\leqslant \sum_{|\alpha|=k+1}\big\|\partial^\alpha K_t * F_\alpha\big\|_{L^q(\mathbb{R}^d)}\\ &\leqslant \sum_{|\alpha|=k+1}\big\|\partial^\alpha K_t\big\|_{L^q(\mathbb{R}^d)}\|F_\alpha\|_{L^1(\mathbb{R}^d)}\\ &\lesssim t^{-\frac{d}{s}(1-\frac{1}{q})}t^{-\frac{(k+1)}{s}}\|u_0\|_{L^1(\mathbb{R}^d,|x|^{k+1})}.\end{aligned}$$

This ends the proof. □

2.3. *Asymptotics for the higher order terms*

In this subsection we prove Theorem 1.2 (recall that $1 \leqslant q \leqslant 2$).

Proof of Theorem 1.2. Using the same ideas as in the proof of Lemma 2.2 it remains to prove that for some $d/2 < n \leqslant [s]$ the following holds

$$\big\||x|^n\big(\partial^\alpha K_t - \partial^\alpha G_t^s\big)\big\|_{L^2(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2s}+\frac{n-(|\alpha|+r)}{s}}.$$

Applying Plancherel's identity the proof of the last inequality is reduced to the proof of the following one

$$\int_{\mathbb{R}^d}\big|\partial^n_{\xi_j}\big[\xi^\alpha\big(\widehat{K_t}-\widehat{G_t^s}\big)\big]\big|^2\,d\xi \lesssim t^{-\frac{d}{2s}+\frac{n-(|\alpha|+r)}{s}}, \quad j=1,\dots,d,$$

provided that all the above terms make sense. This means that all the partial derivatives $\partial^k_{\xi_j}\widehat{K_t}$ and $\partial^k_{\xi_j}\widehat{G_t^s}$, $j=1,\dots,d$, $k=0,\dots,n$ have to be defined. Thus, we need $n \leqslant [s]$.

We consider the case $j=1$ the other cases being similar. Applying again Leibnitz's rule, we get

$$\begin{aligned}\big|\partial^n_{\xi_1}\big[\xi^\alpha\big(\widehat{K_t}-\widehat{G_t^s}\big)\big]\big|^2 &\lesssim \xi_2^{2\alpha_2}\dots\xi_d^{2\alpha_d}\sum_{k=0}^{\min\{n,\alpha_1\}}\xi_1^{2(\alpha_1-k)}\big|\partial^{n-k}_{\xi_1}\big(\widehat{K_t}-\widehat{G_t^s}\big)\big|^2\\ &\lesssim \sum_{k=0}^{\min\{n,\alpha_1\}}|\xi|^{2(|\alpha|-k)}\big|\partial^{n-k}_{\xi_1}\big(\widehat{K_t}-\widehat{G_t^s}\big)\big|^2.\end{aligned}$$

In the following we prove that

$$\int_{\mathbb{R}^d}|\xi|^{2m_1}\big|\partial^m_{\xi_1}\big(\widehat{K_t}-\widehat{G_t^s}\big)\big|^2\,d\xi \lesssim t^{-\frac{d}{s}+\frac{2(m-m_1-r)}{s}}$$

for all $|\alpha| - \min\{n,\alpha_1\} \leqslant m_1 \leqslant |\alpha|$ and $n-\min\{n,\alpha_1\} \leqslant m \leqslant n$.

Using that the integral outside of a ball of radius R decay exponentially, it remains to analyze the decay of the following integral

$$\int_{|\xi|\leqslant R}|\xi|^{2m_1}\big|\partial^m_{\xi_1}\big(\widehat{K_t}-\widehat{G_t^s}\big)\big|^2 d\xi$$

where R is as before. Using the definition of $\widehat{K_t}$ and G_t^s, we obtain that

$$\partial^m_{\xi_1}\widehat{K_t}(\xi) = e^{t(\widehat{J}(\xi)-1)}\sum_{i_1+2i_2+\dots+mi_m=m}a_{i_1,\dots,i_m}t^{i_1+\dots+i_m}\prod_{j=1}^m\big[\partial^j_{\xi_1}\widehat{J}(\xi)\big]^{i_j}$$

and

$$\partial^m_{\xi_1}\widehat{G_t^s}(\xi) = e^{tp_s(\xi)}\sum_{i_1+2i_2+\dots+mi_m=m}a_{i_1,\dots,i_m}t^{i_1+\dots+i_m}\prod_{j=1}^m\big[\partial^j_{\xi_1}p_s(\xi)\big]^{i_j}$$

where $p_s(\xi) = -|\xi|^s$. Then

$$\big|\partial^m_{\xi_1}\widehat{K_t}(\xi)-\partial^m_{\xi_1}\widehat{G_t^s}(\xi)\big|^2 \lesssim I_1(\xi,t)+I_2(\xi,t)$$

where

$$I_1(\xi,t)=\left|e^{t(\widehat{J}(\xi)-1)}-e^{tp_s(\xi)}\right|^2\sum_{i_1+2i_2+\cdots+mi_m=m}t^{2(i_1+\cdots+i_m)}\prod_{j=1}^{m}\left|\partial_{\xi_1}^{j}p_s(\xi)\right|^{2i_j}$$

and

$$I_2(\xi,t)=e^{2tp_s(\xi)}\sum_{i_1+2i_2+\cdots+mi_m=m}t^{2(i_1+\cdots+i_m)}\left|\prod_{j=1}^{m}\left[\partial_{\xi_1}^{j}\widehat{J}(\xi)\right]^{i_j}-\prod_{j=1}^{m}\left[\partial_{\xi_1}^{j}p_s(\xi)\right]^{i_j}\right|^2.$$

First, let us analyze $I_1(\xi,t)$.

Tacking into account that $|\partial_{\xi_1}^{j}p_s(\xi)|\leqslant|\xi|^{s-j}$ for all $j\leqslant m\leqslant[s]$, $|\xi|\leqslant R$, and that

$$\begin{aligned}\left|e^{t(\widehat{J}(\xi)-1)}-e^{tp_s(\xi)}\right|^2&=e^{-2t|\xi|^s}\left|e^{t(\widehat{J}(\xi)-1+|\xi|^s)}-1\right|^2\\&\lesssim e^{-2t|\xi|^s}\left|t\left(\widehat{J}(\xi)-1+|\xi|^s\right)\right|^2\\&\lesssim t^2e^{-2t|\xi|^s}|\xi|^{2(r+s)}\end{aligned}$$

the same arguments as in the proof of Lemma 2.2 give us the right decay.

It remains to analyze $I_2(\xi,t)$. We make use of the following elementary inequality

$$\left|\prod_{j=1}^{m}a_j-\prod_{j=1}^{m}b_j\right|\leqslant\sum_{j=1}^{m}|b_1\ldots b_{j-1}||a_j-b_j||a_{j+1}\ldots a_m|.$$

Then by Cauchy's inequality we also have

$$\left|\prod_{j=1}^{m}a_j-\prod_{j=1}^{m}b_j\right|^2\lesssim\sum_{j=1}^{m}b_1^2\ldots b_{j-1}^2(a_j-b_j)^2a_{j+1}^2\ldots a_m^2.$$

Applying the last inequality with $a_j=\partial_{\xi_1}^{j}\widehat{J}(\xi)$ and $b_j=\partial_{\xi_1}^{j}p_s(\xi)$ we obtain

$$I_2(\xi,t)\lesssim e^{2tp_s(\xi)}\sum_{i_1+2i_2+\cdots+mi_m=m}t^{2(i_1+\cdots+i_m)}g(i_1,\ldots,i_m,\xi)$$

where

$$g(\mathbf{i},\xi)=\sum_{j=1}^{m}\prod_{k=1}^{j-1}\left|\partial_{\xi_1}^{k}p_s(\xi)\right|^{2i_k}\left(\left[\partial_{\xi_1}^{k}\widehat{J}(\xi)\right]^{i_k}-\left[\partial_{\xi_1}^{k}p_s(\xi)\right]^{i_k}\right)^2\prod_{k=j+1}^{n}\left[\partial_{\xi_1}^{k}\widehat{J}(\xi)\right]^{2i_k}$$

and $\mathbf{i}=(i_1,\ldots,i_m)$.

Choosing eventually a smaller R we can guarantee that for $|\xi|\leqslant R$ and $k\leqslant[s]$ the following inequalities hold:

$$\left|\partial_{\xi_1}^{k}\widehat{J}(\xi)-\partial_{\xi_1}^{k}p_s(\xi)\right|\lesssim|\xi|^{s+r-k},\qquad\left|\partial_{\xi_1}^{k}\widehat{J}(\xi)\right|\lesssim|\xi|^{s-k},\qquad\left|\partial_{\xi_1}^{k}p_s(\xi)\right|\lesssim|\xi|^{s-k}.$$

Hence, we get

$$\begin{aligned}\left|\left[\partial_{\xi_1}^{k}\widehat{J}(\xi)\right]^{i_k}-\left[\partial_{\xi_1}^{k}p_s(\xi)\right]^{i_k}\right|&\leqslant\left|\partial_{\xi_1}^{k}\widehat{J}(\xi)-\partial_{\xi_1}^{k}p_s(\xi)\right|\sum_{l=0}^{i_k-1}\left[\partial_{\xi_1}^{k}\widehat{J}(\xi)\right]^{l}\left[\partial_{\xi_1}^{k}p_s(\xi)\right]^{i_k-l-1}\\&\lesssim|\xi|^{s+r-k}|\xi|^{(i_k-1)(s-k)}\\&=|\xi|^{r}|\xi|^{i_k(s-k)}.\end{aligned}$$

This yields the following estimate on the function $g(i_1,\ldots,i_m,\xi)$:

$$g(i_1,\ldots,i_m,\xi)\leqslant|\xi|^{2r}|\xi|^{2\sum_{j=1}^{m}i_k(s-k)},$$

and consequently

$$\int_{\mathbb{R}^d}I_2(t,\xi)\,d\xi\lesssim\int_{\mathbb{R}^d}e^{-2t|\xi|^s}\sum_{i_1+2i_2+\cdots+mi_m=m}t^{2(i_1+\cdots+i_m)}|\xi|^{2r+2\sum_{j=1}^{m}i_k(s-k)}\,d\xi.$$

Making a change of variable and using similar arguments as in the proof of Lemma 2.2, we obtain the desired result. □

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