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On Mockor's Question

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For certain classes of Prüfer domains A, we study the completion $\hat{A}^{\cdot\mathscr{T}}$ of A with respect to the supremum topology $\mathscr{T} = \sup\{\mathscr{T}_w | w \in \Omega\}$, where Ω is the family of nontrivial valuations on the quotient field which are nonnegative on A and $\mathscr{T}_w$ is a topology induced by a valuation $w \in \Omega$. It is shown that the concepts "SFT Prüfer domain" and "generalized Dedekind domain" are the same. We show that if E is the ring of entire functions, then $\hat{E}^{\cdot\mathscr{T}}$ is a Bezout ring which is not a $\hat{\mathscr{T}}$-Prüfer ring, and if A is an SFT Prüfer domain, then $\hat{A}^{\cdot\mathscr{T}}$ is a Prüfer ring under a certain condition. We also show that under the same conditions as above, $\hat{A}^{\cdot\mathscr{T}}$ is a $\hat{\mathscr{T}}$-Prüfer ring if and only if the number of independent valuation overrings of A is finite. In particular, if A is a Dedekind domain (resp., h-local Prüfer domain), then $\hat{A}^{\cdot\mathscr{T}}$ is a $\hat{\mathscr{T}}$-Prüfer ring if and only if A has only finitely many prime ideals (resp., maximal ideals). These provide an answer to Mockor's question.

1. INTRODUCTION

Let A be an integral domain with quotient field K and let Ω be the family of nontrivial valuations on K which are nonnegative on A. A valuation $w \in \Omega$ with the value group G_w induces a topology $\mathscr{T}_w$ on K with the sets $U_{w,\alpha} = \{x \in K | w(x) > \alpha\}$, $\alpha \in G_w^+ = \{\beta \in G_w | \beta \geq 0\}$, as a

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base of zero neighborhoods in K. It is well known that the completion $\hat{K}^{,\mathcal{T}_w}$ of K with respect to the topology $\mathcal{T}_w$ is a field and the extension $\hat{w}$ of w on $\hat{K}^{,\mathcal{T}_w}$ is a valuation on $\hat{K}^{,\mathcal{T}_w}$ [Bo2]. Let R_w be the valuation ring of w, i.e., $R_w = \{x \in K | w(x) \geq 0\}$, and let $\widehat{R_w}^{,\mathcal{T}_w}$ denote the completion of R_w with respect to the subspace topology induced by $\mathcal{T}_w$. Bourbaki also showed that $\widehat{R_w}^{,\mathcal{T}_w} = R_{\hat{w}}$, the valuation ring of $\hat{w}$, and $G_w = G_{\hat{w}}$, the value group of $\hat{w}$.

In this paper we consider a more general situation. Let $\mathcal{T} = \sup\{\mathcal{T}_w | w \in \Omega\}$, i.e., $\mathcal{T}$ is the topology with the set $\{U_{w,\alpha} | w \in \Omega,\ \alpha \in G_w^+\}$ as a subbase of zero neighborhoods in K. Let $\bar{A}^{,\mathcal{T}}$ be the closure of A in $\hat{K}^{,\mathcal{T}}$, the completion of K with respect to the $\mathcal{T}$ topology. In view of [Bo1, II.3.4, Proposition 8], $\bar{A}^{,\mathcal{T}} = \hat{A}^{,\mathcal{T}}$, which is the completion of A with respect to the subspace topology on A. In [Mo], Mockor studied the ring $\hat{A}^{,\mathcal{T}}$ for a Prüfer domain A. He presented some sufficient conditions for $\hat{A}^{,\mathcal{T}}$ to be a Prüfer ring, and equivalent conditions for $\hat{A}^{,\mathcal{T}}$ to be a $\hat{\mathcal{T}}$-Prüfer ring (see Section 8). However he left it an open question if there exists a Prüfer domain A such that $\hat{A}^{,\mathcal{T}}$ is not a $\hat{\mathcal{T}}$-Prüfer ring or such that $\hat{A}^{,\mathcal{T}}$ is a Prüfer ring but not a $\hat{\mathcal{T}}$-Prüfer ring.

The purpose of this paper is to construct examples of Mockor's question by studying certain classes of Prüfer domains such as h-local Prüfer domains, the ring of entire functions, and SFT Prüfer domains (Sections 5, 6, and 7). In particular, in dealing with an SFT Prüfer domain, we will use the following results [KP1, Theorem 15 and Corollary 17] (although it is stated for finite-dimensional A, its proof is also valid for the infinite-dimensional case). If A is an SFT Prüfer domain and I is a proper ideal of A, then the I-adic completion $\hat{A}^{,I}$ is an SFT Prüfer ring and, moreover, if $\sqrt{I}$ is a prime ideal, then $\hat{A}^{,I}$ is an SFT Prüfer domain and $\operatorname{Spec}(\hat{A}^{,I}) = \{(0)\} \cup \{\hat{Q}^{,I} | Q \in \operatorname{Spec}(A) \text{ and } Q \supseteq I\}$, where $\hat{Q}^{,I}$ is the I-adic completion of Q.

Section 2: In the literature there are two important classes of Prüfer domains: one is the class of SFT Prüfer domains introduced by Arnold in 1972 [Ar1] and the other is the class of generalized Dedekind domains introduced by Popescu in 1984 [Po]. Each of them has a good book that deals with it [Br, FHP]. We show that an integral domain is an SFT Prüfer domain if and only if it is a generalized Dedekind domain. We believe that this result will facilitate the research on SFT Prüfer domains or generalized Dedekind domains. So far only a few examples of SFT Prüfer domains have been given. Facchini's existence theorem [Fa, Theorem 5.3] for generalized Dedekind domains thus provides many examples of SFT Prüfer domains.

Section 3: Let $\mathcal{C}$ be the set of all maximal chains of nonzero prime ideals of an integral domain A and let $C_\alpha \in \mathcal{C}$. We introduce the C_α

topology and the $\mathscr{C}$ topology, which is the supremum of the C_α topologies. We consider three kinds of topologies on an integral domain A, namely, the $\mathscr{T}$ topology, the ideal-adic topology, and the $\mathscr{C}$ topology. We investigate the relation between these topologies on an SFT Prüfer domain. In a particular situation, one topology is more useful than the others in studying the completion of a Prüfer domain.

In Section 4, we study the completions of a Prüfer domain A with respect to the $\mathscr{T}$ topology and the $\mathscr{C}$ topology, and show that $\hat{A}^{,\mathscr{T}} \cong \prod_{w \in \Omega_0} \hat{A}^{,\mathscr{T}_w}$ and $\hat{A}^{,\mathscr{C}} \cong \prod_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$, where Ω_0 is a family of independent valuations that are positive on A and whose equivalence classes constitute the set of all equivalence classes of Ω, and $\{C_\alpha\}_{\alpha \in \Lambda}$ is a representing family of the independent maximal chains in Spec$(A)^*$.

In Section 5, we give a short survey on the completion of an h-local Prüfer domain.

In Section 6, we show that the ring E of entire functions has the completion $\hat{E}^{,\mathscr{T}}$, which is a Bezout ring but not a $\hat{\mathscr{T}}$-Prüfer ring. We show that for a Prüfer domain A, $\hat{A}^{,C_\alpha} \cong \varprojlim_{P \in C_\alpha} \hat{A}^{,P}$.

Section 7: For $C_\alpha \in \mathscr{C}$, we introduce the Jacobson radical $J(C_\alpha)$ of C_α. It is shown that $\hat{A}^{,C_\alpha}$ is an SFT Prüfer domain for an SFT Prüfer domain A and a maximal chain C_α with nonzero Jacobson radical. In this case, $\mathrm{Spec}(\hat{A}^{,C_\alpha}) = \{(0)\} \cup \{\widehat{P_0}^{,C_\alpha} | P_0 \in \mathrm{Spec}(A)^*$ and P_0 contains some $P \in C_\alpha\}$.
If in addition $J(C_\alpha) \neq \{0\}$ for all $C_\alpha \in \mathscr{C}$, then $\hat{A}^{,\mathscr{T}}$ is a Prüfer ring.

It is well known that for a Noetherian domain A and $a_1, \ldots, a_n \in A$, $\hat{A}^{,(a_1,\ldots,a_n)} \cong A[\![X_1,\ldots,X_n]\!]/(X_1 - a_1, \ldots, X_n - a_n)$. We show that it also holds for an SFT Prüfer domain A. As a corollary we obtain that $(X_1 - a_1, \ldots, X_n - a_n)$ is a radical ideal of $A[\![X_1,\ldots,X_n]\!]$ and that $(X_1 - a_1, \ldots, X_n - a_n)$ is a prime ideal of $A[\![X_1,\ldots,X_n]\!] \Leftrightarrow \sqrt{(a_1,\ldots,a_n)}$ is a prime ideal of $A \Leftrightarrow A$ is analytically irreducible with respect to $(a_1,\ldots,a_n)$. An interesting result is that for an SFT Prüfer domain A, a prime ideal P of A, and $a_1,\ldots,a_n \in A$, we have $A[\![X_1,\ldots,X_n]\!]_{P+(X_1,\ldots,X_n)}/(X_1 - a_1, \ldots, X_n - a_n) \cong A_P[\![X_1,\ldots,X_n]\!]/(X_1 - a_1, \ldots, X_n - a_n)$.

In Section 8, constructing examples, we give answers to Mockor's question. Namely, we show that (1) the completion $\hat{A}^{,\mathscr{T}}$ of an h-local Prüfer domain is a $\hat{\mathscr{T}}$-Prüfer ring $\Leftrightarrow$ $|\mathrm{Max}(A)| < \infty$, (2) the completion $\hat{E}^{,\mathscr{T}}$ of the ring E of entire functions is not a $\hat{\mathscr{T}}$-Prüfer ring, (3) the completion $\hat{D}^{,\mathscr{T}}$ of a Dedekind domain is a $\hat{\mathscr{T}}$-Prüfer ring $\Leftrightarrow$ $|\mathrm{Spec}(D)| < \infty$, and (4) the completion $\hat{A}^{,\mathscr{T}}$ of an SFT Prüfer domain with $J(C_\alpha) \neq \{0\}$ for all maximal chains C_α is a $\hat{\mathscr{T}}$-Prüfer ring $\Leftrightarrow$ there exist only finitely many independent valuation overrings of A. In the cases (1), (3), and (4), every nonminimal prime ideal of $\hat{A}^{,\mathscr{T}}$ is of the form $\hat{P}^{,\mathscr{T}}$, where P is a nonzero prime ideal of A.

Throughout this paper A will be a Prüfer domain with quotient field K unless otherwise specified. For undefined terms and notation the reader is referred to [Bo1, Gi, Mo]. We would like to mention that to make this paper self-contained and for the sake of easy reference, we will sometimes paraphrase known results.

2. SFT PRÜFER DOMAINS AND GENERALIZED DEDEKIND DOMAINS

Let A be a commutative ring with identity and let I be an ideal of A. The ideal I will be called an SFT ideal (an ideal of strong finite type) provided there exist a finitely generated ideal $J \subseteq I$ and a positive integer k such that $a^k \in J$ for each $a \in I$. If each ideal of A is an SFT ideal, then we say that A is an SFT ring. This concept was introduced by Arnold in 1972 [Ar1]. It plays an important role in dealing with the formal power series ring. For example, Arnold has shown [Ar1, Ar4] that if A is not an SFT ring, then $\dim A[\![X]\!] = \infty$, and if A is a finite-dimensional SFT Prüfer domain, then $\dim A[\![X_1, \dots, X_n]\!] = n \dim A + 1$. For other results on SFT Prüfer domains, see [AKP, Ar2, Ar3, Ar5, Br, KP1, KP2, Ol].

Since in this paper we are primarily concerned with an SFT Prüfer domain, we list here some properties of an SFT ring.

PROPOSITION 2.1 [Ar1, Proposition 2.2 and 2.5, Corollary 2.7]. *Let A be a commutative ring with identity.*

(1) *A is an SFT ring if and only if each prime ideal is an SFT ideal.*

(2) *An SFT ring A has a Noetherian prime spectrum. In particular, each ideal of A has only finitely many minimal prime divisors.*

(3) *If P is a nonzero SFT prime ideal of an integral domain, then $P \neq P^2$.*

An integral domain is called a valuation domain if for each nonzero element a and b, a divides b or b divides a. An integral domain A is called a Prüfer domain if for each maximal ideal M of A, A_M is a valuation domain.

PROPOSITION 2.2 [Ar2, Proposition 3.1]. *For the Prüfer domain A to be an SFT ring, it is necessary and sufficient that for each nonzero prime ideal P of A, there exists a finitely generated ideal I such that $P^2 \subseteq I \subseteq P$.*

In 1984, 12 years after Arnold had invented the concept "SFT Prüfer domain," Popescu [Po] introduced the concept of a generalized Dedekind domain. He defined a generalized Dedekind domain to be a Prüfer domain A on which for every two distinct localizing systems F_1 and F_2, $A_{F_1} \neq A_{F_2}$. Then he obtained the following result.

PROPOSITION 2.3 [Po, Theorem 2.5]. *Let A be a Prüfer domain. The following assertions are equivalent.*

(1) *A is a generalized Dedekind domain.*

(2) *If P is a nonzero prime ideal of A, then $P \neq P^2$ and P is the radical of a finitely generated ideal.*

Apparently these two concepts were not realized to be the same. We prove that they are in fact the same.

THEOREM 2.4. *The concepts "SFT Prüfer domain" and "generalized Dedekind domain" are the same.*

Proof. It is clear that every SFT Prüfer domain is a generalized Dedekind domain (Proposition 2.1(3), 2.2, and 2.3). Conversely let A be a generalized Dedekind domain and let P be a nonzero prime ideal of A. Then $P \neq P^2$ and $P = \sqrt{I}$ for some finitely generated ideal I. Choose $a \in P \setminus P^2$. Put $J = (I, a)$. Then J is finitely generated, $P = \sqrt{J}$, and $J \not\subseteq P^2$. Since for each maximal ideal M of A, A_M is a valuation domain and P^2 is a P-primary ideal, $P^2 A_M \subseteq J A_M \subseteq P A_M$. Thus $P^2 \subseteq J \subseteq P$ locally and hence globally. Thus these two concepts are the same. ∎

So we can use Fontana, Huckaba, Popescu, and Facchini's results on the generalized Dedekind domains in dealing with the SFT Prüfer domains. In particular, using the following existence theorem due to Facchini, we can obtain a lot of examples of SFT Prüfer domains with the prime spectrum satisfying suitable conditions which we want.

Recall that a tree is a partially ordered set $(X, \leq)$ with the property that for every $x \in X$ the set $B_x = \{y \in X \mid y \leq x\}$ is a chain (i.e., a totally ordered set); it is Noetherian if every ascending chain $x_1 \leq x_2 \leq \cdots$ of elements of X is stationary.

THEOREM 2.5 [Fa, Theorem 5.3]. *Let X be a partially ordered set. The following statements are equivalent:*

(1) *X is a Noetherian tree with a least element.*

(2) *There exists a generalized Dedekind domain A whose prime spectrum* (Spec A, $\subseteq$) *is order isomorphic to X.*

3. TOPOLOGIES

Let A be a Prüfer domain with quotient field K. There are several kinds of topologies which make A a topological ring. The I-adic topology, where I is a proper ideal of A, is the topology with the set $\{I^n \mid n = 1, 2, \dots\}$ as a base of zero neighborhoods in A. The A topology is the topology with the ideals aA, $a \in A^*$ $(= A \setminus \{0\})$, as a base of zero neighborhoods in A. Let

$\Omega(A)$ be the family of nontrivial valuations on K which are nonnegative on A. A valuation $w \in \Omega(A)$ with the value group G_w induces the topology $\mathcal{T}_w$ on K with the sets $U_{w,\alpha} = \{x \in K | w(x) > \alpha\}$, $\alpha \in G_w^+ = \{\beta \in G_w | \beta \geq 0\}$, as a base of zero neighborhoods in K. We can give A the subspace topology of K. We shall call this topology the $\mathcal{T}_w$ topology on A. We denote by $P(w)$ the center of the valuation ring R_w of w on A.

LEMMA 3.1. *Let $w \in \Omega(A)$ be a valuation with valuation ring R_w. Then for each $\alpha \in G_w^+$, there exists $a \in A^*$ such that $\alpha = w(a)$, and hence $U_{w,\alpha} = aP(w)A_{P(w)}$.*

Proof. Since A is a Prüfer domain, $R_w = A_{P(w)}$, and so this is clear. ∎

Now let $\mathcal{T}(A) = \sup\{\mathcal{T}_w | w \in \Omega(A)\}$, i.e., $\mathcal{T}(A)$ is the topology on K with the set $\{U_{w,\alpha} | w \in \Omega(A),\ \alpha \in G_w^+\}$ as a subbase of zero neighborhoods in K. We shall call the subspace topology on A induced by this topology the $\mathcal{T}(A)$ topology on A. If there is no ambiguity, we will use Ω and $\mathcal{T}$ instead of $\Omega(A)$ and $\mathcal{T}(A)$.

Let Spec$(A)^*$ denote the set of all nonzero prime ideals of A. Then (Spec$(A)^*$, $\subseteq$) is a partially ordered set. Let C_α be a chain in Spec$(A)^*$. We define the C_α topology to be the topology on A with the ideals P^n, $P \in C_\alpha$, $n = 1, 2, \ldots$, as a base of zero neighborhoods in A.

We want to compare these topologies with each other.

Recall that an integral domain A is said to be h-local if every nonzero element of A is contained in only finitely many maximal ideals and if every nonzero prime ideal of A is contained in only one maximal ideal. Using Lemma 3.1, we give an easy proof that the $\mathcal{T}$ topology and the A topology are the same in an h-local Prüfer domain A.

LEMMA 3.2 [Mo, Lemma 13]. *Let A be an h-local Prüfer domain. Then the $\mathcal{T}$ topology on A is the same as the A topology.*

Proof. Let $w_i \in \Omega$ and $\alpha_i \in G_{w_i}^+$. Then there exists $a_i \in A^*$ such that $U_{w_i,\alpha_i} = a_i P(w_i) A_{P(w_i)}$. Now $U_{w_1,\alpha_1} \cap \cdots \cap U_{w_n,\alpha_n} \cap A = a_1 P(w_1) A_{P(w_1)} \cap \cdots \cap a_n P(w_n) A_{P(w_n)} \cap A \supseteq a_1 \cdots a_n A$. Conversely, let $a \in A^*$. Then since A is h-local, there exists only a finite number of maximal ideals $M_1, \ldots, M_n$ of A such that $a \in M_i$. Let w_i be the valuation corresponding to the valuation domain A_{M_i}, $i = 1, 2, \ldots, n$. Then $aA = \bigcap_{M \in \mathrm{Max}(A)} aA_M = \bigcap_{i=1}^n (aA_{M_i} \cap A) \supseteq \bigcap_{i=1}^n (aM_i A_{M_i} \cap A) = U_{w_1, w_1(a)} \cap \cdots \cap U_{w_n, w_n(a)} \cap A$.
∎

Let P be a prime ideal of A. We denote by ht P the *height* of P, i.e., the supremum of the length of chains of prime ideals contained in P.

LEMMA 3.3. *Let A be a Prüfer domain such that no minimal prime ideal is idempotent if there is any. Then the $\mathcal{T}$ topology on A is the same as the $\mathcal{C}$ topology.*

Proof. Let $w_i \in \Omega$ and $\alpha_i \in G_{w_i}^+$. Then there exists $a_i \in A^*$ such that $U_{w_i,\alpha_i} = a_i P(w_i) A_{P(w_i)}$, $i = 1, 2, \ldots, n$. If $a_i \notin P(w_i)$, then $a_i P(w_i) A_{P(w_i)} = P(w_i) A_{P(w_i)}$. Assume that $a_i \in P(w_i)$. Then $\sqrt{a_i A_{P(w_i)}} = Q_i A_{P(w_i)}$ for some nonzero prime ideal Q_i contained in $P(w_i)$. If ht $Q_i > 1$, then choose a nonzero prime ideal Q_i' properly contained in Q_i. Then by [Gi, Theorem (17.1)(5)], there exists $k_i \in \mathbb{N}$ such that $Q_i'^{k_i} A_{P(w_i)} \subseteq a_i A_{P(w_i)}$. Hence $Q_i'^{k_i+1} A_{P(w_i)} \subseteq a_i P(w_i) A_{P(w_i)}$. If ht $Q_i = 1$, then $Q_i \neq Q_i^2$. By [Gi, Theorem (17.3)], there exists $k_i \in \mathbb{N}$ such that $Q_i^{k_i} A_{P(w_i)} \subseteq a_i A_{P(w_i)}$. Hence $Q_i^{k_i+1} A_{P(w_i)} \subseteq a_i P(w_i) A_{P(w_i)}$. Thus in either case, there exist $Q_i \in \operatorname{Spec}(A)^*$ contained in $P(w_i)$ and $k_i \in \mathbb{N}$ such that $U_{w_1,\alpha_1} \cap \cdots \cap U_{w_n,\alpha_n} \cap A \supseteq Q_1^{k_1} \cap \cdots \cap Q_n^{k_n}$. Conversely, suppose $Q_i \in \operatorname{Spec}(A)^*$, $k_i \in \mathbb{N}$. Choose $a \in Q_1^{k_1} \cap \cdots \cap Q_n^{k_n} \setminus \{0\}$. Let w_i be the valuation corresponding to A_{Q_i}. Then $Q_1^{k_1} \cap \cdots \cap Q_n^{k_n} = Q_1^{k_1} A_{Q_1} \cap \cdots \cap Q_n^{k_n} A_{Q_n} \cap A \supseteq a Q_1 A_{Q_1} \cap \cdots \cap a Q_n A_{Q_n} \cap A = U_{w_1, w_1(a)} \cap \cdots \cap U_{w_n, w_n(a)} \cap A$. ∎

Let E be the ring of entire functions. It is well known that E is a Bezout domain, i.e., every finitely generated ideal of E is principal. Henriksen [He] has shown that if M is a maximal fixed ideal, then M is principal and ht $M = 1$ and if M is a maximal free ideal, then ht $M = \infty$. In fact, if P is an any prime free ideal of E, then ht $P = \infty$. Thus since E satisfies the conditions in Lemma 3.3, the $\mathscr{T}$ topology on E is the same as the $\mathscr{C}$ topology.

LEMMA 3.4. *Let A be an SFT Prüfer domain. Then the $\mathscr{T}$ topology on A, the A topology, and the $\mathscr{C}$ topology coincide.*

Proof. Let $w_i \in \Omega$ and $\alpha_i \in G_{w_i}^+$. Then there exists $a_i \in A^*$ such that $U_{w_i,\alpha_i} = a_i P(w_i) A_{P(w_i)}$. Thus $U_{w_1,\alpha_1} \cap \cdots \cap U_{w_n,\alpha_n} \cap A = a_1 P(w_1) A_{P(w_1)} \cap \cdots \cap a_n P(w_n) A_{P(w_n)} \cap A \supseteq a_1 \cdots a_n A$. Now let $a \in A^*$. Since A is an SFT Prüfer domain, by Proposition 2.1(2), (a) has only finitely many minimal prime divisors, say, $P_1, \ldots, P_n$. Thus we have $\sqrt{(a)} = P_1 \cap \cdots \cap P_n = P_1 \cdots P_n$. By Proposition 2.2, there exists $k \in \mathbb{N}$ such that $(a) \supseteq (P_1 \cdots P_n)^k = P_1^k \cap \cdots \cap P_n^k$. Finally, given $P_i \in \operatorname{Spec}(A)^*$ and $k_i \in \mathbb{N}$, choose $a \in P_1^{k_1} \cap \cdots \cap P_n^{k_n} \setminus \{0\}$. Let w_i be the valuation corresponding to A_{P_i}. Then $P_1^{k_1} \cap \cdots \cap P_n^{k_n} = P_1^{k_1} A_{P_1} \cap \cdots \cap P_n^{k_n} A_{P_n} \cap A \supseteq a P_1 A_{P_1} \cap \cdots \cap a P_n A_{P_n} \cap A = U_{w_1, w_1(a)} \cap \cdots \cap U_{w_n, w_n(a)} \cap A$. Thus these three topologies coincide. ∎

4. COMPLETIONS

Let X be a topological ring. X is said to be *complete* if every Cauchy filter converges. For details, see [Bo1].

DEFINITION 4.1. A completion of X is a pair $(\hat{X}, f)$, where $\hat{X}$ is a Hausdorff complete topological ring and $f\colon X \to \hat{X}$ is a continuous homomorphism satisfying the following conditions:

(a) $\operatorname{Ker} f = \overline{\{0\}}$, the closure of $\{0\}$ in X.

(b) The quotient topology of $f(X)$ coincides with the topology induced by $\hat{X}$.

(c) $f(X)$ is dense in $\hat{X}$.

It is well known that a completion exists and is unique in the following sense. Let $(\hat{X}, f)$ and $(\hat{Y}, g)$ be two completions of X. Then there is a unique isomorphism $\varphi\colon \hat{X} \to \hat{Y}$ which is also a homeomorphism such that $\varphi \circ f = g$ [Bo1]. Henceforth, we shall say that $\hat{X}$ is *the* completion of X and f is the canonical mapping of X into its completion.

Now we wish to consider the completions of a Prüfer domain A with respect to the topologies defined in Section 3.

Recall that for $v, w \in \Omega$, v and w (or R_v and R_w) are said to be *independent* if there exists no nontrivial valuation overring containing both R_v and R_w. Otherwise, v and w are said to be *dependent.* We say that a subset Ω' of Ω is independent if every two elements of Ω' are independent. We have the following approximation theorem for independent valuations. To make this paper self-contained, we state and prove the following well-known result.

PROPOSITION 4.2 [Gr, Proposition 24]. *Let A be a Prüfer domain. Let $w_1, w_2, \ldots, w_n \in \Omega$ be independent valuations with the value groups $G_{w_1}, G_{w_2}, \ldots, G_{w_n}$, respectively. Given $\beta_1 \in G_{w_1}^+, \ldots, \beta_n \in G_{w_n}^+$ and $t_1, \ldots, t_n \in A$, there exists $t \in A$ such that $w_i(t - t_i) = \beta_i$, $i = 1, 2, \ldots, n$.*

Proof. Let $I_i = \{x \in A \mid w_i(x) > \beta_i\}$. Then I_i is a nonzero proper ideal of A such that $I_i = I_i A_{P(w_i)} \cap A$. Since $A_{P(w_i)}$ is a valuation domain, $\sqrt{I_i A_{P(wi)}} = Q_i A_{P(w_i)}$ for some nonzero prime ideal Q_i contained in $P(w_i)$. So $\sqrt{I_i} = \sqrt{I_i A_{P(w_i)} \cap A} = \sqrt{I_i A_{P(w_i)}} \cap A = Q_i$. Since $w_1, \ldots, w_n$ are independent, $I_1, \ldots, I_n$ are relatively prime. Suppose not. Then there exist $i \neq j$ and a maximal ideal M of A such that $I_i + I_j \subseteq M$. It follows that $\sqrt{I_i} + \sqrt{I_j} \subseteq M$, i.e., $Q_i + Q_j \subseteq M$. Since A is a Prüfer domain, Q_i and Q_j are comparable. Assume that $Q_i \subseteq Q_j$. Then A_{Q_i} is a nontrivial valuation overring containing both $A_{P(w_i)}$ and $A_{P(w_j)}$, a contradiction. Applying the Chinese remainder theorem, we can find $y \in A$ such that $y - t_i \in I_i$, i.e., $w_i(y - t_i) > \beta_i$, $i = 1, 2, \ldots, n$. Since $\beta_i \in G_{w_i}^+$ and $R_{w_i} = A_{P(w_i)}$, there exist $x_i \in A$ such that $w_i(x_i) = \beta_i$, $i = 1, 2, \ldots, n$. Applying the Chinese remainder theorem again, we can find $d \in A$ such that $d - x_i \in I_i$, i.e., $w_i(d - x_i) > \beta_i$, $i = 1, 2, \ldots, n$. Note that $w_i(d) = w_i(x_i +$

$(d - x_i)) = \beta_i$, $i = 1, 2, \ldots, n$. Put $t = d + y \in A$. Then $w_i(t - t_i) = w_i(d + (y - t_i)) = \beta_i$, $i = 1, 2, \ldots, n$. ∎

Define an equivalence relation $\sim$ on the family Ω by $v \sim w$ if and only if v and w are dependent. Let Ω_0 be a family of representatives of the equivalence classes.

LEMMA 4.3. $\mathscr{T} = \sup\{\mathscr{T}_w | w \in \Omega\} = \sup\{\mathscr{T}_w | w \in \Omega_0\}$.

Proof. Let $w \in \Omega$ and $\alpha \in G_w^+$. Then by Lemma 3.1, there exists $a \in A^*$ such that $U_{w,\alpha} = aP(w)A_{P(w)}$. Let $v \in \Omega_0$ be the valuation that is dependent on w. Then there exists a nonzero prime ideal Q such that A_Q is a valuation overring containing both $A_{P(v)}$ and $A_{P(w)}$. So we have $U_{w,\alpha} = aP(w)A_{P(w)} \supseteq aQA_{P(w)} = aQA_Q = aQA_{P(v)} \supseteq abA_{P(v)} \supseteq abP(v)A_{P(v)} = U_{v, v(ab)}$ for any element $b \in Q \setminus \{0\}$. ∎

Let w be a valuation on the field K. Then by Lemma 4.3, the $\mathscr{T}(R_w)$-topology on K is the same as the $\mathscr{T}_w$-topology on K. Bourbaki considered the completion $\hat{K}^{,\mathscr{T}_w}$ of K with respect to the topology $\mathscr{T}_w$. Since we will often use results in Bourbaki, we include some of them here for easy reference.

Let G_w be the value group of w. Define on the set $G'_w = G_w \cup \{\infty\}$ a topology by setting $\overline{X}$ (= the closure of X) $= X \cup \{\infty\}$ for every nonempty subset X of G'_w, and $\overline{\varnothing} = \varnothing$. Then clearly $w\colon K \to G'_w$ is continuous and it induces the continuous extension $\hat{w}\colon \hat{K}^{,\mathscr{T}_w} \to G'_w$.

THEOREM 4.4 [Bo2, VI.5.3, Proposition 5]. *Let K be a field and let w be a valuation on K. Then we have the following statements.*

(1) *$\hat{K}^{,\mathscr{T}_w}$ is a topological field.*

(2) *The continuous extension $\hat{w}$ of w to $\hat{K}^{,\mathscr{T}_w}$ is a valuation and $G_{\hat{w}} = G_w$.*

(3) *The topology on $\hat{K}^{,\mathscr{T}_w}$ is the topology with the set $\{U_{\hat{w},\alpha} | \alpha \in G_{\hat{w}}^+\}$ as a base of zero neighborhoods in $\hat{K}^{,\mathscr{T}_w}$.*

(4) *$U_{\hat{w},\alpha} = \overline{U_{w,\alpha}}$, the closure of $U_{w,\alpha}$ in $\hat{K}^{,\mathscr{T}_w}$.*

(5) *$R_{\hat{w}} = \widehat{R_w}^{,\mathscr{T}_w}$ the completion of R_w with respect to the $\mathscr{T}_w$ topology on R_w.*

(6) $\hat{K}^{,\mathscr{T}_w} = \widehat{R_w}^{,\mathscr{T}_w}_{R_w \setminus \{0\}}$.

COROLLARY 4.5. $\operatorname{Spec}(\widehat{R_w}^{,\mathscr{T}_w}) = \{\hat{Q}^{,\mathscr{T}_w} | Q \in \operatorname{Spec}(R_w)\}$, *where $\hat{Q}^{,\mathscr{T}_w}$ is the completion of Q with respect to the subspace topology induced by R_w.*

Proof. Let $Q \in \operatorname{Spec}(R_w)^*$. We give Q the subspace topology and give R_w/Q the quotient topology induced by R_w. Then since these are linear topologies, by [Ma, Theorem 8.1], $0 \to \hat{Q}^{,\mathscr{T}_w} \to \widehat{R_w}^{,\mathscr{T}_w} \to \widehat{R_w/Q}$ is exact, so that $\widehat{R_w}^{,\mathscr{T}_w}/\hat{Q}^{,\mathscr{T}_w} \hookrightarrow \widehat{R_w/Q}$ naturally. Since R_w is a valuation domain, by

Lemmas 3.2 and 4.3, $\mathscr{T}$-topology on $R_w = \mathscr{T}_w$-topology on $R_w = R_w$-topology. Since $Q \neq (0)$, Q is open in R_w, and then $\widehat{R_w/Q}$ has the discrete topology, so that $\widehat{R_w/Q} \cong R_w/Q$. Thus we have $\widehat{R_w}^{,\mathscr{T}_w}/\hat{Q}^{,\mathscr{T}_w} \hookrightarrow R_w/Q$. Since the embedding is clearly onto, $\widehat{R_w}^{,\mathscr{T}_w}/\hat{Q}^{,\mathscr{T}_w} \cong R_w/Q$. Thus $\hat{Q}^{,\mathscr{T}_w}$ is a prime ideal of $\widehat{R_w}^{,\mathscr{T}_w}$ such that $\hat{Q}^{,\mathscr{T}_w} \cap R_w = Q$ for each $Q \in \operatorname{Spec}(R_w)^*$. From Theorem 4.4(2) and [Gi, Corollary (17.9)], the conclusion follows: Let $Q_0 \in \operatorname{Spec}(\widehat{R_w}) \Rightarrow \hat{w}(\widehat{R_w} \setminus Q_0) = w(R_w \setminus Q) = \hat{w}(\widehat{R_w} \setminus \hat{Q})$ for some $Q \in \operatorname{Spec}(R_w) \Rightarrow Q_0 = \hat{Q}$. ∎

We denote by $(A, \mathscr{T})$ and $(A, \mathscr{T}_w)$ the topological space A with the topologies $\mathscr{T}$ and $\mathscr{T}_w$, respectively. Let us denote by $\hat{A}^{,\mathscr{T}}$ and $\hat{A}^{,\mathscr{T}_w}$ the completions of $(A, \mathscr{T})$ and $(A, \mathscr{T}_w)$, respectively.

PROPOSITION 4.6. $\hat{A}^{,\mathscr{T}} \cong \prod_{w \in \Omega_0} \hat{A}^{,\mathscr{T}_w}$.

Proof. Since $\hat{A}^{,\mathscr{T}_w}$ is a Hausdorff complete topological ring, by [Bo1, II.3.5, Proposition 10], so is the product space $\prod_{w \in \Omega_0} \hat{A}^{,\mathscr{T}_w}$. Let $f_w\colon (A, \mathscr{T}_w) \to \hat{A}^{,\mathscr{T}_w}$ be the canonical mapping. Then by Lemma 4.3, the mapping $f = \prod_{w \in \Omega_0} f_w\colon (A, \mathscr{T}) \to \prod_{w \in \Omega_0} \hat{A}^{,\mathscr{T}_w}$ defined by $a \mapsto \prod_{w \in \Omega_0} f_w(a)$ is a continuous homomorphism and obviously the conditions (a) and (b) in Definition 4.1 are satisfied. Now we claim that $f(A)$ is dense in $\prod_{w \in \Omega_0} \hat{A}^{,\mathscr{T}_w}$. Since $(A, \mathscr{T}_w)$ and $(A, \mathscr{T})$ are Hausdorff, we may identify $(A, \mathscr{T}_w)$ with $f_w(A)$, and $(A, \mathscr{T})$ with $f(A)$. Let V be an open neighborhood in $\prod_{w \in \Omega_0} \hat{A}^{,\mathscr{T}_w}$. Since by [Bo1, III.3.4, Proposition 7], the topology on $\hat{A}^{,\mathscr{T}_w}$ is the topology with the sets $\overline{U_{w,\alpha} \cap A}$ (= the closure of $U_{w,\alpha} \cap A$ in $\hat{A}^{,\mathscr{T}_w}$), $\alpha \in G_w^+$, as a base of zero neighborhoods in $\hat{A}^{,\mathscr{T}_w}$, there exist $\prod_{w \in \Omega_0} y_w \in V$ and $U_{w_1,\alpha_1}, \ldots, U_{w_n,\alpha_n}$, $w_i \in \Omega_0$, $\alpha_i \in G_{w_i}^+$ such that $\prod_{w \in \Omega_0}(y_w + V_w) \subseteq V$, where $V_w = \hat{A}^{,\mathscr{T}_w}$ for $w \neq w_i$ and $V_{w_i} = \overline{U_{w_i,\alpha_i} \cap A}$ for $i = 1, 2, \ldots, n$. Note that $\prod_{w \in \Omega_0}(y_w + V_w)$ is open in $\prod_{w \in \Omega_0} \hat{A}^{,\mathscr{T}_w}$. Since $(A, \mathscr{T}_w)$ is a subspace of $(K, \mathscr{T}_w)$, the closure $\bar{A}$ of A in $\hat{K}^{,\mathscr{T}_w}$ is complete by [Bo1, II.3.4, Proposition 8]. Thus $\hat{A}^{,\mathscr{T}_w} \cong \bar{A} \subseteq \hat{K}^{,\mathscr{T}_w}$. Let $\overline{U_{w,\alpha}}$ be the closure of $U_{w,\alpha}$ in $\hat{K}^{,\mathscr{T}_w}$. Then $\overline{U_{w,\alpha} \cap A} = \overline{U_{w,\alpha}} \cap \hat{A}^{,\mathscr{T}_w} = U_{\hat{w},\alpha} \cap \hat{A}^{,\mathscr{T}_w}$ by Theorem 4.4. Since $(A, \mathscr{T}_{w_i})$ is dense in $\hat{A}^{,\mathscr{T}_{w_i}}$, there exists $x_i \in A$ such that $x_i \in y_{w_i} + U_{\widehat{w_i},\alpha_i} \cap \hat{A}^{,\mathscr{T}_{w_i}}$, i.e., $\widehat{w_i}(x_i - y_{w_i}) > \alpha_i$, $i = 1, 2, \ldots, n$. Since $w_1, \ldots, w_n$ are independent, by Proposition 4.2, there exists $a \in A$ such that $w_i(a - x_i) > \alpha_i$, $i = 1, 2, \ldots, n$. Then, $\widehat{w_i}(a - y_{w_i}) = \widehat{w_i}(a - x_i + x_i - y_{w_i}) \geq \min(\widehat{w_i}(a - x_i), \widehat{w_i}(x_i - y_{w_i})) > \alpha_i$, i.e., $a \in y_{w_i} + U_{\widehat{w_i},\alpha_i}$, $i = 1, 2, \ldots, n$. Thus V contains an element a of A. Therefore, $\prod_{w \in \Omega_0} \hat{A}^{,\mathscr{T}_w}$ is a completion of $(A, \mathscr{T})$, i.e., $\prod_{w \in \Omega_0} \hat{A}^{,\mathscr{T}_w} \cong \hat{A}^{,\mathscr{T}}$. ∎

As we can see in the proof of Proposition 4.2, for a Prüfer domain A, v and w being dependent is equivalent to $P(v) \cap P(w)$ containing a nonzero prime ideal of A. We define an equivalence relation $\sim$ on the set

$\mathrm{Spec}(A)^*$ by

$$P_1 \sim P_2 \text{ if and only if } P_1 \cap P_2 \text{ contains a nonzero prime ideal of } A.$$

Let $\mathscr{C}$ be the set of maximal chains of $\mathrm{Spec}(A)^*$. The relation $\sim$ on $\mathscr{C}$ defined by

$$C_\alpha \sim C_\beta \text{ if and only if } P_1 \sim P_2 \text{ for some } P_1 \in C_\alpha \text{ and } P_2 \in C_\beta$$

is also an equivalence relation. In fact,

$$C_\alpha \sim C_\beta \text{ if and only if } P_1 \sim P_2 \text{ for all } P_1 \in C_\alpha \text{ and all } P_2 \in C_\beta.$$

Let $\{\overline{C_\alpha}\}_{\alpha \in \Lambda}$ be the set of all equivalence classes of $\mathscr{C}$. We denote by $(A, \mathscr{C})$ and (A, C_α) the topological space A with the topologies induced by $\mathscr{C}$ and C_α, respectively. Let us denote by $\hat{A}^{,\mathscr{C}}$ and $\hat{A}^{,C_\alpha}$ the completions of $(A, \mathscr{C})$ and (A, C_α), respectively. Then a similar proof to that of Proposition 4.6 implies the following result.

PROPOSITION 4.7. *$\hat{A}^{,\mathscr{C}} \cong \prod_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$, where $\{C_\alpha\}_{\alpha \in \Lambda}$ is a collection of representatives of the equivalence classes of $\mathscr{C}$.*

Proof. Since $\hat{A}^{,C_\alpha}$ is a Hausdorff complete topological ring, by [Bo1, II.3.5, Proposition 10], so is the product space $\prod_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$. Let $f_\alpha \colon (A, C_\alpha) \to \hat{A}^{,C_\alpha}$ be the canonical mapping. Then the mapping $f = \prod_{\alpha \in \Lambda} f_\alpha \colon (A, \mathscr{C}) \to \prod_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$ defined by $a \mapsto \prod_{\alpha \in \Lambda} f_\alpha(a)$ is a continuous homomorphism. Clearly, $\operatorname{Ker} f = \bigcap_{\alpha \in \Lambda} \operatorname{Ker} f_\alpha = \bigcap_{P \in C_\alpha,\, \alpha \in \Lambda,\, n \in \mathbb{N}} P^n = \bigcap_{P \in \mathrm{Spec}(A)^*,\, n \in \mathbb{N}} P^n = \overline{\{0\}}$, the closure of $\{0\}$ in $(A, \mathscr{C})$. We claim that the quotient topology of $f(A)$ coincides with the subspace topology induced by the product space $\prod_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$. Let $P_1, \dots, P_n \in \mathrm{Spec}(A)^*$, $k_1, \dots, k_n \in \mathbb{N}$. Then there exists $\alpha_i \in \Lambda$ such that $P_i \sim P_i'$ for some $P_i' \in C_{\alpha_i}$, i.e., $P_i \cap P_i'$ contains a nonzero prime ideal of A. This implies that P_i contains some $Q_i \in C_{\alpha_i}$ since C_{α_i} is a maximal chain in $\mathrm{Spec}(A)^*$. Thus $Q_1^{k_1} \cap \cdots \cap Q_n^{k_n} \subseteq P_1^{k_1} \cap \cdots \cap P_n^{k_n}$. Since $Q_1^{k_1} \cap \cdots \cap Q_n^{k_n}$ is obviously contained in the inverse image under f of the topology of the product space $\prod_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$, our claim has been verified. Now by Definition 4.1, it remains to show that $f(A)$ is dense in $\prod_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$. Let V be an open neighborhood in $\prod_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$. Since by [Bo1, III.3.4, Proposition 7], the sets $\overline{f_\alpha(P^k)}$ (= the closure of $f_\alpha(P^k)$ in $\hat{A}^{,C_\alpha}$), $P \in C_\alpha$, $k \in \mathbb{N}$, is a base of zero neighborhoods in $\hat{A}^{,C_\alpha}$, there exist $\prod_{\alpha \in \Lambda} y_\alpha \in V$ and $P_1 \in C_{\alpha_1}, \dots, P_n \in C_{\alpha_n}$, $k_1, \dots, k_n \in \mathbb{N}$ such that $\prod_{\alpha \in \Lambda}(y_\alpha + V_\alpha) \subseteq V$, where $V_\alpha = \hat{A}^{,C_\alpha}$ for $\alpha \neq \alpha_i$, $V_{\alpha_i} = \overline{f_{\alpha_i}(P_i^{k_i})}$ for $i = 1, 2, \dots, n$. Since $f_\alpha(A)$ is dense in $\hat{A}^{,C_\alpha}$, there exist $x_1, \dots, x_n \in A$ such that $f_{\alpha_i}(x_i) \in y_{\alpha_i} + \overline{f_{\alpha_i}(P_i^{k_i})}$. Since $C_{\alpha_1}, \dots, C_{\alpha_n}$ are independent, $P_1, \dots, P_n$ are independent. Let w_i be the valuation corresponding to A_{P_i}. Then $w_1, \dots, w_n$ are independent, and so by Proposition 4.2 and Lemma 3.1, there exists $a \in A$ such that

$a - x_i \in P_i^{k_i}$, $i = 1, 2, \ldots, n$. Therefore, $f_{\alpha_i}(a) - y_{\alpha_i} = (f_{\alpha_i}(a) - f_{\alpha_i}(x_i)) + (f_{\alpha_i}(x_i) - y_{\alpha_i}) \in \overline{f_{\alpha_i}(P_i^{k_i})}$, $i = 1, 2, \ldots, n$. Thus V contains an element $f(a)$ of $f(A)$. Therefore, $f(A)$ is dense in $\prod_{\alpha \in \Lambda} \hat{A}^{, C_\alpha}$, and so $\hat{A}^{, \mathscr{C}} \cong \prod_{\alpha \in \Lambda} \hat{A}^{, C_\alpha}$. ∎

Remark 4.8. For each $w \in \Omega_0$, let C_w be a maximal chain in Spec$(A)^*$ containing $P(w)$. Then $\{\overline{C_w}\}_{w \in \Omega_o}$ is the set of all equivalence classes of $\mathscr{C}$. Therefore, $\hat{A}^{, \mathscr{C}} \cong \prod_{w \in \Omega_0} \hat{A}^{, C_w}$.

Recall that a Prüfer ring (resp., Bezout ring) is a ring in which every finitely generated regular ideal is invertible (resp., principal). By [Hu, Theorem 6.2], R is a Prüfer ring if and only if $(R_{[M]}, [M]R_{[M]})$ is a valuation pair for each regular maximal ideal M of R.

Later it will turn out that $\hat{A}^{, C_\alpha}$ is a Prüfer domain, a valuation domain, or a Bezout domain in several important cases. This together with Propositions 4.6 and 4.7 naturally leads us to consider the direct product of Prüfer domains and Bezout domains. We show that the direct product of Prüfer domains (resp., Bezout domains) is a Prüfer ring (resp., Bezout ring).

PROPOSITION 4.9. *Let $B = \prod_\alpha B_\alpha$. If B_α is a Prüfer domain (resp., Bezout domain) for each α, then B is a Prüfer ring (resp., Bezout ring).*

Proof. Note that the total quotient ring $T(B)$ of B is isomorphic to the direct product of the quotient fields K_α of B_α. Let $I = (a_1, \ldots, a_n)$ be a regular ideal of $\prod_\alpha B_\alpha$. Write $a_1 = \prod_\alpha a_{1,\alpha}, \ldots, a_n = \prod_\alpha a_{n,\alpha}$. Since I is regular, I contains a regular element. So we may assume that a_1 is regular. It follows that $a_{1,\alpha} \neq 0$ for all α. For each α, consider the nonzero finitely generated ideal $(a_{1,\alpha}, \ldots, a_{n,\alpha})$ of B_α. Since B_α is a Prüfer domain (resp., Bezout domain), it is invertible (resp., principal). Therefore, there exist $x_{1,\alpha}, \ldots, x_{n,\alpha} \in K_\alpha$ such that $\sum_{i=1}^n a_{i,\alpha} x_{i,\alpha} = 1$ and $a_{i,\alpha} x_{j,\alpha} \in B_\alpha$ for all $i, j = 1, 2, \ldots, n$ (resp., there exists $a_\alpha \in B_\alpha$ such that $(a_{1,\alpha}, \ldots, a_{n,\alpha}) = (a_\alpha)$). Let $x_1 = \prod_\alpha x_{1,\alpha}, \ldots, x_n = \prod_\alpha x_{n,\alpha} \in \prod_\alpha K_\alpha = T(B)$ (resp., let $a = \prod_\alpha a_\alpha$). Then $\sum_{i=1}^n a_i x_i = 1$ and $a_i x_j \in \prod_\alpha B_\alpha$ for all $i, j = 1, 2, \ldots, n$. Thus I is invertible (resp., $I = (a)$, i.e., I is principal). Therefore, B is a Prüfer ring (resp., Bezout ring). ∎

5. $\hat{A}^{, \mathscr{T}}$, A AN h-LOCAL PRÜFER DOMAIN

Let A be an h-local Prüfer domain. Denote by $\hat{A}^{, A}$ the completion of A in the A-topology and $\widehat{A_M}^{, A_M}$ the completion of A_M in the A_M-topology, $M \in \mathrm{Max}(A)$. Although it is already known that $\hat{A}^{, A} \cong \prod_{M \in \mathrm{Max}(A)} \widehat{A_M}^{, A_M}$ and that the $\mathscr{T}$ topology on A is the same as the A topology [M, Mo], for the sake of completeness, we present its proof.

THEOREM 5.1. *Let A be an h-local Prüfer domain. Then*

(1) $\hat{A}^{\cdot\mathscr{T}_w} = \widehat{A_{M_0}}^{\cdot A_{M_0}}$, *where M_0 is the maximal ideal of A containing $P(w)$;*

(2) $\hat{A}^{\cdot\mathscr{T}}$ *is a Bezout ring.*

Proof. (1) As in Lemma 3.2, we can show that $(A, \mathscr{T}_w)$ is a subspace of A_{M_0} with the A_{M_0}-topology. If we show that A is dense in $\widehat{A_{M_0}}^{\cdot A_{M_0}}$, then $\hat{A}^{\cdot\mathscr{T}_w} \cong \widehat{A_{M_0}}^{\cdot A_{M_0}}$. In fact, it suffices to show that A is dense in A_{M_0}. Given $\frac{a}{s} \in A_{M_0}$, $a \in A$, $s \in A \setminus M_0$, and bA_{M_0}, where $b \in A^*$, we must show that there exists $c \in A$ such that $c \in \frac{a}{s} + bA_{M_0}$. If $a = 0$, we may choose $c = 0$. So we may assume that $a \neq 0$. Let $\Omega_0 = \{v \in \Omega \mid v$ is the valuation corresponding to A_M, $M \in \mathrm{Max}(A)\}$. Since A is h-local, Ω_0 is a representing family of the independent valuations of Ω. Since every nonzero element of A is contained in only a finite number of maximal ideals, the set $\{v \in \Omega_0 \mid v(\frac{a}{s}) < 0\}$ is finite. Let w_0 be the valuation corresponding to A_{M_0}. By Proposition 4.2, there exists $t \in A$ such that $w_0(t) = w_0(b)$, $v(t) = 0$ for each $v \in \Omega_0$ such that $v(\frac{a}{s}) < 0$. Consider the finitely generated fractional ideal $(\frac{a}{s}t, \frac{s}{a})$ of A. Since A is a Prüfer domain, it has the inverse J. Let $v \in \Omega_0$. If $v(\frac{s}{a}) > 0$, then $v(\frac{a}{s}) < 0$ so that $v(t) = 0$. This implies that $v(\frac{a}{s}t) = v(\frac{a}{s}) < 0$. Therefore, $v(J) \geq 0$, i.e., $v(x) \geq 0$ for all $x \in J$. Thus $J \subseteq \bigcap_{M \in \mathrm{Max}(A)} A_M = A$. Now from $(\frac{a}{s}t, \frac{s}{a})J = A$, it follows that for some $c_1, c_2 \in J$, we have $\frac{a}{s}tc_1 + \frac{s}{a}c_2 = 1$, i.e., $c_2 - \frac{a}{s} = -(\frac{a}{s})^2 tc_1$, so that $w_0(c_2 - \frac{a}{s}) = 2w_0(\frac{a}{s}) + w_0(t) + w_0(c_1) \geq w_0(t) = w_0(b)$, i.e., $c_2 - \frac{a}{s} \in bA_{M_0}$.

(2) By Proposition 4.6, $\hat{A}^{\cdot\mathscr{T}} \cong \prod_{w \in \Omega_0} \hat{A}^{\cdot\mathscr{T}_w}$ and by (1), $\hat{A}^{\cdot\mathscr{T}_w} \cong \widehat{A_{M_0}}^{\cdot A_{M_0}}$, where M_0 is the maximal ideal containing $P(w)$. Let w_0 be the valuation corresponding to A_{M_0}. Since every valuation domain is an h-local Prüfer domain, by Lemmas 3.2 and 4.3, $\widehat{A_{M_0}}^{\cdot A_{M_0}} \cong \widehat{A_{M_0}}^{\cdot\mathscr{T}_{w_0}}$. By Theorem 4.4(5), $\widehat{A_{M_0}}^{\cdot\mathscr{T}_{w_0}}$ is a valuation domain. Therefore, applying Proposition 4.9, we conclude that $\hat{A}^{\cdot\mathscr{T}}$ is a Bezout ring. ∎

6. $\hat{E}^{\cdot\mathscr{T}}$, E THE RING OF ENTIRE FUNCTIONS

Let A be a Prüfer domain and let C_α be a maximal chain in $\mathrm{Spec}(A)^*$. Then the C_α topology is the linear topology defined by $\{P^n \mid P \in C_\alpha, n \in \mathbb{N}\}$. Therefore, by [Ma, p. 55], the inverse limit $\varprojlim_{P \in C_\alpha, n \in \mathbb{N}} A/P^n$ is the completion of (A, C_α). For each $P \in C_\alpha$, we denote by $\hat{A}^{\cdot P}$ the P-adic completion of A, i.e., $\hat{A}^{\cdot P} \cong \varprojlim_{n \in \mathbb{N}} A/P^n$. We show that

$$\varprojlim_{P \in C_\alpha}(\varprojlim_{n \in \mathbb{N}} A/P^n) \cong \varprojlim_{P \in C_\alpha,\, n \in \mathbb{N}} A/P^n.$$

LEMMA 6.1. $\hat{A}^{,C_\alpha} \cong \varprojlim_{P\in C_\alpha}\hat{A}^{,P}$.

Proof. Since for each $Q \subseteq P$ in C_α, there exists a natural map from A/Q^n to A/P^n, $n \in \mathbb{N}$, there exists a natural map f_{PQ}: $\hat{A}^{,Q} \to \hat{A}^{,P}$. Therefore, we can construct the inverse system $(\hat{A}^{,P}, f_{PQ})$. Since $\hat{A}^{,P}$ is a Hausdorff complete topological ring, by [Bo1, II.3.5, Corollary to Proposition 10], so is $\varprojlim_{P\in C_\alpha}\hat{A}^{,P}$. For each $P \in C_\alpha$, let f_P: $A \to \hat{A}^{,P}$ be the natural map. Then f_P is the canonical mapping of A into its P-adic completion $\hat{A}^{,P}$ and $f_{PQ} \circ f_Q = f_P$ whenever $Q \subseteq P$. Then the mapping $f = \prod_{P\in C_\alpha} f_P$: $(A, C_\alpha) \to \varprojlim_{P\in C_\alpha}\hat{A}^{,P}$ $(\subseteq \prod_{P\in C_\alpha}\hat{A}^{,P})$ defined by $a \mapsto \prod_{P\in C_\alpha} f_P(a)$ is a well-defined continuous homomorphism, $\operatorname{Ker} f = \bigcap_{P\in C_\alpha}\operatorname{Ker} f_P = \bigcap_{P\in C_\alpha,\, n\in\mathbb{N}} P^n = \overline{\{0\}}$, the closure of $\{0\}$ in (A, C_α), and obviously the C_α topology on A is the inverse image under f of the topology of $\varprojlim_{P\in C_\alpha}\hat{A}^{,P}$. Let π_P: $\varprojlim_{P\in C_\alpha}\hat{A}^{,P} \to \hat{A}^{,P}$ be the natural projection. Then $\pi_P \circ f = f_P$. By [Bo1, I.4.4, Corollary to Proposition 9], $\overline{f(A)}$ (= the closure of $f(A)$ in $\varprojlim_{P\in C_\alpha}\hat{A}^{,P}$) = $\varprojlim_{P\in C_\alpha}\overline{\pi_P(f(A))}$($\overline{\pi_P(f(A))}$ denotes the closure of $\pi_P(f(A))$ in $\hat{A}^{,P}$) = $\varprojlim_{P\in C_\alpha}\overline{f_P(A)} = \varprojlim_{P\in C_\alpha}\hat{A}^{,P}$. Thus $f(A)$ is dense in $\varprojlim_{P\in C_\alpha}\hat{A}^{,P}$. Therefore, $\hat{A}^{,C_\alpha} \cong \varprojlim_{P\in C_\alpha}\hat{A}^{,P}$. ∎

LEMMA 6.2. *Let P be a nonzero prime ideal of A. Then $\hat{A}^{,P} = \bigcap_{P\subseteq M\in \operatorname{Max}(A)}\widehat{A_M}^{,PA_M}$, where $\widehat{A_M}^{,PA_M}$ is the PA_M-adic completion of A_M.*

Proof. Since $P^nA_P = P^nA_M$ [Gi, Theorem (17.6)(b)] and $P^nA_M \cap A = P^n$, $n \in \mathbb{N}$, we obtain the natural embeddings $\hat{A}^{,P} \hookrightarrow \widehat{A_M}^{,PA_M} \hookrightarrow \widehat{A_P}^{,PA_P}$ for all $M \in \operatorname{Max}(A)$ containing P, so that the intersection $\bigcap_{P\subseteq M\in\operatorname{Max}(A)}\widehat{A_M}^{,PA_M}$ is meaningful. Let $R = A/P^n$. Then by [Hu, Theorem 6.1], $R = \bigcap_{N\in\operatorname{Max}(R)} R_{(N)}$, where $R_{(N)} = \{\frac{a}{b} \in T(R) \mid a, b \in R,\ b \notin N$, and b is regular$\}$. Since $\operatorname{Max}(R) = \{M/P^n \mid M \in \operatorname{Max}(A)$ such that $M \supseteq P\}$ and $Z(R)$ (= the set of zero divisors of R) = P/P^n,

$$A/P^n = R = \bigcap_{N\in\operatorname{Max}(R)} R_{(N)} = \bigcap_{P\subseteq M\in\operatorname{Max}(A)} R_{(M/P^n)}$$

$$= \bigcap_{P\subseteq M\in\operatorname{Max}(A)} R_{M/P^n} = \bigcap_{P\subseteq M\in\operatorname{Max}(A)} A_M/P^nA_M.$$

Therefore,

$$\hat{A}^{,P} \cong \varprojlim_{n\in\mathbb{N}} A/P^n \subseteq \prod_{n\in\mathbb{N}} A/P^n = \prod_{n\in\mathbb{N}}\left(\bigcap_{P\subseteq M\in\operatorname{Max}(A)} A_M/P^nA_M\right)$$

$$= \bigcap_{P\subseteq M\in\operatorname{Max}(A)}\left(\prod_{n\in\mathbb{N}} A_M/P^nA_M\right).$$

Let $x \in \widehat{A_M}^{,PA_M}$ for all $M \in \operatorname{Max}(A)$ containing P. Then $x \in \bigcap_{P\subseteq M\in\operatorname{Max}(A)}(\prod_{n\in N} A_M/P^nA_M) = \prod_{n\in\mathbb{N}} A/P^n$. Consider the following commutative diagram:

$$\begin{array}{ccc} \varprojlim_{n\in\mathbb{N}} A_M/P^nA_M & \hookrightarrow & \prod_{n\in\mathbb{N}} A_M/P^nA_M \\ \uparrow & & \uparrow \\ \varprojlim_{n\in\mathbb{N}} A/P^n & \hookrightarrow & \prod_{n\in\mathbb{N}} A/P^n \end{array}$$

Since $(\varprojlim_{n\in\mathbb{N}} A_M/P^nA_M) \cap (\prod_{n\in\mathbb{N}} A/P^n) = \varprojlim_{n\in\mathbb{N}} A/P^n$, $x \in \varprojlim_{n\in\mathbb{N}} A/P^n \cong \hat{A}^{,P}$. ∎

Let C_α be a maximal chain in Spec$(A)^*$. Let Max(C_α) denote the set $\{M \in \mathrm{Max}(A) \mid M$ contains some $P \in C_\alpha\}$ and $J(C_\alpha) = \bigcap_{M\in\mathrm{Max}(C_\alpha)} M$. We call $J(C_\alpha)$ the Jacobson radical of C_α.

LEMMA 6.3. *$\hat{A}^{,C_\alpha} = \bigcap_{M\in\mathrm{Max}(C_\alpha)} \widehat{A_M}^{,C_{\alpha,M}}$, where $C_{\alpha,M} = \{PA_M \mid P \in C_\alpha$ and $P \subseteq M\}$ (which is a chain in Spec$(A_M)^*$).*

Proof. If C_α has minimal element P, then the C_α topology on A is the same as the P-adic topology and the $C_{\alpha,M}$ topology on A_M is the same as the PA_M-adic topology. Therefore in this case Lemma 6.3 is just Lemma 6.2. Assume that C_α has no minimal element. Let $M_1, M_2 \in \mathrm{Max}(C_\alpha)$. Then there exists $P_0 \in C_\alpha$ such that $P_0 \subseteq M_1 \cap M_2$. Let w_i (resp., v) be the valuation corresponding to A_{M_i} (resp., A_{P_0}), $i = 1,2$. Since A_{M_i} is a valuation domain and ht$(PA_{M_i}) = \infty$ for all $PA_{M_i} \in \mathrm{Spec}(A_{M_i})^*$, $\widehat{A_{M_i}}^{,C_{\alpha,M_i}} = \widehat{A_{M_i}}^{,\mathcal{T}(A_{M_i})}$ by Lemma 3.3. In view of Lemma 4.3, it is equal to $\widehat{A_{M_i}}^{,\mathcal{T}_{w_i}}$. Since by Theorem 4.4, $\hat{K}^{,\mathcal{T}_{w_i}}$ is the quotient field of $\widehat{A_{M_i}}^{,\mathcal{T}_{w_i}}$ and by Lemma 3.3, $\hat{K}^{,\mathcal{T}_{w_i}} = \hat{K}^{,\mathcal{T}_v}$, $\widehat{A_{M_1}}^{,C_{\alpha,M_1}}$ and $\widehat{A_{M_2}}^{,C_{\alpha,M_2}}$ have the same quotient field $\hat{K}^{,\mathcal{T}_v}$. Thus, the intersection $\bigcap_{M\in\mathrm{Max}(C_\alpha)} \widehat{A_M}^{,C_{\alpha,M}}$ is meaningful. Note that $\widehat{A_M}^{,C_{\alpha,M}} \cong \varprojlim_{PA_M\in C_{\alpha,M}} \widehat{A_M}^{,PA_M} \cong \varprojlim_{P\in C_\alpha} \widehat{A_M}^{,PA_M}$ $(\subseteq \prod_{P\in C_\alpha} \widehat{A_M}^{,PA_M})$. Let $x \in \bigcap_{M\in\mathrm{Max}(C_\alpha)} \widehat{A_M}^{,C_{\alpha,M}}$. Choose $M_0 \in \mathrm{Max}(C_\alpha)$. Since $x \in \widehat{A_{M_0}}^{,C_{\alpha,M_0}}$, we can write $x = \prod_{P\in C_\alpha} x_P \in \varprojlim_{P\in C_\alpha} \widehat{A_{M_0}}^{,PA_{M_0}}$. For each $P \in C_\alpha$, $x_P \in \widehat{A_M}^{,PA_M}$ for all $M \in \mathrm{Max}(C_\alpha)$. In particular, $x_P \in \bigcap_{P\subseteq M\in\mathrm{Max}(C_\alpha)} \widehat{A_M}^{,PA_M} = \hat{A}^{,P}$ by Lemma 6.2. Thus $x \in \prod_{P\in C_\alpha} \hat{A}^{,P}$. Let M be the maximal ideal contained in C_α, i.e., $\{M\} = C_\alpha \cap \mathrm{Max}(C_\alpha)$. Consider the following commutative diagram:

$$\begin{array}{ccc} \varprojlim_{P\in C_\alpha} \widehat{A_M}^{,PA_M} & \hookrightarrow & \prod_{P\in C_\alpha} \widehat{A_M}^{,PA_M} \\ \uparrow & & \uparrow \\ \varprojlim_{P\in C_\alpha} \hat{A}^{,P} & \hookrightarrow & \prod_{P\in C_\alpha} \hat{A}^{,P} \end{array}$$

Since $\left(\varprojlim_{P\in C_\alpha}\widehat{A_M}^{,PA_M}\right) \cap (\prod_{P\in C_\alpha}\hat{A}^{,P}) = \varprojlim_{P\in C_\alpha}\hat{A}^{,P}$, $x = \prod_{P\in C_\alpha}x_P \in \varprojlim_{P\in C_\alpha}\hat{A}^{,P} \cong \hat{A}^{,C_\alpha}$ by Lemma 6.1. ∎

THEOREM 6.4. *Let E be the ring of entire functions and let C_α be a maximal chain in* Spec$(E)^*$. *Then*

(1) $\hat{E}^{,C_\alpha} \cong \widehat{E_{M_\alpha}}^{,\mathcal{T}_{v_\alpha}}$, *where* $\{M_\alpha\} = \mathrm{Max}(C_\alpha)$ *and v_α is the valuation corresponding to E_{M_α}, and*

(2) $\hat{E}^{,\mathcal{T}}$ *is a Bezout ring.*

Proof. (1) $\hat{E}^{,C_\alpha} \cong \varprojlim_{P\in C_\alpha}\hat{E}^{,P}$ by Lemma 6.1. Since every nonzero prime ideal of E is contained in a unique maximal ideal [He, Theorem 6], Max(C_α) has only one element $= M_\alpha$ so that by Lemmas 6.2 and 6.1, $\varprojlim_{P\in C_\alpha}\hat{E}^{,P} = \varprojlim_{P\in C_\alpha}\widehat{E_{M_\alpha}}^{,PE_{M_\alpha}} \cong \widehat{E_{M_\alpha}}^{,C_\alpha,M_\alpha}$, which is isomorphic to $\widehat{E_{M_\alpha}}^{,\mathcal{T}_{v_\alpha}}$ by Lemmas 3.3 and 4.3.

(2) By Lemma 3.3, Proposition 4.7, and part (1), $\hat{E}^{,\mathcal{T}} = \hat{E}^{,\mathscr{C}} \cong \prod_{\alpha\in\Lambda}\hat{E}^{,C_\alpha} = \prod_{\alpha\in\Lambda}\widehat{E_{M_\alpha}}^{,\mathcal{T}_{v_\alpha}}$. Since by Theorem 4.4, each $\widehat{E_{M_\alpha}}^{,\mathcal{T}_{v_\alpha}}$ is a valuation domain, Proposition 4.9 implies that $\hat{E}^{,\mathcal{T}}$ is a Bezout ring. (Note that $|\Lambda| = |\mathrm{Max}(E)|$.) ∎

7. $\hat{A}^{,\mathcal{T}}$, A AN SFT PRÜFER DOMAIN

Let A be an SFT Prüfer domain and let C_α be a maximal chain in Spec$(A)^*$. Since $\bigcap_{P\in C_\alpha,\, n\in\mathbb{N}} P^n = \{0\}$, the topological ring (A, C_α) is Hausdorff, and so we may assume that $A \subseteq \hat{A}^{,C_\alpha}$. For each subring B (not necessarily with identity) of A, let us denote by $\bar{B}^{,C_\alpha}$ the closure of B in $\hat{A}^{,C_\alpha}$. In fact, by [Bo1, II.3.4, Proposition 8], $\bar{B}^{,C_\alpha}$ can be regarded as the completion $\hat{B}^{,C_\alpha}$ of B with respect to the subspace topology induced by (A, C_α). Henceforth we will use similar notation for other topologies.

Although [KP1, Corollary 17] is stated for the finite-dimensional case, its proof is also valid for the infinite-dimensional case. Thus every nonzero prime ideal of $\hat{A}^{,P}$ is of the form $\hat{Q}^{,P}$ for some prime ideal Q of A such that $Q \supseteq P$. This result is crucial in proving the next result.

LEMMA 7.1. *Let N be a prime ideal of $\hat{A}^{,C_\alpha}$ such that $N \cap A \neq (0)$. Then $\hat{A}^{,C_\alpha}_N$ is a valuation domain.*

Proof. Let $P_0 = N \cap A$. Since A is an SFT Prüfer domain, there exists a finitely generated ideal J of A such that $P_0^2 \subseteq J \subseteq P_0$ (Proposition 2.2). We claim that P_0 contains some $P \in C_\alpha$. Note that $J(\varprojlim_{P\in C_\alpha}\hat{A}^{,P}) \subseteq \varprojlim_{P\in C_\alpha} J\hat{A}^{,P}$ for any ideal J of A, where $J\hat{A}^{,P}$ means $f_P(J)\hat{A}^{,P}$, f_P the canonical mapping of A into the P-adic completion $\hat{A}^{,P}$. Since J is finitely generated, J is invertible. Let J^{-1} be its inverse. Choose $a \in J \setminus \{0\}$. Then

aJ^{-1} is an ideal of A. So

$$a\left(\varprojlim_{P\in C_\alpha}\hat{A}^{,P}\right) = aJ^{-1}J\left(\varprojlim_{P\in C_\alpha}\hat{A}^{,P}\right) \subseteq aJ^{-1}\left(\varprojlim_{P\in C_\alpha}J\hat{A}^{,P}\right) \subseteq \varprojlim_{P\in C_\alpha} aJ^{-1}J\hat{A}^{,P}$$
$$= \varprojlim_{P\in C_\alpha} a\hat{A}^{,P} = a\left(\varprojlim_{P\in C_\alpha}\hat{A}^{,P}\right)$$

(For the last equality, note the following. Since $a \neq 0$, $a \notin \bigcap_{n\in\mathbb{N}} P^n$ for some $P \in C_\alpha$ and hence a is regular in $\hat{A}^{,P}$. In fact, a is regular in every $\hat{A}^{,P_0}$ such that $P_0 \in C_\alpha$ and $P_0 \subseteq P$.) Thus $aJ^{-1}J(\varprojlim_{P\in C_\alpha}\hat{A}^{,P}) = aJ^{-1}(\varprojlim_{P\in C_\alpha}J\hat{A}^{,P})$. Multiplying both sides by $a^{-1}J$, we get $J(\varprojlim_{P\in C_\alpha}\hat{A}^{,P}) = \varprojlim_{P\in C_\alpha}J\hat{A}^{,P}$. Note that $\mathrm{Max}(\hat{A}^{,P}) = \{\hat{M}^{,P} | M \in \mathrm{Max}(A)$ and M contains $P\}$ [KP1, Corollary 17(2)]. If $J \not\subseteq M$ for any $M \in \mathrm{Max}(C_\alpha)$, then $J\hat{A}^{,P} = \hat{A}^{,P}$ for all $P \in C_\alpha$, which implies that $J(\varprojlim_{P\in C_\alpha}\hat{A}^{,P}) = \varprojlim_{P\in C_\alpha}\hat{A}^{,P}$. Now

$$N \supseteq P_0\left(\varprojlim_{P\in C_\alpha}\hat{A}^{,P}\right) \supseteq J\left(\varprojlim_{P\in C_\alpha}\hat{A}^{,P}\right) = \varprojlim_{P\in C_\alpha}\hat{A}^{,P},$$

a contradiction. Therefore, $J \subseteq M$ for some $M \in \mathrm{Max}(C_\alpha)$, i.e., $P_0 \subseteq M$. So P_0 and P are comparable for some $P \in C_\alpha$. If $P_0 \subseteq P$, then since C_α is a maximal chain in $\mathrm{Spec}(A)^*$, $P_0 \in C_\alpha$. Thus $P \subseteq P_0$ for some $P \in C_\alpha$.

Let v be the valuation corresponding to A_{P_0}. Then since P_0 contains some P in C_α, the C_α-topology is the same as the C_v-topology (see Remark 4.8). As in Lemma 3.4, we can show that the C_v-topology is the same as the $\mathcal{T}_v$-topology on A. Hence $\hat{A}^{,C_\alpha} \subseteq \hat{K}^{,\mathcal{T}_v}$. Since by Theorem 4.4, $\hat{K}^{,\mathcal{T}_v}$ is the quotient field of $\widehat{A_{P_0}}^{,\mathcal{T}_v}$, $A_{P_0} \subseteq \hat{A}_N^{,C_\alpha} \subseteq \hat{K}^{,\mathcal{T}_v}$. We give A_{P_0} and $\hat{A}_N^{,C_\alpha}$ the subspace topologies of $\hat{K}^{,\mathcal{T}_v}$. Then taking the completions, we get $\widehat{A_{P_0}}^{,\mathcal{T}_v} \subseteq \widehat{\hat{A}_N^{,C_\alpha}} \subseteq \hat{K}^{,\mathcal{T}_v}$. Since $\hat{K}^{,\mathcal{T}_v}$ is the quotient field of $\widehat{A_{P_0}}^{,\mathcal{T}_v}$, $\widehat{\hat{A}_N^{,C_\alpha}}$ is a valuation overring of $\widehat{A_{P_0}}^{,\mathcal{T}_v}$. Since by Corollary 4.5, $\mathrm{Spec}(\widehat{A_{P_0}}^{,\mathcal{T}_v}) = \{\widehat{QA_{P_0}}^{,\mathcal{T}_v} \mid Q$ is a prime ideal of A contained in $P_0\}$, $\widehat{\hat{A}_N^{,C_\alpha}} = \widehat{A_{P_0}}^{,\mathcal{T}_v}{}_{\widehat{QA_{P_0}}^{,\mathcal{T}_v}}$ for some $Q \subseteq P_0$. We claim that $Q = P_0$. Let P be an arbitrary nonzero prime ideal of A contained in P_0. Since A is an SFT Prüfer domain, there exists a finitely generated ideal I of A such that $P^2 \subseteq I \subseteq P$. So the P-adic topology coincides with the I-adic topology. Since $I \subseteq \sqrt{J} = P_0$, there exists $l \geq 1$ such that $I^l \subseteq J$. Therefore, $I^l\hat{A}^{,I} \subseteq J\hat{A}^{,I} = J\hat{A}^{,P}$. Since I is finitely generated, by [Na, Theorem 17.4], $I^l\hat{A}^{,I} = \widehat{I^{l}}^{,I}$. Note that $\widehat{I^{l}}^{,I}$ is open in $\hat{A}^{,I}$ [Bo1, III.3.4, Proposition 7]. Since $J\hat{A}^{,P}$ is an ideal of $\hat{A}^{,P}$ containing an open set $\widehat{I^{l}}^{,I} (= \widehat{I^{l}}^{,P})$, $J\hat{A}^{,P}$ is closed in $\hat{A}^{,P}$, i.e., $J\hat{A}^{,P} = \hat{J}^{,P}$.

Therefore,

$$J\hat{A}^{,C_\alpha} = J\Big(\varprojlim_{P\in C_\alpha} \hat{A}^{,P}\Big) \cong J\Big(\varprojlim_{P\in C_\alpha,\, P\subseteq P_0} \hat{A}^{,P}\Big)$$
$$= \varprojlim_{P\in C_\alpha,\, P\subseteq P_0} J\hat{A}^{,P} = \varprojlim_{P\in C_\alpha,\, P\subseteq P_0} \hat{J}^{,P} \cong \hat{J}^{,C_\alpha}.$$

Since $P_0^2 \subseteq J \subseteq P_0$, $\widehat{P_0^{2}}^{,C_\alpha} \subseteq \hat{J}^{,C_\alpha} = J\hat{A}^{,C_\alpha} \subseteq P_0\hat{A}^{,C_\alpha} \subseteq N$. For each $n \in \mathbb{N}$, we give A/P_0^n the quotient topology of (A, C_α). Then by [Ma, Theorem 8.1], $0 \to \widehat{P_0^{n}}^{,C_\alpha} \to \hat{A}^{,C_\alpha} \to \widehat{A/P_0^n}$ is exact, so that $\hat{A}^{,C_\alpha}/\widehat{P_0^{n}}^{,C_\alpha} \hookrightarrow \widehat{A/P_0^n} \cong A/P_0^n$ naturally. Since the embedding is clearly onto, we have $\hat{A}^{,C_\alpha}/\widehat{P_0^{n}}^{,C_\alpha} \cong A/P_0^n$. From this, it follows that $\widehat{P_0^{2}}^{,C_\alpha}$ is a $\widehat{P_0}^{,C_\alpha}$-primary ideal since P_0^n is a P_0-primary ideal of A, and so $\widehat{P_0}^{,C_\alpha} \subseteq N$. Now, again by [Ma, Theorem 8.1], we have $\widehat{A_{P_0}}^{,\mathscr{T}_v}/\widehat{P^nA_{P_0}}^{,\mathscr{T}_v} \cong A_{P_0}/P^nA_{P_0} \supseteq A/P^n \cong \hat{A}^{,C_\alpha}/\widehat{P^{n}}^{,C_\alpha}$, and hence $\widehat{P^nA_{P_0}}^{,\mathscr{T}_v} \cap \hat{A}^{,C_\alpha} = \widehat{P^{n}}^{,C_\alpha}$, from which it follows that $\widehat{P^nA_{P_0}}^{,\mathscr{T}_v} \cap \hat{A}_N^{,C_\alpha} \subseteq (\widehat{P^nA_{P_0}}^{,\mathscr{T}_v} \cap \hat{A}^{,C_\alpha})\hat{A}_N^{,C_\alpha} = \widehat{P^{n}}^{,C_\alpha}\hat{A}_N^{,C_\alpha}$. Since $N\hat{A}_N^{,C_\alpha} \supseteq \widehat{P_0}^{,C_\alpha}\hat{A}_N^{,C_\alpha} \supseteq \widehat{P_0A_{P_0}}^{,\mathscr{T}_v} \cap \hat{A}_N^{,C_\alpha}$ and $\widehat{P_0A_{P_0}}^{,\mathscr{T}_v}$ is open in $\hat{K}^{,\mathscr{T}_v}$, $N\hat{A}_N^{,C_\alpha}$ is open in $\hat{A}_N^{,C_\alpha}$. We give $\hat{A}_N^{,C_\alpha}/N\hat{A}_N^{,C_\alpha}$ the quotient topology of $\hat{A}_N^{,C_\alpha}$ and then $\hat{A}_N^{,C_\alpha}/N\hat{A}_N^{,C_\alpha}$ has the discrete topology, so that $\hat{A}_N^{,C_\alpha}/N\hat{A}_N^{,C_\alpha} \cong \widehat{\hat{A}_N^{,C_\alpha}/N\hat{A}_N^{,C_\alpha}}$ naturally. From the exact sequence $0 \to \widehat{N\hat{A}_N^{,C_\alpha}} \to \widehat{\hat{A}_N^{,C_\alpha}} \to \widehat{\hat{A}_N^{,C_\alpha}/N\hat{A}_N^{,C_\alpha}}$ [Ma, Theorem 8.1], we obtain the natural embedding $\widehat{\hat{A}_N^{,C_\alpha}}/\widehat{N\hat{A}_N^{,C_\alpha}} \hookrightarrow \hat{A}_N^{,C_\alpha}/N\hat{A}_N^{,C_\alpha}$. Since the embedding is clearly onto, $\widehat{\hat{A}_N^{,C_\alpha}}/\widehat{N\hat{A}_N^{,C_\alpha}} \cong \hat{A}_N^{,C_\alpha}/N\hat{A}_N^{,C_\alpha}$ (and it is a field). Thus $\widehat{N\hat{A}_N^{,C_\alpha}}$ is the unique maximal ideal of $\widehat{\hat{A}_N^{,C_\alpha}}$, i.e., $\widehat{N\hat{A}_N^{,C_\alpha}} = \widehat{QA_{P_0}}^{,\mathscr{T}_v}\widehat{A_{P_0}}^{,\mathscr{T}_v}{}_{\widehat{QA_{P_0}}^{,\mathscr{T}_v}}$. Therefore, $P_0 = N \cap A = N\hat{A}_N^{,C_\alpha} \cap A = \widehat{N\hat{A}_N^{,C_\alpha}} \cap A = \widehat{QA_{P_0}}^{,\mathscr{T}_v}\widehat{A_{P_0}}^{,\mathscr{T}_v}{}_{\widehat{QA_{P_0}}^{,\mathscr{T}_v}} \cap A = \widehat{QA_{P_0}}^{,\mathscr{T}_v} \cap A = Q$, where the last equality follows from the fact that $\widehat{A_{P_0}}^{,\mathscr{T}_v}/\widehat{QA_{P_0}}^{,\mathscr{T}_v} \cong A_{P_0}/QA_{P_0}$. Thus $P_0 = Q$.

After all, $\widehat{\hat{A}_N^{,C_\alpha}} = \widehat{A_{P_0}}^{,\mathscr{T}_v}$. Now let $R = \hat{A}_N^{,C_\alpha}$. Since for each nonzero prime ideal P contained in P_0, $\widehat{P^nA_{P_0}}^{,\mathscr{T}_v} \cap R = (\widehat{P^nA_{P_0}}^{,\mathscr{T}_v} \cap \hat{A}^{,C_\alpha})R = \widehat{P^{n}}^{,C_\alpha}R$, $\{\widehat{P^{n}}^{,C_\alpha}R \mid (0) \neq P \subseteq P_0,\ n \in \mathbb{N}\}$ is a base of zero neighborhoods in R. We claim that R is a valuation domain. It suffices to show that $a\hat{R} \cap R = aR$ for all $a \in R^*$ since $\hat{R}$ is a valuation domain. The case when a is a unit being trivial, we may assume that a is a nonunit, i.e., $a \in NR$. Since $\hat{R}$ is a Hausdorff complete space, $\bigcap_{(0)\neq P\subseteq P_0} \widehat{P^nA_{P_0}}^{,\mathscr{T}_v} = (0)$. Since $\hat{R}$ is a valuation domain, there exist a nonzero prime ideal P contained in P_0 and $n \in \mathbb{N}$ such that $\widehat{P^nA_{P_0}}^{,\mathscr{T}_v} \subseteq a\hat{R}$. So $\widehat{P^{n}}^{,C_\alpha}R = \widehat{P^nA_{P_0}}^{,\mathscr{T}_v} \cap R \subseteq a\hat{R} \cap R$. Since $a\hat{R} \cap R \subseteq \bigcap_{(0)\neq Q\subseteq P_0,\, l\in\mathbb{N}} (aR + \widehat{Q^{l}}^{,C_\alpha}R)$, $aR + \widehat{P^{n}}^{,C_\alpha}R = aR$

$+ \widehat{P^{n+1},^{C_\alpha}}R = \cdots$. Since A is an SFT Prüfer domain, there exists a finitely generated ideal I of A such that $P^2 \subseteq I \subseteq P$. So $I^n R \subseteq P^n R \subseteq \widehat{P^n},^{C_\alpha} R \subseteq \widehat{P^{2(n+1)}},^{C_\alpha} + aR \subseteq \widehat{I^{n+1}},^{C_\alpha} R + aR = (I^{n+1}\hat{A},^{C_\alpha})R + aR = I^{n+1}R + aR$. Thus $aR + I^n R = aR + I^{n+1}R = \cdots$. Let $\overline{R} = R/aR$ and $\overline{IR} = (IR + aR)/aR$. Then $\overline{IR}$ is a finitely generated ideal of $\overline{R}$, $\overline{IR} \subseteq J(\overline{R}) = NR/aR$, and $\overline{IR}^n = \overline{IR}^{n+1}$. By Nakayama's lemma, $\overline{IR}^n = \{\overline{0}\}$, i.e., $I^n R \subseteq aR$. Thus $a\hat{R} \cap R \subseteq aR + \widehat{P^{2n}},^{C_\alpha} R \subseteq aR + I^n R = aR$ and hence $a\hat{R} \cap R = aR$. ∎

COROLLARY 7.2. *Let N be a prime ideal of $\hat{A},^{C_\alpha}$ such that $N \cap A \neq (0)$. Then the following statements hold.*

(1) *$N = \widehat{P_0},^{C_\alpha}$, where $P_0 = N \cap A$, and P_0 contains some $P \in C_\alpha$.*

(2) *$\hat{A},^{C_\alpha}_{\widehat{P_0},^{C_\alpha}} = \widehat{A_{P_0}},^{\mathcal{F}_v} \cap$ q.f.$(\hat{A},^{C_\alpha})$, where q.f.$(\hat{A},^{C_\alpha})$ denotes the quotient field of $\hat{A},^{C_\alpha}$ and v is the valuation corresponding to A_{P_0}.*

(3) *Every prime ideal of $\hat{A},^{C_\alpha}$ contained in $\widehat{P_0},^{C_\alpha}$ is of the form $\hat{P},^{C_\alpha}$, where $P \in$ Spec(A) and $P \subseteq P_0$.*

(4) *$\hat{A},^{C_\alpha}_N$ is an SFT valuation domain.*

Proof. (1) From the proof of Lemma 7.1, we have $N = N\hat{A},^{C_\alpha}_N \cap \hat{A},^{C_\alpha} = \widehat{N\hat{A},^{C_\alpha}_N} \cap \hat{A},^{C_\alpha} = \widehat{P_0 A_{P_0}},^{\mathcal{F}_v} \cap \hat{A},^{C_\alpha} = \widehat{P_0},^{C_\alpha}$.

(2) Let $R = \hat{A},^{C_\alpha}_{\widehat{P_0},^{C_\alpha}}$ and $\hat{R} = \widehat{A_{P_0}},^{\mathcal{F}_v}$. Since $a\hat{R} \cap R = aR$ for all $a \in R^*$, $\hat{R} \cap$ q.f. $(R) = R$.

(3) By Corollary 4.5, Spec$(\widehat{A_{P_0}},^{\mathcal{F}_v}) = \{\widehat{PA_{P_0}},^{\mathcal{F}_v} | P \in$ Spec(A) contained in $P_0\}$, so that by (2) and [Gi, Theorem (19.16)], every prime ideal of $\hat{A},^{C_\alpha}$ contained in $\widehat{P_0},^{C_\alpha}$ is of the form $\widehat{PA_{P_0}},^{\mathcal{F}_v} \cap \hat{A},^{C_\alpha} = \hat{P},^{C_\alpha}$, $P \subseteq P_0$.

(4) Since A is an SFT Prüfer domain, A_{P_0} is an SFT valuation domain and hence it is discrete, i.e., each branched prime ideal of A_{P_0} is not idempotent. (For the definition of "branched," see [Gi, p. 189].) Since by Theorem 4.4, $\widehat{A_{P_0}},^{\mathcal{F}_v}$ is a valuation domain with the value group $G_{\hat{v}} = G_v$, by [Gi, Exercise 22, p. 205], $\widehat{A_{P_0}},^{\mathcal{F}_v}$ is also discrete, and hence by (2) and [Gi, Theorem (19.16)(b)], so is $\hat{A},^{C_\alpha}_N$. Now we claim that every nonzero prime ideal of the valuation domain $\hat{A},^{C_\alpha}_N$ is branched, i.e., it is the radical of a principal ideal [Gi, Theorem (17.3)]. By (3), every nonzero prime ideal of $\hat{A},^{C_\alpha}_N$ is of the form $\hat{P},^{C_\alpha}\hat{A},^{C_\alpha}_N$ for some $P \in$ Spec$(A)^*$ such that $p \subseteq P_0$. Since A is an SFT Prüfer domain, there exists a finitely generated ideal J such that $P^2 \subseteq J \subseteq P$. In the proof of Lemma 7.1, we have shown that $\widehat{P^2},^{C_\alpha} \subseteq \hat{J},^{C_\alpha} = J\hat{A},^{C_\alpha} \subseteq \hat{P},^{C_\alpha}$ and $\widehat{P^2},^{C_\alpha}$ is a $\hat{P},^{C_\alpha}$-primary ideal of $\hat{A},^{C_\alpha}$, so that $\hat{P},^{C_\alpha}\hat{A},^{C_\alpha}_N = \sqrt{J\hat{A},^{C_\alpha}_N}$. Since J is finitely generated and $\hat{A},^{C_\alpha}_N$ is a valuation domain, $J\hat{A},^{C_\alpha}_N$ is principal. So no nonzero prime

ideal of $\hat{A}_N^{,C_\alpha}$ is idempotent. Thus since every nonzero prime ideal of $\hat{A}_N^{,C_\alpha}$ is a radical of a finitely generated ideal and it is not idempotent, the conclusion follows from Proposition 2.3 and Theorem 2.4. ∎

Recall that $J(C_\alpha) = \bigcap_{M \in \operatorname{Max}(C_\alpha)} M$. Since in an h-local Prüfer domain and the ring of entire functions, the condition $J(C_\alpha) \neq \{0\}$ is obviously satisfied, we are naturally led to consider the SFT Prüfer domain A with the condition $J(C_\alpha) \neq \{0\}$ for all maximal chains in $\operatorname{Spec}(A)^*$.

LEMMA 7.3. *$J(C_\alpha) \neq \{0\}$ if and only if there exists a prime ideal $P_0 \in C_\alpha$ such that $P_0 \subseteq J(C_\alpha)$.*

Proof. Assume that $J(C_\alpha) \neq \{0\}$. Since $J(C_\alpha) = \bigcap_{M \in \operatorname{Max}(C_\alpha)} M$, $J(C_\alpha)$ is a nonzero radical ideal of A. Since A is an SFT Prüfer domain, by Proposition 2.1(2), $J(C_\alpha)$ has only finitely many minimal prime divisors, say $P_1, P_2, \dots, P_n$. By rearranging, we may assume that there exists k $(1 \leq k \leq n)$ such that for $i \leq k$, $P_i \subseteq M$ for some $M \in \operatorname{Max}(C_\alpha)$ and for $i > k$, $P_i \not\subseteq M$ for any $M \in \operatorname{Max}(C_\alpha)$. Let $\beta_i = \{M \in \operatorname{Max}(C_\alpha) | M \supseteq P_i\}$, $i = 1, \dots, k$. Then $\operatorname{Max}(C_\alpha) = \bigcup_{i=1}^k \beta_i$. Choose $M_i \in \beta_i$, $i = 1, \dots, k$. By the definition of $\operatorname{Max}(C_\alpha)$, $M_i \sim M_j$ for all $i, j = 1, \dots, k$. Since $P_i \sim M_i$ for all $i = 1, \dots, k$, $P_i \sim P_j$ for all $i, j = 1, \dots, k$. Therefore, there exists a prime ideal $P_0 \in C_\alpha$ such that $P_0 \subseteq \bigcap_{i=1}^k P_i \subseteq \bigcap_{i=1}^k (\bigcap_{M \in \beta_i} M) = \bigcap_{M \in \operatorname{Max}(C_\alpha)} M = J(C_\alpha)$. ∎

LEMMA 7.4. *If $J(C_\alpha) \neq \{0\}$, then* $\text{q.f.}(\hat{A}^{,C_\alpha}) = \hat{A}_{A\setminus\{0\}}^{,C_\alpha}$.

Proof. By Lemma 7.3, there exists a prime ideal $P_0 \in C_\alpha$ such that $P_0 \subseteq J(C_\alpha)$. Recall that $\hat{A}^{,C_\alpha} = \bigcap_{M \in \operatorname{Max}(C_\alpha)} \widehat{A_M}^{,C_{\alpha,M}}$ (Lemma 6.3) and $\widehat{A_M}^{,C_{\alpha,M}} \subseteq \widehat{A_{P_0}}^{,C_{\alpha,P_0}}$. Choose $a \in P_0 \setminus \{0\}$. Then for each prime ideal $P \in C_\alpha$ contained in P_0 and for each $M \in \operatorname{Max}(C_\alpha)$,

$$
\begin{aligned}
a\widehat{A_{P_0}}^{,PA_{P_0}} &= a\Big(\varprojlim_{n\in\mathbb{N}} A_{P_0}/P^nA_{P_0}\Big) = \varprojlim_{n\in\mathbb{N}} a\big(A_{P_0}/P^nA_{P_0}\big) \\
&\subseteq \varprojlim_{n\in\mathbb{N}} P_0A_{P_0}/P^nA_{P_0} = \varprojlim_{n\in\mathbb{N}} P_0A_M/P^nA_M \\
&\subseteq \varprojlim_{n\in N} A_M/P^nA_M = \widehat{A_M}^{,PA_M}.
\end{aligned}
$$

From this and Lemma 6.1, we have $a\widehat{A_{P_0}}^{,C_{\alpha,P_0}} \cong a(\varprojlim_{PA_{P_0}\in C_{\alpha,P_0}} \widehat{A_{P_0}}^{,PA_{P_0}}) = \varprojlim_{PA_{P_0}\in C_{\alpha,P_0}} a\widehat{A_{P_0}}^{,PA_{P_0}} \subseteq \varprojlim_{P\in C_\alpha, P\subseteq P_0} \widehat{A_M}^{,PA_M} \cong \widehat{A_M}^{,C_{\alpha,M}}$, $M \in \operatorname{Max}(C_\alpha)$. Now $a\widehat{A_{P_0}}^{,C_{\alpha,P_0}} \subseteq \bigcap_{M \in \operatorname{Max}(C_\alpha)} \widehat{A_M}^{,C_{\alpha,M}} = \hat{A}^{,C_\alpha}$, which implies that $\hat{A}_{A\setminus\{0\}}^{,C_\alpha} = \widehat{A_{P_0}}_{A\setminus\{0\}}^{,C_{\alpha,P_0}} = \widehat{A_{P_0}}_{A_{P_0}\setminus\{0\}}^{,C_{\alpha,P_0}}$. Since A is an SFT Prüfer domain, $\widehat{A_{P_0}}^{,C_{\alpha,P_0}} = \widehat{A_{P_0}}^{,\mathscr{T}_v}$ (Lemma 3.4), where v is the valuation corresponding to A_{P_0}.

Since by Theorem 4.4(6), $\text{q.f.}(\widehat{A_{P_0}}^{,\mathcal{F}_v}) = \widehat{A_{P_0}}^{,\mathcal{F}_v}_{A_{P_0}\setminus\{0\}}$, we conclude that $\text{q.f.}(\widehat{A_{P_0}}^{,C_\alpha,P_0}) = \text{q.f.}(\hat{A}^{,C_\alpha}) = \hat{A}^{,C_\alpha}_{A\setminus\{0\}}$. ∎

THEOREM 7.5. *Let A be an SFT Prüfer domain such that $J(C_\alpha) \neq \{0\}$ for all maximal chains C_α in* Spec$(A)^*$. *Then*

(1) Spec$(\hat{A}^{,C_\alpha}) = \{(0)\} \cup \{\widehat{P_0}^{,C_\alpha} | P_0$ *is a prime ideal of A containing some $P \in C_\alpha$*$\}$ *and* Max$(\hat{A}^{,C_\alpha}) = \{\hat{M}^{,C_\alpha} | M \in$ Max$(C_\alpha)\}$;

(2) $\hat{A}^{,C_\alpha}_{\widehat{P_0}^{,C_\alpha}} = \widehat{A_{P_0}}^{,\mathcal{F}_v}$ *for every* $\widehat{P_0}^{,C_\alpha} \in$ Spec$(\hat{A}^{,C_\alpha})^*$, *where v is the valuation corresponding to* A_{P_0};

(3) $\hat{A}^{,C_\alpha}$ *is an SFT Prüfer domain;*

(4) $\hat{A}^{,\mathcal{F}}$ *is a Prüfer ring. Moreover it is an SFT-ring if and only if* $|\Omega_0| < \infty$, *i.e., the number of independent valuation overrings of A is finite.*

Proof. Let N be a prime ideal of $\hat{A}^{,C_\alpha}$ such that $N \cap A = (0)$. Then $N\hat{A}^{,C_\alpha}_{A\setminus\{0\}}$ is a prime ideal of $\hat{A}^{,C_\alpha}_{A\setminus\{0\}}$. Since by Lemma 7.4, $\hat{A}^{,C_\alpha}_{A\setminus\{0\}} = \text{q.f.}(\hat{A}^{,C_\alpha})$, $N\hat{A}^{,C_\alpha}_{A\setminus\{0\}} = (0)$, i.e., $N = (0)$. Therefore, (1) and (2) immediately follow from Corollary 7.2 (for (2), note that $\widehat{A_{P_0}}^{,\mathcal{F}_v} \subseteq \text{q.f.}(\widehat{A_{P_0}}^{,\mathcal{F}_v}) = \hat{A}^{,C_\alpha}_{A\setminus\{0\}}$ as is shown in the proof of Lemma 7.4).

(3) From (1) and (2), it follows that $\hat{A}^{,C_\alpha}$ is a Prüfer domain. Now let N be a nonzero prime ideal of $\hat{A}^{,C_\alpha}$. Then $N = \widehat{P_0}^{,C_\alpha}$, $P_0 = N \cap A$ $(\neq (0))$. Since A is an SFT Prüfer domain, there exists a finitely generated ideal J such that $P_0^2 \subseteq J \subseteq P_0$. In the proof of Lemma 7.1, we have shown that $\widehat{P_0^2}^{,C_\alpha} \subseteq \hat{J}^{,C_\alpha} = J\hat{A}^{,C_\alpha} \subseteq \widehat{P_0}^{,C_\alpha}$ and $\widehat{P_0^2}^{,C_\alpha}$ is a $\widehat{P_0}^{,C_\alpha}$-primary ideal, so that $\widehat{P_0}^{,C_\alpha} = \sqrt{J\hat{A}^{,C_\alpha}}$. Since by Corollary 7.2(4), $\hat{A}^{,C_\alpha}_N$ is an SFT valuation domain, $N\hat{A}^{,C_\alpha}_N \neq (N\hat{A}^{,C_\alpha}_N)^2$. This implies $N \neq N^2$, i.e., $\widehat{P_0}^{,C_\alpha} \neq (\widehat{P_0}^{,C_\alpha})^2$. Then by Proposition 2.3 and Theorem 2.4, $\hat{A}^{,C_\alpha}$ is an SFT Prüfer domain.

(4) By Lemma 3.4 and Proposition 4.7, $\hat{A}^{,\mathcal{F}} = \hat{A}^{,\mathscr{C}} \cong \prod_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$. Since each $\hat{A}^{,C_\alpha}$ is a Prüfer domain, by Proposition 4.9, $\hat{A}^{,\mathcal{F}}$ is a Prüfer ring.

Now for the second claim, consider the ideal $\sum_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$ of $\prod_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$, where Λ is the index set for a representing family of the independent maximal chains in Spec$(A)^*$, i.e., $\{C_\alpha\}_{\alpha \in \Lambda}$ is a collection of representatives of the equivalence classes of $\mathscr{C}$. Note that by Remark 4.8, $|\Lambda| = |\Omega_0|$. For each element $x \in \prod_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$, write $x = \prod_{\alpha \in \Lambda} x_\alpha$. If $\sum_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$ is an SFT ideal, then there exist a finitely generated ideal $(x_1, \dots, x_n) \subseteq \sum_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$ and a positive integer k such that $x^k \in (x_1, \dots, x_n)$ for all $x \in \sum_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$. Since $x_i \in \sum_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$, there exists a finite subset Λ_i of Λ such that $x_{i,\alpha} = 0$ for all $\alpha \in \Lambda \setminus \Lambda_i$. Therefore, $x_\alpha^k = 0$ for all $x \in \sum_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$, $\alpha \in \Lambda \setminus \bigcup_{i=1}^n \Lambda_i$. Since $\hat{A}^{,C_\alpha}$ is an integral domain, this implies that $x_\alpha = 0$ for all $x \in \sum_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$, $\alpha \in \Lambda \setminus \bigcup_{i=1}^n \Lambda_i$. This is impossible if $\Lambda \neq \bigcup_{i=1}^n \Lambda_i$. Thus in order that $\prod_{\alpha \in \Lambda} \hat{A}^{,C_\alpha}$ is an SFT ring, we must have $|\Lambda| < \infty$.

Conversely, assume that $|\Lambda| < \infty$, say $\Lambda = \{\alpha_1, \ldots, \alpha_n\}$. Then $\hat{A}^{\cdot\mathcal{F}} = \hat{A}^{\cdot\mathcal{C}} \cong \hat{A}^{\cdot C_{\alpha_1}} \oplus \cdots \oplus \hat{A}^{\cdot C_{\alpha_n}}$. Since by (3), $\hat{A}^{\cdot C_{\alpha_i}}$ is an SFT Prüfer domain and every prime ideal of $\hat{A}^{\cdot\mathcal{F}}$ is of the form $\hat{A}^{\cdot C_{\alpha_1}} \oplus \cdots \oplus \hat{A}^{\cdot C_{\alpha_{i-1}}} \oplus Q_i \oplus \hat{A}^{\cdot C_{\alpha_{i+1}}} \oplus \cdots \oplus \hat{A}^{\cdot C_{\alpha_n}}$, where Q_i is a prime ideal of $\hat{A}^{\cdot C_{\alpha_i}}$, $\hat{A}^{\cdot\mathcal{F}}$ is an SFT ring by Proposition 2.1(1). ∎

For an integral domain A and a prime ideal P of A, $A[\![X_1, \ldots, X_n]\!]_{P+(X_1,\ldots,X_n)} \neq A_P[\![X_1, \ldots, X_n]\!]$ unless P is the unique maximal ideal of A. Interestingly it turns out that

$$A[\![X_1, \ldots, X_n]\!]_{P+(X_1,\ldots,X_n)}/(X_1 - a_1, \ldots, X_n - a_n)$$
$$= A_P[\![X_1, \ldots, X_n]\!]/(X_1 - a_1, \ldots, X_n - a_n)$$

for all $a_1, \ldots, a_n \in A$ provided that A is an SFT Prüfer domain. First we show that for an SFT Prüfer domain A and $a_1, \ldots, a_n \in A$, $\hat{A}^{\cdot(a_1,\ldots,a_n)} \cong A[\![X_1, \ldots, X_n]\!]/(X_1 - a_1, \ldots, X_n - a_n)$, which is well known in the case when A is a Noetherian domain.

THEOREM 7.6. *Let A be an SFT Prüfer domain and $a_1, \ldots, a_n \in A$. Then the $(a_1, \ldots, a_n)$-adic completion $\hat{A}^{\cdot(a_1,\ldots,a_n)}$ of A is isomorphic to the ring $A[\![X_1, \ldots, X_n]\!]/(X_1 - a_1, \ldots, X_n - a_n)$.*

Proof. Since by [Na, Theorem 17.5],

$$\hat{A}^{\cdot(a_1,\ldots,a_n)} \cong A[\![X_1, \ldots, X_n]\!]/\overline{(X_1 - a_1, \ldots, a_n)},$$

where $\overline{(X_1 - a_1, \ldots, X_n - a_n)}$ is the closure of the ideal $(X_1 - a_1, \ldots, X_n - a_n)$ in $A[\![X_1, \ldots, X_n]\!]$ with respect to the $(X_1, \ldots, X_n)$-adic topology, it suffices to show that

$$\overline{(X_1 - a_1, \ldots, X_n - a_n)} = (X_1 - a_1, \ldots, X_n - a_n).$$

Since

$$\begin{aligned}
&\overline{(X_1 - a_1, \ldots, X_n - a_n)} \\
&\quad = \bigcap_{m=1}^{\infty} \left((X_1 - a_1, \ldots, X_n - a_n) + (X_1, \ldots, X_n)^m\right) \\
&\quad = \bigcap_{m=1}^{\infty} \left((X_1 - a_1, \ldots, X_n - a_n) + (a_1, \ldots, a_n)^m\right),
\end{aligned}$$

we have

$$\begin{aligned}
&\overline{(X_1 - a_1, \ldots, X_n - a_n)}/(X_1 - a_1, \ldots, X_n - a_n) \\
&\quad = \bigcap_{m=1}^{\infty} (a_1, \ldots, a_n)^m \left(A[\![X_1, \ldots, X_n]\!]/(X_1 - a_1, \ldots, X_n - a_n)\right).
\end{aligned}$$

So it suffices to show that $\bigcap_{m=1}^{\infty}(a_1,\dots,a_n)^m(A[\![X_1,\dots,X_n]\!]/(X_1-a_1,\dots,X_n-a_n))=\{0\}$. We will use induction on n. The case $n=1$ is clear by [GS, Proposition 3.4]. Suppose

$$\bigcap_{m=1}^{\infty}(a_1,\dots,a_{n-1})^m(A[\![X_1,\dots,X_{n-1}]\!]/(X_1-a_1,\dots,X_{n-1}-a_{n-1}))$$
$$=\{0\}.$$

Now we consider

$$\bigcap_{m=1}^{\infty}(a_1,\dots,a_n)^m(A[\![X_1,\dots,X_n]\!]/(X_1-a_1,\dots,X_n-a_n)).$$

Let $R=A[\![X_1,\dots,X_n]\!]/(X_1-a_1,\dots,X_n-a_n)$. Then

$$R\cong(A[\![X_1]\!]/(X_1-a_1))[\![X_2,\dots,X_n]\!]/(X_2-a_2,\dots,X_n-a_n)$$
$$\cong\hat{A}^{\cdot(a_1)}[\![X_2,\dots,X_n]\!]/(X_2-a_2,\dots,X_n-a_n).$$

If $a_1=0$, then since $\hat{A}^{\cdot(a_1)}=A$, $R\cong A[\![X_2,\dots,X_n]\!]/(X_2-a_2,\dots,X_n-a_n)$. Therefore, by induction hypothesis, the conclusion follows. If a_1 is a unit in A, then X_1-a_1 is also a unit in $A[\![X_1,\dots,X_n]\!]$, which implies $R=\{0\}$. In this case, clearly $\bigcap_{m=1}^{\infty}(a_1,\dots,a_n)^mR=\{0\}$. So we may assume that a_1 is a nonzero nonunit element in A. Let $\{P_1,\dots,P_k\}$ be the set of minimal prime divisors of a_1A. Then $\hat{A}^{\cdot(a_1)}\cong\hat{A}^{\cdot P_1}\oplus\cdots\oplus\hat{A}^{\cdot P_k}$ (see [KP1, Theorem 15]). Therefore,

$$R\cong\hat{A}^{\cdot P_1}[\![X_2,\dots,X_n]\!]/(X_2-a_2,\dots,X_n-a_n)\oplus\cdots$$
$$\oplus\hat{A}^{\cdot P_k}[\![X_2,\dots,X_n]\!]/(X_2-a_2,\dots,X_n-a_n).$$

To prove $\bigcap_{m=1}^{\infty}(a_1,\dots,a_n)^mR=\{0\}$, it suffices to show that $\bigcap_{m=1}^{\infty}(a_1,\dots,a_n)^m(\hat{A}^{\cdot P_i}[\![X_2,\dots,X_n]\!]/(X_2-a_2,\dots,X_n-a_n))=\{0\}$ for all $i=1,2,\dots,k$. Note that $\hat{A}^{\cdot P_i}$ is an SFT Prüfer domain with $\operatorname{Spec}(\hat{A}^{\cdot P_i})=\{(0)\}\cup\{\hat{Q}^{\cdot P_i}\mid Q\in\operatorname{Spec}(A)\text{ and }Q\supseteq P_i\}$ ([KP1, Theorem 15 and Corollary 17], where the proof of Corollary 17 is also valid for the infinite-dimensional case). Therefore, by induction hypothesis, $\hat{A}^{\cdot P_i}[\![X_2,\dots,X_n]\!]/(X_2-a_2,\dots,X_n-a_n)\cong\widehat{\hat{A}^{\cdot P_i}}^{\cdot(a_2,\dots,a_n)\hat{A}^{\cdot P_i}}$.

Case I. $(a_2,\dots,a_n)\hat{A}^{\cdot P_i}=\{0\}$. Then

$$\bigcap_{m=1}^{\infty}(a_1,\dots,a_n)^m\widehat{\hat{A}^{\cdot P_i}}^{\cdot(a_2,\dots,a_n)\hat{A}^{\cdot P_i}}$$
$$=\bigcap_{m=1}^{\infty}(a_1,\dots,a_n)^m\hat{A}^{\cdot P_i}$$
$$=\bigcap_{m=1}^{\infty}a_1^m\hat{A}^{\cdot P_i}\subseteq\bigcap_{m=1}^{\infty}P_i^m\hat{A}^{\cdot P_i}\subseteq\bigcap_{m=1}^{\infty}\widehat{P_i^m}^{\cdot P_i}=\{0\}$$

since $\hat{A}^{,P_i}$ is complete with respect to the linear topology determined by $\{\widehat{P_i^{m},^{P_i}}\}_{m\in\mathbb{N}}$.

Case II. $(a_2,\ldots,a_n)\hat{A}^{,P_i} = \hat{A}^{,P_i}$. Then $\widehat{\hat{A}^{,P_i},^{(a_2,\ldots,a_n)\hat{A}^{,P_i}}} = \{0\}$ and hence $\bigcap_{m=1}^{\infty}(a_1,\ldots,a_n)^m\widehat{\hat{A}^{,P_i},^{(a_2,\ldots,a_n)\hat{A}^{,P_i}}} = \{0\}$.

Case III. $(a_2,\ldots,a_n)\hat{A}^{,P_i}$ is a nonzero proper ideal of $\hat{A}^{,P_i}$. Since $\hat{A}^{,P_i}$ is an SFT Prüfer domain, $\widehat{\hat{A}^{,P_i},^{(a_2,\ldots,a_n)\hat{A}^{,P_i}}} \cong \widehat{\hat{A}^{,P_i},^{\widehat{Q_{i1}},^{P_i}}} \oplus \cdots \oplus \widehat{\hat{A}^{,P_i},^{\widehat{Q_{ik_i}},^{P_i}}}$, where $\{\widehat{Q_{i1}},^{P_i},\ldots,\widehat{Q_{ik_i}},^{P_i}\}$ is the set of minimal prime divisors of $(a_2,\ldots,a_n)\hat{A}^{,P_i}$. Note that $Q_{ij}\in\operatorname{Spec}(A)$ and $Q_{ij}\supseteq P_i \ni a_1$ for all $i = 1,2,\ldots,k$; and $j = 1,2,\ldots,k_i$. Therefore, $\bigcap_{m=1}^{\infty}(a_1,\ldots,a_n)^m\widehat{\hat{A}^{,P_i},^{\widehat{Q_{ij}},^{P_i}}} \subseteq \bigcap_{m=1}^{\infty}Q_{ij}^m\widehat{\hat{A}^{,P_i},^{\widehat{Q_{ij}},^{P_i}}} \subseteq \bigcap_{m=1}^{\infty}\widehat{(\widehat{Q_{ij}},^{P_i})^{m},^{\widehat{Q_{ij}},^{P_i}}} = \{0\}$ since $\widehat{\hat{A}^{,P_i},^{\widehat{Q_{ij}},^{P_i}}}$ is complete with respect to the linear topology determined by $\{\widehat{(\widehat{Q_{ij}},^{P_i})^{m}},^{\widehat{Q_{ij}},^{P_i}}\}_{m\in\mathbb{N}}$. This implies that $\bigcap_{m=1}^{\infty}(a_1,\ldots,a_n)^m\widehat{\hat{A}^{,P_i},^{(a_2,\ldots,a_n)\hat{A}^{,P_i}}} = \{0\}$. Thus the conclusion follows. ∎

Next we provide an equivalent condition for an SFT Prüfer domain to be analytically irreducible (with respect to a given ideal-adic topology).

COROLLARY 7.7. *Let A be an SFT Prüfer domain and $a_1,\ldots,a_n\in A$. Then*

(1) *$(X_1-a_1,\ldots,X_n-a_n)$ is a prime ideal of $A[\![X_1,\ldots,X_n]\!]$ if and only if $\sqrt{(a_1,\ldots,a_n)}$ is a prime ideal of A.*

(2) *$(X_1-a_1,\ldots,X_n-a_n)$ is a radical ideal of $A[\![X_1,\ldots,X_n]\!]$.*

Proof. (1) follows from Theorem 7.6 and [KP1, Theorem 15]. Since by [KP1, Theorem 15], $\hat{A}^{,(a_1,\ldots,a_n)}$ is a direct product of a finite number of SFT Prüfer domains, (2) follows from (1). ∎

PROPOSITION 7.8. *Let A be an SFT Prüfer domain, P be a prime ideal of A, and $a_1,\ldots,a_n\in A$. Then $A[\![X_1,\ldots,X_n]\!]_{P+(X_1,\ldots,X_n)}/(X_1-a_1,\ldots, X_n-a_n)\cong A_P[\![X_1,\ldots,X_n]\!]/(X_1-a_1,\ldots,X_n-a_n)$.*

Proof. If $(a_1,\ldots,a_n)\not\subseteq P$, then both sides are $\{0\}$. So we may assume that $(a_1,\ldots,a_n)\subseteq P$. If $P=(0)$, then both sides are K, the quotient field of A. So we may assume that $P\neq(0)$. Note

$$\begin{aligned} &A[\![X_1,\ldots,X_n]\!]_{P+(X_1,\ldots,X_n)}/(X_1-a_1,\ldots,X_n-a_n)\\ &\cong\left(\frac{A[\![X_1,\ldots,X_n]\!]}{X_1-a_1,\ldots,X_n-a_n}\right)_{P+(X_1,\ldots,X_n)/(X_1-a_1,\ldots,X_n-a_n)}\\ &\cong \hat{A}^{,(a_1,\ldots,a_n)}_{\hat{P},(a_1,\ldots,a_n)}\end{aligned}$$

by Theorem 7.6. Let $\{P_1, \ldots, P_k\}$ be the set of minimal prime divisors of $(a_1, \ldots, a_n)$ and $P_1 \subseteq P$. Then by [KP1, Theorem 15], $\hat{A}^{,(a_1,\ldots,a_n)} \cong \hat{A}^{,P_1} \oplus \cdots \oplus \hat{A}^{,P_n}$ and $\hat{P}^{,(a_1,\ldots,a_n)} = P\hat{A}^{,(a_1,\ldots,a_n)} \cong P\hat{A}^{,P_1} \oplus P\hat{A}^{,P_2} \oplus \cdots \oplus P\hat{A}^{,P_n} \cong \hat{P}^{,P_1} \oplus \hat{A}^{,P_2} \oplus \cdots \oplus \hat{A}^{,P_k}$ (note that $P_i \not\subseteq P$ and $P \not\subseteq P_i$ for all $i = 2, \ldots, k$, so that $P + P_i = A$). Therefore, $\hat{A}^{,(a_1,\ldots,a_n)}_{\hat{P},(a_1,\ldots,a_n)} \cong \hat{A}^{,P_1}_{\hat{P},\hat{P}_1}$. On the other hand, $A_P[\![X_1, \ldots, X_n]\!]/(X_1 - a_1, \ldots, X_n - a_n) \cong \widehat{A_P}^{,(a_1,\ldots,a_n)A_P} \cong \widehat{A_P}^{,P_1A_P}$ by Theorem 7.6 and the fact that A_P is an SFT Prüfer domain. Thus to prove the proposition, it suffices to show that $\hat{A}^{,P_1}_{\hat{P},\hat{P}_1} \cong \widehat{A_P}^{,P_1A_P}$. Following Arnold's notation, put $\mathscr{B}(P_1) := \bigcap_{m=1}^{\infty} P_1^m$, which is the prime ideal of A just below P_1, and let $\bar{A} = A/\mathscr{B}(P_1)$, $\bar{P}_1 = P_1/\mathscr{B}(P_1)$, and $\bar{P} = P/\mathscr{B}(P_1)$. Note that $\bar{A}$ is an SFT Prüfer domain, $\bar{P}_1$ is a height 1 prime ideal of $\bar{A}$,

$$\hat{A}^{,P_1}_{\hat{P},\hat{P}_1} \cong \hat{\bar{A}}^{,\bar{P}_1}_{\hat{\bar{P}},\hat{\bar{P}}_1}, \text{ and } \widehat{A_P}^{,P_1A_P} \cong \widehat{\bar{A}_{\bar{P}}}^{,\bar{P}_1\bar{A}_{\bar{P}}}.$$

Therefore, we may assume that P_1 is a height 1 prime ideal of A. Let C_α be a maximal chain in $\operatorname{Spec}(A)^*$ containing P_1. Then clearly $\hat{A}^{,C_\alpha} = \hat{A}^{,P_1}$ and hence by Theorem 7.5, $\hat{A}^{,P_1}_{\hat{P},\hat{P}_1} = \widehat{A_P}^{,P_1A_P}$. Now the conclusion follows. ∎

8. $\hat{\mathscr{T}}$-PRÜFER RING

Mockor [Mo] introduced the notion of an $\mathscr{F}$-Prüfer ring. He defined an $\mathscr{F}$-Prüfer ring to be a commutative ring R with identity and the total quotient ring $T(R)$ in which, for every maximal regular ideal M of R, $(R_{[M]}, [M]R_{[M]})$ is a valuation pair associated with a valuation w on $T(R)$ such that w is continuous in $\mathscr{F}$, where $\mathscr{F}$ is a topology on the ring $T(R)$.

Again, let A be a Prüfer domain with quotient field K and let Ω be the family of nontrivial valuations on K which are nonnegative on A and put $\mathscr{T} = \sup\{\mathscr{T}_w | w \in \Omega\}$. If $(\hat{K}^{,\mathscr{T}_w}, \widehat{\mathscr{T}_w})$ and $(\hat{K}^{,\mathscr{T}}, \hat{\mathscr{T}})$ are the completions of $(K, \mathscr{T}_w)$ and $(K, \mathscr{T})$, respectively, we denote by $\hat{w}$ and $\tilde{w}$ the continuous extensions of w on $\hat{K}^{,\mathscr{T}_w}$ and $\hat{K}^{,\mathscr{T}}$, respectively. It is well known that $\hat{w}$ is a valuation on the field $\hat{K}^{,\mathscr{T}_w}$ and $\widehat{\mathscr{T}_w} = \mathscr{T}_{\hat{w}}$ (Theorem 4.4). In [Mo], Mockor proved that $\tilde{w}$ is a (Manis) valuation on $\hat{K}^{,\mathscr{T}}$ for any $w \in \Omega$ and that $\hat{\mathscr{T}} = \sup\{\mathscr{T}_{\tilde{w}} | w \in \Omega\}$.

Mockor [Mo] asked if there exists a Prüfer domain A such that $\hat{A}$ is not a $\hat{\mathscr{T}}$-Prüfer ring or such that $\hat{A}^{,\mathscr{T}}$ is a Prüfer ring but not a $\hat{\mathscr{T}}$-Prüfer ring. In this section, we answer Mockor's question by constructing some examples. We show that (1) the completion $\hat{A}^{,\mathscr{T}}$ of an h-local Prüfer domain is a $\hat{\mathscr{T}}$-Prüfer ring $\Leftrightarrow$ $|\operatorname{Max}(A)| < \infty$, (2) the completion $\hat{E}^{,\mathscr{T}}$ of the ring E of entire functions is not a $\hat{\mathscr{T}}$-Prüfer ring, (3) the completion $\hat{D}^{,\mathscr{T}}$ of a Dedekind domain is a $\hat{\mathscr{T}}$-Prüfer ring $\Leftrightarrow$ $|\operatorname{Spec}(D)| < \infty$, and (4) the completion $\hat{A}^{,\mathcal{T}}$ of an SFT Prüfer domain with $J(C_\alpha) \neq \{0\}$ for all maximal chains C_α is a $\hat{\mathcal{T}}$-Prüfer ring $\Leftrightarrow$ there exist only finitely many independent valuation overrings of A. In the cases (1), (3), and (4), every nonminimal prime ideal of $\hat{A}^{,\mathcal{T}}$ is of the form $\hat{P}^{,\mathcal{T}}$, where P is a nonzero prime ideal of A. To show these, we begin with quoting Mockor's result.

THEOREM 8.1 [Mo, Theorem 14]. *Let A be a Prüfer domain. Then the following conditions are equivalent.*

(1) *$\hat{A}^{,\mathcal{T}}$ is a $\hat{\mathcal{T}}$-Prüfer ring.*

(2) *Every maximal regular ideal of $\hat{A}^{,\mathcal{T}}$ is open in $\hat{A}^{,\mathcal{T}}$ and $R_{\hat{w}} = (\hat{A}^{,\mathcal{T}_w})_{P(\hat{w})}$ for every $w \in \Omega$, where $P(\hat{w})$ is the center of $R_{\hat{w}}$ on $\hat{A}^{,\mathcal{T}_w}$.*

PROPOSITION 8.2. *Let A be an h-local Prüfer domain, the ring of entire functions, or an SFT Prüfer domain such that $J(C_\alpha) \neq \{0\}$ for all maximal chains C_α in* Spec$(A)^*$. *Then $R_{\hat{w}} = \hat{A}^{,\mathcal{T}_w}_{P(\hat{w})}$ for every $w \in \Omega$.*

Proof. Let A be an h-local Prüfer domain. Since $A \subseteq R_w \subseteq K$, $\hat{A}^{,\mathcal{T}_w} \subseteq \widehat{R_w}^{,\mathcal{T}_w} = R_{\hat{w}} \subseteq \hat{K}^{,\mathcal{T}_w}$. By Theorem 5.1, $\hat{A}^{,\mathcal{T}_w} = \widehat{A_{M_0}}^{,\mathcal{T}_{w_0}}$, where M_0 is the unique maximal ideal of A containing $P(w)$ and w_0 is the valuation corresponding to A_{M_0}. By applying Lemma 4.3 and Theorem 4.4(6) to A_{M_0} with $\Omega_0(A_{M_0}) = \{w\}$ and $\Omega_0(A_{M_0}) = \{w_0\}$, we deduce

$$\hat{K}^{,\mathcal{T}_w} = \hat{K}^{,\mathcal{T}_{w_0}} = \text{q.f.}\left(\widehat{A_{M_0}}^{,\mathcal{T}_{w_0}}\right).$$

Thus since $\hat{K}^{,\mathcal{T}_w}$ is the quotient field of $\hat{A}^{,\mathcal{T}_w}$, $R_{\hat{w}}$ is an overring of the valuation domain $\hat{A}^{,\mathcal{T}_w}$, and so $R_{\hat{w}} = \hat{A}^{,\mathcal{T}_w}_{P(\hat{w})}$.

Now let E be the ring of entire functions. Then by Theorem 6.4, $\hat{E}^{,\mathcal{T}_w} = \widehat{E_{M_0}}^{,\mathcal{T}_{w_0}}$, where M_0 is the unique maximal ideal of A containing $P(w)$ and w_0 is the valuation corresponding to E_{M_0}. From the same argument as above, it follows that $R_{\hat{w}} = \hat{E}^{,\mathcal{T}_w}_{P(\hat{w})}$.

The case when A is an SFT Prüfer domain follows directly from Theorems 4.4(5) and 7.5(2). ∎

Thus in an h-local Prüfer domain, the ring of entire functions and an SFT Prüfer domain such that $J(C_\alpha) \neq \{0\}$ for all maximal chains in Spec$(A)^*$, the second condition in Theorem 8.1(2) is satisfied. Therefore, to determine whether $\hat{A}^{,\mathcal{T}}$ is a $\hat{\mathcal{T}}$-Prüfer ring, it suffices to check whether every maximal regular ideal of $\hat{A}^{,\mathcal{T}}$ is open in $\hat{A}^{,\mathcal{T}}$.

LEMMA 8.3. *Let A be a Prüfer domain and let Ω_0 be a representing family of the independent valuations in Ω. Assume that each nonzero element a of A is a nonunit in $\hat{A}^{,\mathcal{T}_w}$ for only finitely many w's in Ω_0. Then if $|\Omega_0| = \infty$, $\hat{A}^{,\mathcal{T}}$ is not a $\hat{\mathcal{T}}$-Prüfer ring.*

Proof. Recall that $\hat{A}^{,\mathcal{T}} \cong \prod_{w \in \Omega_0} \hat{A}^{,\mathcal{T}_w}$ (Proposition 4.6). Note that if $|\Omega_0| = \infty$, then $\sum_{w \in \Omega_0} \hat{A}^{,\mathcal{T}_w}$ is a proper ideal of $\prod_{w \in \Omega_0} \hat{A}^{,\mathcal{T}_w}$. Since w is a nontrivial valuation on K, $\hat{A}^{,\mathcal{T}_w}$ is an integral domain which is not a field. (Note that $\hat{A}^{,\mathcal{T}_w} \subseteq \widehat{A_{P(w)}}^{,\mathcal{T}_w} \subseteq \hat{K}^{,\mathcal{T}_w}$ and $\widehat{P(w)A_{P(w)}}^{,\mathcal{T}_w} \cap A = P(w)$.) For each $w \in \Omega_0$, choose a nonzero nonunit element a_w in $\hat{A}^{,\mathcal{T}_w}$. Then

$$\sum_{w \in \Omega_0} \hat{A}^{,\mathcal{T}_w} + \left(\prod_{w \in \Omega_0} a_w \right) \neq \prod_{w \in \Omega_0} \hat{A}^{,\mathcal{T}_w}.$$

For otherwise, there exist $\prod_{w \in \Omega_0} x_w \in \sum_{w \in \Omega_0} \hat{A}^{,\mathcal{T}_w}$ and $\prod_{w \in \Omega_0} y_w \in \prod_{w \in \Omega_0} \hat{A}^{,\mathcal{T}_w}$ such that $1 = \prod_{w \in \Omega_0} x_w + (\prod_{w \in \Omega_0} a_w)(\prod_{w \in \Omega_0} y_w)$. Since $x_w = 0$ for almost all (that is, for all but a finite number of) $w \in \Omega_0$, $a_w y_w = 1$ for almost all $w \in \Omega_0$. This contradicts our choice of a_w's.

Now let N be a maximal ideal of $\hat{A}^{,\mathcal{T}}$ containing $\sum_{w \in \Omega_0} \hat{A}^{,\mathcal{T}_w} + (\prod_{w \in \Omega_0} a_w)$. Since $\prod_{w \in \Omega_0} a_w$ is a regular element of $\hat{A}^{,\mathcal{T}}$, N is a regular maximal ideal of $\hat{A}^{,\mathcal{T}}$. Now we claim that $\sum_{w \in \Omega_0} \hat{A}^{,\mathcal{T}_w} + (\prod_{w \in \Omega_0} a) = \hat{A}^{,\mathcal{T}}$ for all $a \in A \setminus \{0\}$. Let $\{w_1, w_2, \dots, w_n\}$ be the finite subset of Ω_0 consisting of those elements w such that a is a nonunit in $\hat{A}^{,\mathcal{T}_w}$. For each $w \neq w_i$, let b_w be the inverse of a in $\hat{A}^{,\mathcal{T}_w}$. Put $x = \prod_{w \in \Omega_0} x_w$, where $x_w = b_w$ for $w \neq w_i$, $x_{w_i} = 0$ for $i = 1, 2, \dots, n$, and $y = \prod_{w \in \Omega_0} y_w$, where $y_w = 0$ for $w \neq w_i$, $y_{w_i} = 1$ for $i = 1, 2, \dots, n$. Then $x \in \prod_{w \in \Omega_0} \hat{A}^{,\mathcal{T}_w}$, $y \in \sum_{w \in \Omega_0} \hat{A}^{,\mathcal{T}_w}$, and $y + x(\prod_{w \in \Omega_0} a) = 1$. Therefore, $N \cap A = (0)$. If N is open in $\hat{A}^{,\mathcal{T}}$, then $N \cap A$ is also open in A, but since (0) is not open in A, N is not open in $\hat{A}^{,\mathcal{T}}$. Then by Theorem 8.1, $\hat{A}^{,\mathcal{T}}$ is not a $\hat{\mathcal{T}}$-Prüfer ring. ∎

THEOREM 8.4. *Let A be an h-local Prüfer domain. Then $\hat{A}^{,\mathcal{T}}$ is a $\hat{\mathcal{T}}$-Prüfer ring if and only if* $|\mathrm{Max}(A)| < \infty$.

Proof. Note that $\Omega_0 = \{w \in \Omega \mid w$ is the valuation corresponding to A_{M_w}, $M_w \in \mathrm{Max}(A)\}$ is a representing family of the independent valuations in Ω. By Theorem 5.1, $\hat{A}^{,\mathcal{T}_w} = \widehat{A_{M_w}}^{,\mathcal{T}_w}$ and by Theorem 4.4 and Corollary 4.5, $\widehat{A_{M_w}}^{,\mathcal{T}_w}$ is a valuation domain with the maximal ideal $\widehat{M_w A_{M_w}}^{,\mathcal{T}_w}$. Since $\widehat{M_w A_{M_w}}^{,\mathcal{T}_w} \cap A = M_w$, each nonzero element a of A is a nonunit in $\hat{A}^{,\mathcal{T}_w}$ for only finitely many w's in Ω_0. So by Lemma 8.3, the "only if" part follows.

Now assume that $\mathrm{Max}(A) = \{M_1, \dots, M_n\}$. Then $\hat{A}^{,\mathcal{T}} \cong \hat{A}^{,\mathcal{T}_{w_1}} \oplus \cdots \oplus \hat{A}^{,\mathcal{T}_{w_n}}$, where w_j is the valuation corresponding to A_{M_j}, $j = 1, 2, \dots, n$. Let N be a (regular) maximal ideal of $\hat{A}^{,\mathcal{T}}$. Then $N = \hat{A}^{,\mathcal{T}_{w_1}} \oplus \cdots \oplus \hat{A}^{,\mathcal{T}_{w_{i-1}}} \oplus N_i \oplus \hat{A}^{,\mathcal{T}_{w_{i+1}}} \oplus \cdots \oplus \hat{A}^{,\mathcal{T}_{w_n}}$, where N_i is a maximal ideal of $\hat{A}^{,\mathcal{T}_{w_i}}$. Note that $\hat{A}^{,\mathcal{T}}_{[N]} \cong \hat{K}^{,\mathcal{T}_{w_1}} \oplus \cdots \oplus \hat{K}^{,\mathcal{T}_{w_{i-1}}} \oplus \hat{A}^{,\mathcal{T}_{w_i}}_{N_i} \oplus \hat{K}^{,\mathcal{T}_{w_{i+1}}} \oplus \cdots \oplus \hat{K}^{,\mathcal{T}_{w_n}}$ and $\hat{A}^{,\mathcal{T}_{w_i}}_{N_i} = \hat{A}^{,\mathcal{T}_{w_i}}$. By the same argument as in Proposition 4.6 (or see [Mo, Lemma 8]), we can show that $\hat{K}^{,\mathcal{T}} \cong \hat{K}^{,\mathcal{T}_{w_1}} \oplus \cdots \oplus \hat{K}^{,\mathcal{T}_{w_n}}$. Let v_i: $\hat{K}^{,\mathcal{T}} \cong \prod_{j=1}^{n} \hat{K}^{,\mathcal{T}_{w_j}} \rightarrow$

$G_{w_i} \cup \{\infty\}$ be given by $x = (x_1, \dots, x_n) \mapsto \widehat{w_i}(x_i)$. Then v_i is a (Manis) valuation on $\hat{K}^{,\mathcal{T}}$ and a continuous extension of w_i. So $v_i = \tilde{w}_i$. Since $R_{v_i} = \hat{K}^{,\mathcal{T}_{w_1}} \oplus \cdots \oplus \hat{K}^{,\mathcal{T}_{w_{i-1}}} \oplus R_{\widehat{w_i}} \oplus \hat{K}^{,\mathcal{T}_{w_{i+1}}} \oplus \cdots \oplus \hat{K}^{,\mathcal{T}_{w_n}}$, which is equal to $\hat{A}^{,\mathcal{T}}_{[N]}$, and $v_i = \tilde{w}_i$ is continuous in $\hat{\mathcal{T}}$, $\hat{A}^{,\mathcal{T}}$ is a $\hat{\mathcal{T}}$-Prüfer ring. ∎

Since every Dedekind domain is an h-local Prüfer domain, from the above theorem, we obtain

COROLLARY 8.5. *Let A be a Dedekind domain. Then $\hat{A}^{,\mathcal{T}}$ is a $\hat{\mathcal{T}}$-Prüfer ring if and only if A has only finitely many prime ideals.*

THEOREM 8.6. *Let E be the ring of entire functions. Then $\hat{E}^{,\mathcal{T}}$ is a Prüfer ring but not a $\hat{\mathcal{T}}$-Prüfer ring.*

Proof. Let $\Omega_0 = \{w \in \Omega(E) \mid P(w) \in \operatorname{Max}(E)\}$, $\Omega_1 = \{w \in \Omega_0 \mid P(w)$ is a maximal fixed ideal$\}$, and $\Omega_2 = \{w \in \Omega_0 \mid P(w)$ is a maximal free ideal$\}$. Since $\Omega_0 = \Omega_1 \dot{\cup} \Omega_2$, $\hat{E}^{,\mathcal{T}} \cong \prod_{w \in \Omega_0} \hat{E}^{,\mathcal{T}_w} = (\prod_{w \in \Omega_1} \hat{E}^{,\mathcal{T}_w}) \oplus (\prod_{w \in \Omega_2} \hat{E}^{,\mathcal{T}_w})$ (Proposition 4.6). Recall that $\{(X - \alpha) | \alpha \in \mathbb{C}\}$ is the set of all fixed maximal ideals of E and for each $f \in E \setminus \{0\}$, $Z(f)$, which is the set of zeros of f, is a countable discrete set with no limit point in the open complex plane [Gi, p. 146]. Let $I = \{\prod_{w \in \Omega_1} y_w \in \prod_{w \in \Omega_1} \hat{E}^{,\mathcal{T}_w} | y_w = 0$ except for countably many w's$\}$. Then since $|\Omega_1| = c$, where c is the cardinal number of the continuum, I is a proper ideal of $\prod_{w \in \Omega_1} \hat{E}^{,\mathcal{T}_w}$. For each $w \in \Omega_1$, let $P(w) = (X - \alpha_w)$, $\alpha_w \in \mathbb{C}$. Then clearly $I + (\prod_{w \in \Omega_1} (X - \alpha_w)) \neq \prod_{w \in \Omega_1} \hat{E}^{,\mathcal{T}_w}$. Let N be a maximal ideal of $\prod_{w \in \Omega_1} \hat{E}^{,\mathcal{T}_w}$ containing $I + (\prod_{w \in \Omega_1} (X - \alpha_w))$. Since $\prod_{w \in \Omega_1} (X - \alpha_w)$ is a regular element of $\prod_{w \in \Omega_1} \hat{E}^{,\mathcal{T}_w}$, N is regular. Since for each $f \in E \setminus \{0\}$, $Z(f)$ is a countable set, f is a nonunit in $\hat{E}^{,\mathcal{T}_w}$ only for countably many w's in Ω_1 and hence as in Lemma 8.3, we can show that $I + (\prod_{w \in \Omega_1} f) = \prod_{w \in \Omega_1} \hat{E}^{,\mathcal{T}_w}$ for all $f \in E \setminus \{0\}$. This implies that $N \cap E = (0)$. Let $N_0 = N \oplus (\prod_{w \in \Omega_2} \hat{E}^{,\mathcal{T}_w})$. Then N_0 is a regular maximal ideal of $\hat{E}^{,\mathcal{T}}$ such that $N_0 \cap E = (0)$. Then by Theorems 8.1 and 6.4, $\hat{E}^{,\mathcal{T}}$ is a Prüfer ring which is not a $\hat{\mathcal{T}}$-Prüfer ring. ∎

THEOREM 8.7. *Let A be an SFT Prüfer domain such that for each maximal chain C_α in* Spec$(A)^*$, *$J(C_\alpha) \neq \{0\}$. Then $\hat{A}^{,\mathcal{T}}$ is a $\hat{\mathcal{T}}$-Prüfer ring if and only if $|\Omega_0| < \infty$.*

Proof. For each $w \in \Omega_0$, let $P(w)$ be the center of R_w on A. As in Remark 4.8, let C_w be a maximal chain in Spec$(A)^*$ containing $P(w)$. Then since A is an SFT Prüfer domain, $\hat{A}^{,\mathcal{T}_w} = \hat{A}^{,C_w}$ for all $w \in \Omega_0$ (Lemma 3.4). Since $J(C_w) \neq \{0\}$, Spec$(\hat{A}^{,C_w}) = \{(0)\} \cup \{\widehat{P_0}^{,C_w} | P_0$ contains some $P \in C_w\}$ by Theorem 7.5. Note that $\widehat{P_0}^{,C_w} \cap A = P_0$. Now let a be a nonzero element of A. Since A is an SFT ring, (a) has only finitely many minimal prime divisors, say $P_1, \ldots, P_n$ (Proposition 2.1). For each i, let $w_i \in \Omega_0$ be the valuation such that $P_i \sim P(w_i)$. We claim that a is a unit in $\hat{A}^{,C_w}$ for all $w \neq w_i$. Assume the contrary. Then a is a nonunit in $\hat{A}^{,C_w}$ for some $w \neq w_i$ and hence $a \in \hat{M}^{,C_w}$ for some $\hat{M}^{,C_w} \in \operatorname{Max}(\hat{A}^{,C_w})$. Since $a \in M = \hat{M}^{,C_w} \cap A$, $P_i \subseteq M$ for some i. This implies that $P_i \sim M$. Moreover since $M \sim P(w)$, $P(w_i) \sim P(w)$. By the definition of Ω_0, $w = w_i$, a contradiction. Therefore, by Lemma 8.3, the "only if" part follows.

Now let $\Omega_0 = \{w_1, \ldots, w_n\}$. Since $J(C_{w_j}) \neq \{0\}$ for all $j = 1, \ldots, n$, $\hat{K}^{,\mathcal{T}_{w_j}} = \text{q.f.}(\hat{A}^{,C_{w_j}})$ by Lemma 7.4. Let N_i be a maximal ideal of $\hat{A}^{,C_{w_i}}$. Then by Theorem 7.5(2), $N_i = \hat{M}^{,C_{w_i}}$ where $M = N_i \cap A$ and $\hat{A}^{,C_{w_i}}_{N_i} = \widehat{A_M}^{,\mathcal{T}_{w_i}} = R_{\widehat{w_i}}$. The same argument as in the proof of Theorem 8.4 allows us to conclude that $\hat{A}^{,\mathcal{T}}$ is a $\mathcal{T}$-Prüfer ring. ∎

Remark 8.8. (1) It is easy to see that the collection $\{J(C_\alpha)\}_{\alpha \in \Lambda}$ is independent of a particular representing family $\{C_\alpha\}_{\alpha \in \Lambda}$ of the maximal chains in $\operatorname{Spec}(A)^*$. Then what matters in Theorem 8.7 is whether there exists a representing family $\{C_\alpha\}_{\alpha \in \Lambda}$ of the maximal chains in $\operatorname{Spec}(A)^*$ such that $J(C_\alpha) \neq \{0\}$ for each $\alpha \in \Lambda$.

(2) Facchini's existence theorem (Theorem 2.5) and Lemma 7.3 provide a lot of SFT Prüfer domains satisfying the various conditions such as $J(C_\alpha) = \{0\}$, $J(C_\alpha) \neq \{0\}$, $|\Omega_0| = \infty$, and $|\Omega_0| < \infty$.

(3) Since every Dedekind domain is just a one-dimensional SFT Prüfer domain, Corollary 8.5 also follows from Theorem 8.7.

(4) Let A be a h-local Prüfer domain, the ring of entire functions, or an SFT Prüfer domain such that $J(C_\alpha) \neq \{0\}$ for all maximal chains C_α in $\operatorname{Spec}(A)^*$. In each case, we can see that $\operatorname{Spec}(\hat{A}^{,\mathcal{T}_w}) = \{(0)\} \cup \{\hat{P}^{,\mathcal{T}_w} | P \in \operatorname{Spec}(A)^* \text{ such that } P \sim P(w)\}$, $w \in \Omega$ (Corollary 4.5, Theorem 5.1, Theorem 6.4, and Theorem 7.5). Since by Proposition 4.6, $\hat{A}^{,\mathcal{T}} \cong \prod_{w \in \Omega_0} \hat{A}^{,\mathcal{T}_w}$, if $|\Omega_0| < \infty$, then we can easily describe $\operatorname{Spec}(\hat{A}^{,\mathcal{T}})$ as follows. Let $\Omega_0 = \{w_1, \ldots, w_n\}$. Then $\operatorname{Spec}(\hat{A}^{,\mathcal{T}}) = \{\hat{A}^{,\mathcal{T}_{w_1}} \oplus \cdots \oplus \hat{A}^{,\mathcal{T}_{w_{i-1}}} \oplus Q \oplus \hat{A}^{,\mathcal{T}_{w_{i+1}}} \oplus \cdots \oplus \hat{A}^{,\mathcal{T}_{w_n}} | Q \in \operatorname{Spec}(\hat{A}^{,\mathcal{T}_{w_i}}),\ i = 1, 2, \ldots, n\}$. Let $Q \in \operatorname{Spec}(\hat{A}^{,\mathcal{T}_{w_i}})^*$. Then $Q = \hat{P}^{,\mathcal{T}_{w_i}}$ for some $P \in \operatorname{Spec}(A)^*$ such that $P \sim P(w_i)$. By the definition of Ω_0, $P \nsim P(w_j)$ for all $j \neq i$, and so $\hat{P}^{,\mathcal{T}_{w_j}} = \hat{A}^{,\mathcal{T}_{w_j}}$ for all $j \neq i$. Therefore, $\hat{A}^{,\mathcal{T}_{w_1}} \oplus \cdots \oplus \hat{A}^{,\mathcal{T}_{w_{i-1}}} \oplus Q \oplus \hat{A}^{,\mathcal{T}_{w_{i+1}}} \oplus \cdots \oplus \hat{A}^{,\mathcal{T}_{w_n}} = \prod_{j=1}^{n} \hat{P}^{,\mathcal{T}_{w_j}} \cong \hat{P}^{,\mathcal{T}}$. Thus every nonminimal prime ideal of $\hat{A}^{,\mathcal{T}}$ is of the form $\hat{P}^{,\mathcal{T}}$, where P is a nonzero prime ideal of A. However, for the case $|\Omega_0| = \infty$, we have been unable to describe $\operatorname{Spec}(\hat{A}^{,\mathcal{T}})$.

We could neither describe $\operatorname{Spec}(\hat{A}^{,\mathcal{T}})$ when A is an SFT Prüfer domain such that $J(C_\alpha) = \{0\}$ for all $C_\alpha \in \mathcal{C}$, nor $\operatorname{Spec}(\hat{A}^{,C_\alpha})$, and we do not know if $\hat{A}^{,\mathcal{T}}$ is a Prüfer ring.

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