The $m$-step competition graphs of doubly partial orders

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The competition graph of a doubly partial order is an interval graph. The competition-common enemy graph, a variant of the competition graph, of a doubly partial order is also an interval graph if it does not contain a 4-cycle as an induced subgraph. It is natural to ask whether or not the same phenomenon occurs for other interesting variants of the competition graph. In this paper, we study the $m$-step competition graph, a generalization of the competition graph, of a doubly partial order. We show that the $m$-step competition graph of a doubly partial order is an interval graph for every positive integer $m$. We also show that given a positive integer $m$, an interval graph with sufficiently many isolated vertices is the $m$-step competition graph of a doubly partial order.

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1. Introduction

Given a digraph $D$, the competition graph $C(D)$ of $D$ has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exists a common prey of $u$ and $v$ in $D$. If $(u, v)$ is an arc of a digraph $D$, then we call $v$ a prey of $u$ in $D$ and call $u$ a predator of $v$ in $D$. The notion of the competition graph is due to Cohen [1] and has arisen from ecology. Competition graphs also have applications in coding, radio transmission, and modeling of complex economic systems. (See [2,3] for a summary of these applications.) Since Cohen introduced the notion of the competition graph, various variations have been defined and studied by many authors (see the survey articles by Kim [4] and Lundgren [5]). The competition-common enemy graph (CCE graph) of a digraph $D$ introduced by Scott [6] has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exist both a common prey and a common predator of $u$ and $v$ in $D$. The niche graph of a digraph $D$ introduced by Cable et al. [7] has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exists a common prey or a common predator of $u$ and $v$ in $D$.

As a generalization of competition graph, the concept of the $m$-step competition graph of a digraph was introduced by Cho et al. [8]. Given a digraph $D$ and a positive integer $m$, a vertex $y$ is an $m$-step prey of a vertex $x$ if and only if there exist vertices $a_0, a_1, a_2, \ldots, a_m$ such that $a_0 = x$, $a_m = y$ and $(a_{i-1}, a_i) \in A(D)$ for each $i = 1, 2, \ldots, m$. The $m$-step competition graph of a digraph $D$, denoted by $C^m(D)$, has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exists an $m$-step common prey of $u$ and $v$ in $D$. Recently variants of $m$-step competition graphs have been introduced and studied (see [9]). The any step competition graph of a digraph $D$ has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exists a vertex $x$ which is a $k$-step prey of $u$ and an $l$-step prey of $v$ in $D$ for some positive integers $k$ and $l$. The same step competition graph of a digraph $D$ has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exists a vertex $x$ which is a $k$-step common prey of $u$ and $v$ in $D$ for some positive integer $k$.  

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Cohen [1,10] observed empirically that most competition graphs of acyclic digraphs representing food webs are interval graphs. A graph $G$ is an interval graph if we can assign to each vertex $v$ in $V(G)$ a real interval $J(v) \subseteq \mathbb{R}$ such that whenever $v \neq w$, $vw \in E(G)$ if and only if $J(v) \cap J(w) \neq \emptyset$.

Cohen’s observation and the continued preponderance of examples that are interval graphs led to a large amount of literature devoted to attempts at explaining the observation and at studying the properties of competition graphs. Roberts [11] showed that every graph can be made to be the competition graph of an acyclic digraph by adding isolated vertices. (Add a vertex $i_{ab}$ corresponding to each edge $a = ab$ of $G$, and draw arcs from $a$ and $b$ to $i_{ab}$.) He then asked for a characterization of acyclic digraphs whose competition graphs are interval graphs. The study of acyclic digraphs whose competition graphs are interval graphs led to several new problems and applications (see [12–15]). Recently, Cho and Kim [16] found an interesting class of acyclic digraphs called ‘doubly partial orders’ with interval competition graphs. A digraph $D$ is called a doubly partial order if there exists a finite subset $V$ of $\mathbb{R}^2$ such that $V(D) = V$ and

$$A(D) = \{((x_1, x_2), (v_1, v_2)) \mid (v_1, v_2), (x_1, x_2) \in V, v_1 < x_1 \text{ and } v_2 < x_2\}.$$  

We may embed the competition graph of a doubly partial order $D$ in $\mathbb{R}^2$ by locating each vertex at the same position as in $D$. We will always embed the vertices of a doubly partial order $D$ (as well as the vertices of its competition graph) into $\mathbb{R}^2$ in a natural way.

The following theorems clarify the relations between interval graphs and competition graphs of doubly partial orders.

**Theorem 1** ([16]). The competition graph of a doubly partial order is an interval graph.

**Theorem 2** ([16]). An interval graph with sufficiently many isolated vertices is the competition graph of a doubly partial order.

Then competition-common enemy graphs and niche graphs of doubly partial orders were studied.

**Theorem 3** ([17]). The CCE graph of a doubly partial order is an interval graph unless it contains a 4-cycle as an induced subgraph.

**Theorem 4** ([17]). An interval graph with sufficiently many isolated vertices is the CCE graph of a doubly partial order.

The above results on CCE graphs were generalized by Lu and Wu [18] and Wu and Lu [19]. It turns out that the niche graph of a doubly partial order may not be any interval graph.

**Theorem 5** ([20]). For any integer $n \geq 4$, there is a doubly partial order whose niche graph contains an $n$-cycle as an induced subgraph.

Since a doubly partial order $D$ is transitive, a $k$-step prey of a vertex $v$ is also a 1-step prey of the vertex $v$ in $D$ and therefore the competition graph, the any step competition graph, and the same step competition graph of $D$ are the same. Thus we obtain the following corollaries from Theorems 1 and 2.

**Corollary 6.** For a doubly partial order $D$, the any step competition graph and the same step competition graph of $D$ are interval graphs.

**Corollary 7.** An interval graph with sufficiently many isolated vertices is the any step competition graph (resp. same step competition graph) of a doubly partial order.

However, the $m$-step competition graph of a doubly partial order $D$ is not necessarily equal to the competition graph of $D$ for $m \geq 2$ even if $E(C(D)) \supseteq E(C^m(D))$. Therefore, it is natural to ask whether the $m$-step competition graph of a doubly partial order is an interval graph.

In this paper, we study the $m$-step competition graphs of doubly partial orders for a positive integer $m$ and obtain the following main results, which generalize the results for competition graphs of doubly partial orders.

**Theorem 8.** For any positive integer $m$, the $m$-step competition graph of a doubly partial order is an interval graph.

**Theorem 9.** For any positive integer $m$, an interval graph with sufficiently many isolated vertices is the $m$-step competition graph of a doubly partial order.

As we mentioned earlier for competition graphs, we will always assume that the $m$-step competition graph of a doubly partial order is embedded in $\mathbb{R}^2$ in a natural way.
2. Proofs

In this section, we prove Theorems 8 and 9. For simplicity, we use the notation $u \nleq^D v$ if $u$ is an $m$-step prey of $v$ in a doubly partial order $D$. Especially we use $u \nleq^D v$ for $u \nleq^D v$.

Let $D$ be a doubly partial order and $m$ be a positive integer. We define $J_D^m(x, y)$ to be the smallest interval containing the set

\[ \{b - a \mid (a, b) \text{ is an } m \text{-step prey of } (x, y)\}, \]

that is,

\[ J_D^m(x, y) := \{\max[b - a \mid (a, b) \nleq^D (x, y)], \min[b - a \mid (a, b) \nleq^D (x, y)]\}. \]

If there is no possibility of confusion, we use $J^m(x, y)$ instead of $J_D^m(x, y)$. If a vertex $(x, y)$ has no $m$-step prey, then we let $J^m(x, y) = \emptyset$.

Now we present a proof of Theorem 8.

**Proof of Theorem 8.** Let $D$ be a doubly partial order. We will show that $J^m(x, y)$ is an interval assignment of a vertex $(x, y)$ so that $C^m(D)$ is an interval graph. Suppose that $(x, y)$ and $(z, w)$ are adjacent in $C^m(D)$. Then they have an $m$-step common prey $(a, b)$ and $b - a \in J^m(x, y) \cap J^m(z, w)$. Therefore, $J^m(x, y) \cap J^m(z, w) \neq \emptyset$.

Now we will show that the interval assignments corresponding to non-adjacent vertices in $C^m(D)$ do not overlap. Take two distinct vertices $(x, y)$ and $(z, w)$ which are non-adjacent in $C^m(D)$. Without loss of generality, we may assume that $x \leq z$. If one of $(x, y)$ and $(z, w)$ is an isolated vertex in $C^m(D)$, then one of $J^m(x, y)$ and $J^m(z, w)$ is an empty set, and so $J^m(x, y) \cap J^m(z, w) = \emptyset$. Suppose that neither $(x, y)$ nor $(z, w)$ is an isolated vertex in $C^m(D)$. Then both $(x, y)$ and $(z, w)$ have an $m$-step prey in $D$. If $y \leq w$, then an $m$-step prey of $(x, y)$ is that of $(z, w)$ and so they must be adjacent in $C^m(D)$, contradiction. Therefore,

\[ y > w. \]

If $x = z$, then an $m$-step prey of $(z, w)$ is that of $(x, y)$ and so they must be adjacent in $C^m(D)$, contradiction. Thus,

\[ x < z. \]

To reach a contradiction, suppose that $J^m(x, y) \cap J^m(z, w) \neq \emptyset$. Take a real number $t \in J^m(x, y) \cap J^m(z, w)$. By definition of the interval assignment, there exist $m$-step preys $(a, b), (c, d)$ of $(x, y)$ such that

\[ b - a \leq t \leq d - c \]

and $m$-step preys $(e, f), (g, h)$ of $(z, w)$ such that

\[ f - e \leq t \leq h - g. \]

From these two inequalities above,

\[ b - a \leq h - g. \]

If $(g, h) \nleq (a, b)$ or $(a, b) \nleq (g, h)$ or $(g, h) = (a, b)$, then $(g, h)$ and $(a, b)$ are $m$-step common preys of $(x, y)$ and $(z, w)$, a contradiction. Then $(g, h) \neq (a, b)$ and one of the following holds.

\[ (i) a \leq g \quad \text{and} \quad b \geq h \quad \text{or} \quad (ii) a \geq g \quad \text{and} \quad b \leq h. \]

If (i) holds, then we have $b - a > h - g$ since $(g, h) \neq (a, b)$, and it contradicts inequality (3). Therefore, (ii) holds. Then, by (1) and (2), we obtain

\[ z > x > a \geq g \quad \text{and} \quad b \leq h < w < y \]

(refer to Fig. 1 for a location of $(x, y), (z, w), (g, h)$ and $(a, b)$).

Since $(a, b)$ is an $m$-step prey of $(x, y)$, there exist vertices $(a_i, b_i) \; (i = 0, 1, \ldots, m)$ such that

\[ (a_m, b_m) = (a, b) \nleq^D (a_{m-1}, b_{m-1}) \nleq^D (a_{m-2}, b_{m-2}) \nleq^D \cdots \nleq^D (a_1, b_1) \nleq^D (x, y) = (a_0, b_0). \]

Similarly, there exist vertices $(g_i, h_i) \; (i = 0, 1, \ldots, m)$ such that

\[ (g_m, h_m) = (g, h) \nleq^D (g_{m-1}, h_{m-1}) \nleq^D (g_{m-2}, h_{m-2}) \nleq^D \cdots \nleq^D (g_1, h_1) \nleq^D (z, w) = (g_0, h_0). \]

Let $L_1$ be the path in $\mathbb{R}^2$ obtained by joining $(a_j, b_j)$ and $(a_{j+1}, b_{j+1})$ for each $j = 0, 1, \ldots, m - 1$, and let $L_2$ be the path in $\mathbb{R}^2$ obtained by joining $(g_j, h_j)$ and $(g_{j+1}, h_{j+1})$ for each $j = 0, 1, \ldots, m - 1$. 
Let $D$ be a doubly partial order. Given an integer $m \geq 2$, there exists a doubly partial order $D'$ such that $C^m(D')$ is $C^{m-1}(D)$ with sufficiently many isolated vertices.

**Lemma 10.** Let $D$ be a doubly partial order. Given an integer $m \geq 2$, there exists a doubly partial order $D'$ such that $C^m(D')$ is $C^{m-1}(D)$ with sufficiently many isolated vertices.

**Proof.** Let $P$ be the set of all $(m - 1)$-step preys in $D$. If $P = \emptyset$, then both $C^m(D)$ and $C^{m-1}(D)$ are edgeless graphs and so the lemma trivially holds. Therefore it suffices to consider the case $P \neq \emptyset$.

Let $P_1$ and $P_2$ be the set of the first components and the set of the second components of the vertices in $P$, respectively. We define positive real numbers $\epsilon_1$ and $\epsilon_2$ as follows:

$$
\epsilon_1 := \begin{cases} 
\frac{1}{2} \min \{|a - b| \mid a \neq b, \ a, b \in P_1 \} & \text{if } |P_1| \geq 2; \\
1 & \text{otherwise}
\end{cases}
$$
Let $Q$ be the set of the vertices of $P$ having no prey in $D$. We define

$$Q' := \{(x - \epsilon_1, y - \epsilon_2) \mid (x, y) \in Q\}.$$  

Let $D'$ be the doubly partial order on $V(D) \cup Q'$. Since $Q'$ is a parallel translation of $Q$ to a diagonally negative direction, each vertex in $Q'$ has no prey in $D'$.

To complete the proof, we shall show that $C_m'(D')$ is $C_m^{-1}(D)$ together with some isolated vertices. Suppose that $(a, b)$ and $(c, d)$ are adjacent in $C_m^{-1}(D)$. Then $(a, b)$ and $(c, d)$ have an $(m - 1)$-step common prey, say $(x, y)$. Suppose that $(x, y) \notin Q$. Then $(x, y)$ has a prey, which is an $m$-step common prey of $(a, b)$ and $(c, d)$ in $D$. Since $D$ is a subgraph of $D'$, $(a, b)$ and $(c, d)$ have an $m$-step common prey in $D'$. Now suppose that $(x, y) \in Q$. Then the vertex $(x - \epsilon_1, y - \epsilon_2)$ of $Q'$ is an $m$-step common prey of $(a, b)$ and $(c, d)$ in $D'$, and so $(a, b)$ and $(c, d)$ are adjacent in $C_m'(D')$. Therefore, $E(C_m^{-1}(D)) \subseteq E(C_m'(D'))$.

To show that $E(C_m'(D')) \supseteq E(C_m^{-1}(D))$, take two vertices $(a, b)$ and $(c, d)$ in $V(D')$ which are adjacent in $C_m'(D')$. Since $(a, b)$ and $(c, d)$ are adjacent in $C_m'(D')$, they have an $m$-step common prey $(x, y)$ in $D'$. Then there exist vertices $(a_i, b_i), (c_i, d_i)$ $(0 \leq i \leq m - 1)$ in $D'$ such that

$$\begin{align*}
(x, y) &\not\in (a_{m-1}, b_{m-1}) \not\in (a_{m-2}, b_{m-2}) \not\in \cdots \not\in (a_1, b_1) \not\in (a_0, b_0) = (a, b); \\
(x, y) &\not\in (c_{m-1}, d_{m-1}) \not\in (c_{m-2}, d_{m-2}) \not\in \cdots \not\in (c_1, d_1) \not\in (c_0, d_0) = (c, d).
\end{align*}$$

Since each vertex in $Q'$ has no prey in $D'$,

$$\{(a, b), (c, d)\} \cup \{(a_i, b_i), (c_i, d_i) \mid 1 \leq i \leq m - 1\} \subseteq V(D).$$  

(7)

If $(x, y) \in V(D)$, then $(x, y)$ is also an $m$-step common prey of $(a, b)$ and $(c, d)$ in $D$, and so they are adjacent in $C_m^{-1}(D)$.

Now suppose that $(x, y) \notin V(D)$. Then $(x, y) \in Q'$ which implies that there exists a vertex $(z, w)$ in $Q$ such that $(z, w) = (x + \epsilon_1, y + \epsilon_2)$. We will show that $(z, w)$ is an $(m - 1)$-step common prey of $(a, b)$ and $(c, d)$ in $D$. Suppose that $q_{m-1} < z$. Then we have $x < q_{m-1} < z$, or $z - \epsilon_1 < q_{m-1} < z$, which implies that $|z - q_{m-1}| < \epsilon_1$. However, both $z$ and $q_{m-1}$ belong to $P_1$, and $z \neq q_{m-1}$, and we reach a contradiction to the definition of $\epsilon_1$. Therefore, $z \leq q_{m-1}$. Similarly, we can show that $w \leq b_{m-1}$. Then, by (7) and by the fact that $(z, w) \in Q \subseteq V(D),

$$\begin{align*}
(z, w) &\not\in (a_{m-2}, b_{m-2}) \not\in \cdots \not\in (a_1, b_1) \not\in (a_0, b_0) = (a, b). \\
(z, w) &\not\in (c_{m-2}, d_{m-2}) \not\in \cdots \not\in (c_1, d_1) \not\in (c_0, d_0) = (c, d).
\end{align*}$$

Similarly, we can show that

$$\begin{align*}
(z, w) &\not\in (a_{m-2}, b_{m-2}) \not\in \cdots \not\in (a_1, b_1) \not\in (a_0, b_0) = (a, b). \\
(z, w) &\not\in (c_{m-2}, d_{m-2}) \not\in \cdots \not\in (c_1, d_1) \not\in (c_0, d_0) = (c, d).
\end{align*}$$

Thus, $(z, w)$ is an $(m - 1)$-step common prey of $(a, b)$ and $(c, d)$ in $D$, and so $(a, b)$ and $(c, d)$ are adjacent in $C_m^{-1}(D)$.

Consequently, $E(C_m^{-1}(D')) = E(C_m'(D'))$ and we are done. \[\square\]

**Proof of Theorem 9.** By contradiction. Suppose that there exists a positive integer $m$ such that for some interval graph $G$, no matter how many isolated vertices are added to $G$, it cannot be made to be the $m$-step competition graph of a doubly partial order. We denote by $m$ the smallest among such integers and by $G$ an interval graph which cannot be made to be the $m$-step competition graph of a doubly partial order no matter how many isolated vertices are added. By Theorem 2, $m \geq 2$. By the choice of $m$, $G$ together with sufficiently many isolated vertices is the $(m - 1)$-step competition graph of a doubly partial order. That is, $C_m^{-1}(D) = G \cup I_k$ for some doubly partial order $D$ and some nonnegative integer $k$. By Lemma 10, there exists a doubly partial order $D'$ such that $C_m'(D') = C_m^{-1}(D) \cup I_k$ for some positive integer $k$. Therefore $C_m'(D') = G \cup I_{k+1}$. Thus $G$ together with sufficiently many isolated vertices is the $m$-step competition graph of some doubly partial order, which contradicts our assumption. Hence the theorem holds. \[\square\]

**References**


