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# Stability of a quadratic Jensen type functional equation in the spaces of generalized functions

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**Abstract**

Making use of the fundamental solution of the heat equation we find the solution and prove the stability theorem of the quadratic Jensen type functional equation

$$9f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z)=4\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]$$

in the spaces of Schwartz tempered distributions and Fourier hyperfunctions.


*Keywords:* Quadratic functional equation; Jensen functional equation; Stability; Generalized function; Heat kernel; Gauss transform

## 1. Introduction

The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. We usually say that the functional equation $E_1(F) = E_2(F)$ has the Hyers–Ulam–Rassias stability if for an approximate solution $f$ of this equation, i.e., for a function $f$ with $d(E_1(f), E_2(f)) \leqslant \phi$ holds with a given function $\phi$, there exists a function $g$ such that $E_1(g) = E_2(g)$ and $d(f, g) \leqslant \Phi$ for some fixed function $\Phi$ [7,13,16].

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Trif [15] solved the following Jensen type functional equation:

$$3f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z)$$
$$=2\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right] \tag{1}$$

and investigated the Hyers–Ulam–Rassias stability of this equation. (1) has been considered for the first time by Popoviciu [12] in connection with the following inequality: If $I$ is a nonempty interval and $f: I \to \mathbb{R}$ is a convex function, then it holds that

$$3f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z)\geqslant 2\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]$$

for all $x, y, z \in I$. Today this inequality is commonly known as the Popoviciu inequality.

Lee [9] introduced a quadratic Jensen type functional equation

$$9f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z)$$
$$=4\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right] \tag{2}$$

which is somewhat different from Eq. (1). The solutions of (1) and (2) are different and furthermore the methods of the proving their stability theorems are also different, even though the two of them are much alike.

In this paper we reformulate and prove the stability theorem of Eq. (2) in the spaces of some generalized functions such as the space $\mathcal{S}'$ of Schwartz tempered distributions which is the dual space of the Schwartz space $\mathcal{S}$ of rapidly decreasing functions and the space $\mathcal{F}'$ of Fourier hyperfunctions which is the dual space of the Sato space $\mathcal{F}$ of analytic functions of exponential decay. Note that the above Eq. (2) and the related inequality

$$\left|9f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z)\right.$$
$$\left.-4f\left(\frac{x+y}{2}\right)-4f\left(\frac{y+z}{2}\right)-4f\left(\frac{z+x}{2}\right)\right|\leqslant\varepsilon \tag{3}$$

themselves make no sense in the spaces of generalized functions. Making use of the pullbacks of generalized functions as in [1,2,4,6,10] we extend (2) and (3) to the spaces of generalized functions. Let $L$, $P_1$, $P_2$, $P_3$, $Q_1$, $Q_2$ and $Q_3$ be the functions defined by

$$L(x,y,z)=x+y+z,$$
$$P_1(x,y,z)=x, \qquad P_2(x,y,z)=y, \qquad P_3(x,y,z)=z,$$
$$Q_1(x,y,z)=x+y, \qquad Q_2(x,y,z)=y+z, \qquad Q_3(x,y,z)=z+x$$

for all $x, y, z \in \mathbb{R}^n$. Then (2) and (3) can be naturally extended as follows:

$$9u\circ\frac{L}{3}+u\circ P_1+u\circ P_2+u\circ P_3=4\left[u\circ\frac{Q_1}{2}+u\circ\frac{Q_2}{2}+u\circ\frac{Q_3}{2}\right], \tag{4}$$

$$\left\|9u\circ\frac{L}{3}+u\circ P_1+u\circ P_2+u\circ P_3-4u\circ\frac{Q_1}{2}-4u\circ\frac{Q_2}{2}-4u\circ\frac{Q_3}{2}\right\|\leqslant\varepsilon. \tag{5}$$

Here $\circ$ means the pullback of generalized functions and $\|v\| \leqslant \varepsilon$ means that $|\langle v, \varphi\rangle| \leqslant \varepsilon \|\varphi\|_{L^1}$ for all test functions $\varphi$.

In order to solve Eq. (4) we employ the $n$-dimensional heat kernel, that is, the fundamental solution $E(x,t)$ of the heat operator $\partial_t - \Delta_x$ in $\mathbb{R}^n_x \times \mathbb{R}^+_t$ given by

$$E_t(x) = E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t), & t > 0, \\ 0, & t \leqslant 0. \end{cases}$$

Since for each $t > 0$, $E(\cdot, t)$ belongs to the Schwartz space $\mathcal{S}$ and the Sato space $\mathcal{F}$, the convolution

$$Gu(x,t) = (u * E)(x,t) = u_y\big(E(x-y,t)\big), \quad x \in \mathbb{R}^n,\ t > 0, \tag{6}$$

is well defined for each $u \in \mathcal{S}'$ or $u \in \mathcal{F}'$, which is called the Gauss transform of $u$. By virtue of the useful semigroup property $(E_s * E_t)(x) = E_{s+t}(x)$ of the heat kernel, Eq. (4) will be converted into the classical functional equation of the Gauss transforms.

## 2. Preliminaries

We first introduce briefly spaces of some generalized functions such as tempered distributions and Fourier hyperfunctions. Here we use the multi-index notations, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of nonnegative integers and $\partial_j = \partial/\partial x_j$.

**Definition 2.1.** [5,14] We denote by $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions $\varphi$ in $\mathbb{R}^n$ satisfying

$$\|\varphi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \big|x^\alpha \partial^\beta \varphi(x)\big| < \infty \tag{7}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. A linear form $u$ on $\mathcal{S}$ is said to be tempered distribution if there is a constant $C \geqslant 0$ and a nonnegative integer $N$ such that

$$\big|\langle u, \varphi\rangle\big| \leqslant C \sum_{|\alpha|, |\beta| \leqslant N} \sup_{x \in \mathbb{R}^n} \big|x^\alpha \partial^\beta \varphi\big|$$

for all $\varphi \in \mathcal{S}$. The set of all tempered distributions is denoted by $\mathcal{S}'$.

Imposing growth conditions on $\|\cdot\|_{\alpha,\beta}$ in (7) Sato and Kawai introduced the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions as follows.

**Definition 2.2.** [3] We denote by $\mathcal{F}$ or $\mathcal{F}(\mathbb{R}^n)$ the Sato space of all infinitely differentiable function $\varphi$ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{A,B} = \sup_{x,\alpha,\beta} \frac{|x^\alpha \partial^\beta \varphi(x)|}{A^{|\alpha|} B^{|\beta|} \alpha! \beta!} < \infty \tag{8}$$

for some positive constants $A, B$.

We say that $\varphi_j \to 0$ as $j \to \infty$ if $\|\varphi_j\|_{A,B} \to 0$ as $j \to \infty$ for some $A, B > 0$, and denote by $\mathcal{F}'$ the strong dual of $\mathcal{F}$ and call its elements Fourier hyperfunctions.

It can be verified that the semi-norms (8) are equivalent to

$$\|\varphi\|_{h,k} = \sup_{x\in\mathbb{R}^n,\,\alpha\in\mathbb{N}_0^n} \frac{|\partial^\alpha\varphi(x)|\exp k|x|}{h^{|\alpha|}\alpha!} < \infty$$

for some constants $h, k > 0$. It is easy to see the following topological inclusion:

$$\mathcal{F} \hookrightarrow \mathcal{S}, \qquad \mathcal{S}' \hookrightarrow \mathcal{F}'.$$

From now on a test function means an element in the Schwartz space $\mathcal{S}$ or the Sato space $\mathcal{F}$ and a generalized function means a tempered distribution or Fourier hyperfunction.

We briefly introduced the heat kernel method, which represents generalized functions as the initial values of solutions of the heat equation.

**Theorem 2.3.** [11] *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then its Gauss transform $Gu(x,t)$ in* (6) *is a $C^\infty$-solution of the heat equation satisfying*:

(i) *there exist positive constants $C$, $M$ and $N$ such that*

$$\bigl|Gu(x,t)\bigr| \leqslant Ct^{-M}\bigl(1+|x|\bigr)^N \quad \text{in } \mathbb{R}^n \times (0,\delta); \tag{9}$$

(ii) *$Gu(x,t) \to u$ as $t \to 0^+$ in the following sense*; *for every $\varphi \in \mathcal{S}$*

$$\langle u, \varphi\rangle = \lim_{t\to 0^+} \int Gu(x,t)\varphi(x)\,dx.$$

*Conversely, every $C^\infty$-solution $U(x,t)$ of the heat equation satisfying the growth condition* (9) *can be expressed as $U(x,t) = Gu(x,t)$ for some $u \in \mathcal{S}'$.*

Similarly, we can represent Fourier hyperfunctions as initial values of solutions of the heat equation as a special case of the results in [8]. In this case, estimate (9) is replaced by the following: For every $\varepsilon > 0$ there exists a positive constant $C_\varepsilon$ such that

$$\bigl|Gu(x,t)\bigr| \leqslant C_\varepsilon \exp\bigl(\varepsilon\bigl(|x| + 1/t\bigr)\bigr) \quad \text{in } \mathbb{R}^n \times (0,\delta).$$

Convolving $E_r(x)\cdot E_s(y)\cdot E_t(z)$ in both sides of (4) we have the following functional equation:

$$\begin{aligned} &9Gu\left(\frac{x+y+z}{3}, \frac{r+s+t}{9}\right) + Gu(x,r) + Gu(y,s) + Gu(z,t) \\ &\quad = 4\left[Gu\left(\frac{x+y}{2}, \frac{r+s}{4}\right) + Gu\left(\frac{y+z}{2}, \frac{s+t}{4}\right) + Gu\left(\frac{z+x}{2}, \frac{t+r}{4}\right)\right] \end{aligned} \tag{10}$$

for all $x, y, z \in \mathbb{R}^n$, $r, s, t > 0$. Thus (4) is converted into the classical functional equation in the smooth functions.

## 3. Main results

In this section we find the solution of the functional equation (4) and prove the stability theorem in the spaces of generalized functions.

**Lemma 3.1.** *A function* $f:\mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ *satisfies Eq.* (10), $f(-x,t) = f(x,t)$ *and* $f(0,t) = 0$ *for all* $x, y, z \in \mathbb{R}^n$, $r, s, t > 0$ *if and only if there exists a quadratic function* $Q:\mathbb{R}^n \to \mathbb{C}$ *such that*

$$f(x,t) = Q(x)$$

*for all* $x \in \mathbb{R}^n$, $t > 0$.

**Proof.** *Necessity*: This is obvious.

*Sufficiency*: Putting $y = -x$, $z = 0$, $r = s = t$ in (10) yields

$$f(x,t) = 4f\left(\frac{x}{2}, \frac{t}{2}\right) \tag{11}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Putting $y = z = 0$, $r = s = t$ in (10) and using (11) we get

$$9f\left(\frac{x}{3}, \frac{t}{3}\right) = f(x,t)$$

for all $x \in \mathbb{R}^n$, $t > 0$. Thus (10) is converted into

$$\begin{aligned} &f\left(x+y+z, \frac{r+s+t}{3}\right) + f(x,r) + f(y,s) + f(z,t) \\ &\quad = f\left(x+y, \frac{r+s}{2}\right) + f\left(y+z, \frac{s+t}{2}\right) + f\left(z+x, \frac{t+r}{2}\right) \end{aligned} \tag{12}$$

for all $x, y, z \in \mathbb{R}^n$, $r, s, t > 0$. Putting $z = -x$, $r = s = t$ in (12) yields

$$2f(x,t) + 2f(y,t) = f(x+y,t) + f(x-y,t) \tag{13}$$

for all $x, y \in \mathbb{R}^n$, $t > 0$. Putting $y = -x$, $z = 0$ in (12) yields

$$f(x,r) + f(x,s) = f\left(x, \frac{s+t}{2}\right) + f\left(x, \frac{t+r}{2}\right) \tag{14}$$

for all $x \in \mathbb{R}^n$, $r, s, t > 0$. Putting $s = r$ in (14) yields

$$f(x,r) = f\left(x, \frac{r+t}{2}\right)$$

for all $x \in \mathbb{R}^n$, $r, t > 0$. Thus $f(x,t)$ is independent of $t > 0$ and we may write $Q(x) = f(x,1) = f(x,t)$. Since $f$ satisfies (13), $Q(x)$ satisfies the quadratic functional equation

$$2Q(x) + 2Q(y) = Q(x+y) + Q(x-y)$$

for all $x, y \in \mathbb{R}^n$. □

**Lemma 3.2.** *A function* $f:\mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ *satisfies Eq.* (10) *and* $f(-x,t) = -f(x,t)$ *for all* $x, y, z \in \mathbb{R}^n$, $r, s, t > 0$ *if and only if there exists an additive function* $A:\mathbb{R}^n \to \mathbb{C}$ *such that*

$$f(x,t) = A(x)$$

*for all* $x \in \mathbb{R}^n$, $t > 0$.

**Proof.** *Necessity*: This is obvious.

*Sufficiency*: Putting $y = z = 0$ and $r = s = t$ in (10) yields

$$9f\left(\frac{x}{3}, \frac{t}{3}\right) + f(x,t) = 8f\left(\frac{x}{2}, \frac{t}{2}\right) \tag{15}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Putting $z = -y$, $r = s = t$ in (10) yields

$$9f\left(\frac{x}{3}, \frac{t}{3}\right) + f(x,t) = 4f\left(\frac{x+y}{2}, \frac{t}{2}\right) + 4f\left(\frac{x-y}{2}, \frac{t}{2}\right) \tag{16}$$

for all $x, y \in \mathbb{R}^n$, $t > 0$. It follows from (15) and (16) that we obtain

$$2f(x,t) = f(x+y,t) + f(x-y,t) \tag{17}$$

for all $x, y \in \mathbb{R}^n$, $t > 0$. Putting $y = x$ in (17) we get

$$2f(x,t) = f(2x,t) \tag{18}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Putting $x = \frac{x+y}{2}$, $y = \frac{x-y}{2}$ in (17) and using (18) we have

$$f(x+y,t) = f(x,t) + f(y,t) \tag{19}$$

for all $x, y \in \mathbb{R}^n$, $t > 0$. Putting $y = 2x$ in (19) and using (18) we obtain

$$f(3x,t) = 3f(x,t) \tag{20}$$

for all $x \in \mathbb{R}^n$, $t > 0$. It follows from (18) and (20) that (10) is converted into

$$\begin{aligned} &3f\left(x+y+z, \frac{r+s+t}{9}\right) + f(x,r) + f(y,s) + f(z,t) \\ &\quad = 2\left[f\left(x+y, \frac{r+s}{4}\right) + f\left(y+z, \frac{s+t}{4}\right) + f\left(z+x, \frac{t+r}{4}\right)\right] \end{aligned} \tag{21}$$

for all $x, y, z \in \mathbb{R}^n$, $r, s, t > 0$. Putting $y = -x$, $z = 0$ in (21) yields

$$f(x,r) - f(x,s) = 2f\left(x, \frac{t+r}{4}\right) - 2f\left(x, \frac{s+t}{4}\right) \tag{22}$$

for all $x \in \mathbb{R}^n$, $r, s, t > 0$. Putting $t = 3r$ in (22) yields

$$f(x,r) - f(x,s) = 2f(x,r) - 2f\left(x, \frac{s+3r}{4}\right)$$

and so

$$f(x,r) + f(x,s) = 2f\left(x, \frac{s+3r}{4}\right) \tag{23}$$

for all $x \in \mathbb{R}^n$, $r, s > 0$. Putting $t = 3s$ in (22) yields

$$f(x,r) - f(x,s) = 2f\left(x, \frac{3s+r}{4}\right) - 2f(x,s)$$

and so

$$f(x,r) + f(x,s) = 2f\left(x, \frac{3s+r}{4}\right) \tag{24}$$

for all $x \in \mathbb{R}^n$, $r, s > 0$. It follows from (23) and (24) that

$$f(x, s+3r) = f(x, 3s+r)$$

for all $x \in \mathbb{R}^n$, $r, s > 0$. Thus $f(x,r) = f(x,s)$ for all $(r,s) \in R$, where

$$R = \left\{(r,s)\colon \frac{1}{3}r < s < 3r,\ r > 0\right\}.$$

Now it suffices to show that $f(x,r) = f(x,s)$ for all $(r,s) \notin R$. Let $(r,s) \notin R$. Then we can take $t > 0$ so large that $(r+t, s+t) \in R$. It follows from (22) that

$$f(x,r) - f(x,s) = f(x, r+t) - f(x, s+t) = 0.$$

Therefore $f(x,t)$ is independent of $t > 0$ and we may write $A(x) = f(x,1) = f(x,t)$. Since $f$ satisfies (19), $A(x)$ satisfies

$$A(x+y) = A(x) + A(y)$$

for all $x, y \in \mathbb{R}^n$. □

**Theorem 3.3.** *Every solution $u$ in $\mathcal{S}'$ or $\mathcal{F}'$ of Eq.* (4) *has the form*

$$u = \sum_{1 \leqslant i \leqslant j \leqslant n} a_{ij} x_i x_j + b \cdot x + c,$$

*where $b \in \mathbb{R}^n$ and $c \in \mathbb{C}$.*

**Proof.** Convolving the tensor product $E_r(x) \cdot E_s(y) \cdot E_t(z)$ of $n$-dimensional heat kernels in both sides of (4) we have the following classical functional equation in the upper half space $\mathbb{R}^n \times (0, \infty)$:

$$\begin{aligned} &9Gu\left(\frac{x+y+z}{3}, \frac{r+s+t}{9}\right) + Gu(x,r) + Gu(y,s) + Gu(z,t) \\ &\quad = 4\left[Gu\left(\frac{x+y}{2}, \frac{r+s}{4}\right) + Gu\left(\frac{y+z}{2}, \frac{s+t}{4}\right) + Gu\left(\frac{z+x}{2}, \frac{t+r}{4}\right)\right] \end{aligned} \tag{25}$$

for all $x, y, z \in \mathbb{R}^n$, $r, s, t > 0$.

Let $f_1 : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ be the function defined by $f_1(x,t) = (Gu(x,t) + Gu(-x,t))/2 - Gu(0,t)$ for all $x \in \mathbb{R}^n$, $t > 0$. Then $f_1(0,t) = 0$, $f_1(x,t) = f_1(-x,t)$ and

$$\begin{aligned} &9f_1\left(\frac{x+y+z}{3}, \frac{r+s+t}{9}\right) + f_1(x,r) + f_1(y,s) + f_1(z,t) \\ &\quad = 4\left[f_1\left(\frac{x+y}{2}, \frac{r+s}{4}\right) + f_1\left(\frac{y+z}{2}, \frac{s+t}{4}\right) + f_1\left(\frac{z+x}{2}, \frac{t+r}{4}\right)\right] \end{aligned}$$

for all $x, y, z \in \mathbb{R}^n$, $r, s, t > 0$. By Lemma 3.1, there exists a quadratic function $Q : \mathbb{R}^n \to \mathbb{C}$ such that $f_1(x,t) = Q(x)$ for all $x \in \mathbb{R}^n$, $t > 0$.

Let $f_2 : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ be the function defined by $f_2(x,t) = (Gu(x,t) - Gu(-x,t))/2$ for all $x \in \mathbb{R}^n$, $t > 0$. Then $f_2(0,t) = 0$, $f_2(-x,t) = -f_2(x,t)$ and

$$\begin{aligned} &9f_2\left(\frac{x+y+z}{3}, \frac{r+s+t}{9}\right) + f_2(x,r) + f_2(y,s) + f_2(z,t) \\ &\quad = 4\left[f_2\left(\frac{x+y}{2}, \frac{r+s}{4}\right) + f_2\left(\frac{y+z}{2}, \frac{s+t}{4}\right) + f_2\left(\frac{z+x}{2}, \frac{t+r}{4}\right)\right] \end{aligned}$$

for all $x, y, z \in \mathbb{R}^n$, $r, s, t > 0$. By Lemma 3.2, there exists an additive function $A : \mathbb{R}^n \to \mathbb{C}$ such that $f_2(x,t) = A(x)$ for all $x \in \mathbb{R}^n$, $t > 0$.

Since $Gu(x,t) = f_1(x,t) + f_2(x,t) + Gu(0,t)$, we have

$$Gu(x,t) = Q(x) + A(x) + Gu(0,t)$$

for all $x \in \mathbb{R}^n$, $t > 0$. Note that the Gauss transform $Gu$ is a smooth function. Thus as in [2] $Gu(x,t)$ is of the form

$$Gu(x,t) = \sum_{1 \leqslant i \leqslant j \leqslant n} a_{ij} x_i x_j + b \cdot x + Gu(0,t).$$

Moreover, $Gu(0,t) := g(t)$ satisfies

$$9g\left(\frac{r+s+t}{9}\right) + g(r) + g(s) + g(t) = 4\left[g\left(\frac{r+s}{4}\right) + g\left(\frac{s+t}{4}\right) + g\left(\frac{t+r}{4}\right)\right] \tag{26}$$

for all $r, s, t > 0$. By differentiating (26) with respect to $r$, we find that

$$g'\left(\frac{r+s+t}{9}\right) + g'(r) = g'\left(\frac{r+s}{4}\right) + g'\left(\frac{t+r}{4}\right) \tag{27}$$

for all $r, s, t > 0$. Similarly, differentiation of (26) with respect to $s$ yields

$$g'\left(\frac{r+s+t}{9}\right) + g'(s) = g'\left(\frac{r+s}{4}\right) + g'\left(\frac{s+t}{4}\right) \tag{28}$$

for all $r, s, t > 0$. It follows from (27) and (28) that

$$g'(r) - g'(s) = g'\left(\frac{t+r}{4}\right) - g'\left(\frac{s+t}{4}\right) \tag{29}$$

for all $r, s, t > 0$. Putting $t = 3s$ in (29) we have

$$g'(r) = g'\left(\frac{3s+r}{4}\right) \tag{30}$$

for all $r, s, t > 0$. Thus $g(t) = Gu(0,t)$ is of the form

$$Gu(0,t) = ct + d$$

for some $c, d \in \mathbb{C}$. Therefore the solution $Gu(x,t)$ of Eq. (25) is of the form

$$Gu(x,t) = \sum_{1 \leqslant i \leqslant j \leqslant n} a_{ij} x_i x_j + b \cdot x + ct + d \tag{31}$$

for some $b \in \mathbb{R}^n$, $c, d \in \mathbb{C}$. Letting $t \to 0$ in (31) we have the conclusion. □

Now, we are going to prove the stability theorem of Eq. (4) in the spaces of generalized functions.

**Theorem 3.4.** *Let $u$ be a tempered distribution or a Fourier hyperfunction in $\mathbb{R}^n$ satisfying* (5). *Then there exists unique quadratic form*

$$Q(x) = \sum_{1 \leqslant i \leqslant j \leqslant n} a_{ij} x_i x_j,$$

*$b \in \mathbb{R}^n$ and $c \in \mathbb{C}$, such that*

$$\|u - Q(x) - b \cdot x - c\| \leqslant \frac{2}{3}\varepsilon.$$

**Proof.** Convolving the tensor product $E_r(x) \cdot E_s(y) \cdot E_t(z)$ of $n$-dimensional heat kernels in both sides of (5) we have the following classical functional inequality

$$\left|9Gu\left(\frac{x+y+z}{3}, \frac{r+s+t}{9}\right) + Gu(x,r) + Gu(y,s) + Gu(z,t) - 4Gu\left(\frac{x+y}{2}, \frac{r+s}{4}\right) - 4Gu\left(\frac{y+z}{2}, \frac{s+t}{4}\right) - 4Gu\left(\frac{z+x}{2}, \frac{t+r}{4}\right)\right| \leqslant \varepsilon$$

for all $x, y, z \in \mathbb{R}^n$, $r, s, t > 0$.

Let $f_1 : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ be the function defined by $f_1(x,t) = (Gu(x,t) + Gu(-x,t))/2 - Gu(0,t)$ for all $x \in \mathbb{R}^n$, $t > 0$. Then $f_1(0,t) = 0$, $f_1(-x,t) = f_1(x,t)$ and

$$\left|9f_1\left(\frac{x+y+z}{3}, \frac{r+s+t}{9}\right) + f_1(x,r) + f_1(y,s) + f_1(z,t) - 4f_1\left(\frac{x+y}{2}, \frac{r+s}{4}\right) - 4f_1\left(\frac{y+z}{2}, \frac{s+t}{4}\right) - 4f_1\left(\frac{z+x}{2}, \frac{t+r}{4}\right)\right| \leqslant \varepsilon \tag{32}$$

for all $x, y, z \in \mathbb{R}^n$, $r, s, t > 0$. Putting $y = -x$, $z = 0$ and $r = s = t$ in (32) and dividing by 2 yields

$$\left|f_1(x,t) - 4f_1\left(\frac{x}{2}, \frac{t}{2}\right)\right| \leqslant \frac{\varepsilon}{2}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Replacing $x, t$ by $2x, 2t$, respectively, and dividing by 4 yields

$$\left|\frac{f_1(2x, 2t)}{4} - f_1(x,t)\right| \leqslant \frac{\varepsilon}{8}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Making use of the induction argument and triangle inequality it follows that

$$\left|\frac{f_1(2^n x, 2^n t)}{4^n} - f_1(x,t)\right| \leqslant \frac{\varepsilon}{6} \tag{33}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Replacing $x, t$ by $2^m x$, $2^m t$, respectively, and dividing the result by $4^m$ we obtain

$$\left|\frac{f_1(2^{m+n} x, 2^{m+n} t)}{4^{m+n}} - \frac{f_1(2^m x, 2^m t)}{4^m}\right| \leqslant \frac{\varepsilon}{6 \cdot 4^m}$$

for all $m, n \in \mathbb{N}$, $x \in \mathbb{R}^n$, $t > 0$. We can see that $g_n(x,t) := \frac{f_1(2^n x, 2^n t)}{4^n}$ is a Cauchy sequence which converges uniformly. Hence $g_1(x,t) := \lim_{n\to\infty} \frac{f_1(2^n x, 2^n t)}{4^n}$ exists. In view of (32) we have

$$9g_1\left(\frac{x+y+z}{3}, \frac{r+s+t}{9}\right) + g_1(x,r) + g_1(y,s) + g_1(z,t) = 4\left[g_1\left(\frac{x+y}{2}, \frac{r+s}{4}\right) + g_1\left(\frac{y+z}{2}, \frac{s+t}{4}\right) + g_1\left(\frac{z+x}{2}, \frac{t+r}{4}\right)\right]$$

for all $x, y, z \in \mathbb{R}^n$, $r, s, t > 0$. Since $f_1(0,t) = 0$ and $f_1(-x,t) = f_1(x,t)$, we have $g_1(0,t) = 0$ and $g_1(-x,t) = g_1(x,t)$ for all $x \in \mathbb{R}^n$, $t > 0$. By Lemma 3.1, there exists a quadratic function $Q : \mathbb{R}^n \to \mathbb{C}$ such that

$$g_1(x,t) = Q(x)$$

for all $x \in \mathbb{R}^n$, $t > 0$. On the other hand, the function $Q$ inherits its measurability from $f_1$. Thus, as in [2], $Q(x)$ is of the form

$$Q(x) = \sum_{1 \leqslant i \leqslant j \leqslant n} a_{ij} x_i x_j.$$

Letting $n \to \infty$ in (33) we have

$$\left|Q(x) - f_1(x,t)\right| \leqslant \frac{\varepsilon}{6} \tag{34}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Suppose that another quadratic function $H : \mathbb{R}^n \to \mathbb{C}$ satisfies (34), then

$$\begin{aligned} r^2\left|H(x) - Q(x)\right| &= \left|H(rx) - Q(rx)\right| \\ &\leqslant \left|H(rx) - f_1(rx,t)\right| + \left|f_1(rx,t) - Q(rx)\right| \leqslant \frac{\varepsilon}{3} \end{aligned}$$

for all $r \in \mathbb{Q}$, $x \in \mathbb{R}^n$, $t > 0$. Letting $r \to \infty$ we must have $H = Q$.

Let $f_2 : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be the function defined by $f_2(x,t) = (Gu(x,t) - Gu(-x,t))/2$ for all $x \in \mathbb{R}^n$, $t > 0$. Then $f_2(0,t) = 0$, $f_2(-x,t) = -f_2(x,t)$ and

$$\begin{aligned} &\left|9 f_2\left(\frac{x+y+z}{3}, \frac{r+s+t}{9}\right) + f_2(x,r) + f_2(y,s) + f_2(z,t)\right. \\ &\quad \left. - 4 f_2\left(\frac{x+y}{2}, \frac{r+s}{4}\right) - 4 f_2\left(\frac{y+z}{2}, \frac{s+t}{4}\right) - 4 f_2\left(\frac{z+x}{2}, \frac{t+r}{4}\right)\right| \leqslant \varepsilon \end{aligned} \tag{35}$$

for all $x, y, z \in \mathbb{R}^n$, $r, s, t > 0$. Putting $y = z = 0$ and $r = s = t$ in (35) yields

$$\left|9 f_2\left(\frac{x}{3}, \frac{t}{3}\right) + f_2(x,t) - 8 f_2\left(\frac{x}{2}, \frac{t}{2}\right)\right| \leqslant \varepsilon \tag{36}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Putting $z = -y$ and $r = s = t$ in (35) yields

$$\left|9 f_2\left(\frac{x}{3}, \frac{t}{3}\right) + f_2(x,t) - 4 f_2\left(\frac{x+y}{2}, \frac{t}{2}\right) - 4 f_2\left(\frac{x-y}{2}, \frac{t}{2}\right)\right| \leqslant \varepsilon \tag{37}$$

for all $x, y \in \mathbb{R}^n$, $t > 0$. Putting $y = x$ in (37) yields

$$\left|9 f_2\left(\frac{x}{3}, \frac{t}{3}\right) + f_2(x,t) - 4 f_2\left(x, \frac{t}{2}\right)\right| \leqslant \varepsilon \tag{38}$$

for all $x \in \mathbb{R}^n$, $t > 0$. It follows from (36) and (38) that

$$\left|8 f_2\left(\frac{x}{2}, \frac{t}{2}\right) - 4 f_2\left(x, \frac{t}{2}\right)\right| \leqslant 2\varepsilon$$

for all $x \in \mathbb{R}^n$, $t > 0$. Replacing $x$, $t$ by $2x$, $2t$, respectively, and dividing by 8 yields

$$\left|f_2(x,t) - \frac{f_2(2x,t)}{2}\right| \leqslant \frac{\varepsilon}{4}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Making use of the induction argument and triangle inequality it follows that

$$\left| f_2(x,t) - \frac{f_2(2^n x,t)}{2^n} \right| \leqslant \frac{\varepsilon}{2} \tag{39}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Replacing $x$ by $2^m x$ and dividing the result by $2^m$ we obtain

$$\left| \frac{f_2(2^m x,t)}{2^m} - \frac{f_2(2^{m+n} x,t)}{2^{m+n}} \right| \leqslant \frac{\varepsilon}{2^{m+1}}$$

for all $m, n \in \mathbb{N}$, $x \in \mathbb{R}^n$, $t > 0$. Hence $\frac{f_2(2^n x,t)}{2^n}$ is a Cauchy sequence. Let $g_2(x,t) := \lim_{n\to\infty} \frac{f_2(2^n x,t)}{2^n}$. It follows from (35) that $g_2$ satisfies

$$\begin{aligned} &9g_2\left(\frac{x+y+z}{3}, \frac{r+s+t}{9}\right) + g_2(x,r) + g_1(y,s) + g_2(z,t) \\ &\quad = 4\left[g_2\left(\frac{x+y}{2}, \frac{r+s}{4}\right) + g_2\left(\frac{y+z}{2}, \frac{s+t}{4}\right) + g_2\left(\frac{z+x}{2}, \frac{t+r}{4}\right)\right] \end{aligned}$$

for all $x, y, z \in \mathbb{R}^n$, $r, s, t > 0$. Since $f_2(0,t) = 0$ and $f_2(-x,t) = -f_2(x,t)$, we have $g_2(0,t) = 0$ and $g_2(-x,t) = -g_2(x,t)$ for all $x \in \mathbb{R}^n$, $t > 0$. By Lemma 3.2, there exists an additive function $A : \mathbb{R}^n \to \mathbb{C}$ such that

$$g_2(x,t) = A(x)$$

for all $x \in \mathbb{R}^n$, $t > 0$. Also the function $A$ inherits its measurability from $f_2$ and it is well known that $A(x)$ is of the form

$$A(x) = b \cdot x.$$

Letting $n \to \infty$ in (39) we have

$$\left| f_2(x,t) - A(x) \right| \leqslant \frac{\varepsilon}{2} \tag{40}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Suppose that another additive function $K : \mathbb{R}^n \to \mathbb{C}$ satisfies (40), then

$$\begin{aligned} r\left|K(x) - A(x)\right| &= \left|K(rx) - A(rx)\right| \\ &\leqslant \left|K(rx) - f_2(rx,t)\right| + \left|f_2(rx,t) - A(rx)\right| \leqslant \varepsilon \end{aligned}$$

for all $r \in \mathbb{Q}$, $x \in \mathbb{R}^n$, $t > 0$. Letting $r \to \infty$ we must have $K = A$.

Since $Gu(x,t) = f_1(x,t) + f_2(x,t) + Gu(0,t)$, we have

$$\begin{aligned} &\left|Gu(x,t) - Q(x) - A(x) - Gu(0,t)\right| \\ &\quad \leqslant \left|f_1(x,t) - Q(x)\right| + \left|f_2(x,t) - A(x)\right| \leqslant \frac{2}{3}\varepsilon. \end{aligned} \tag{41}$$

Letting $t \to 0^+$ in (41) we have the conclusion. □

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