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On U-dominant dimension

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Abstract

Let Λ and Γ be artin algebras and ${}_{\Lambda}U_{\Gamma}$ a faithfully balanced selforthogonal bimodule. We show that the U-dominant dimensions of ${}_{\Lambda}U$ and U_{Γ} are identical. As applications of the results obtained, we give some characterizations of the double U-dual functors preserving monomorphisms and being left exact respectively.

Keywords: U-dominant dimension; Flat dimension; Faithfully balanced selforthogonal bimodules; Double U-dual functors

1. Introduction

For a ring Λ, we use $\operatorname{mod}\Lambda$ (respectively $\operatorname{mod}\Lambda^{\mathrm{op}}$) to denote the category of finitely generated left Λ-modules (respectively right Λ-modules).

Definition 1.1. Let Λ and Γ be rings. A bimodule ${}_{\Lambda}T_{\Gamma}$ is called a faithfully balanced selforthogonal bimodule if it satisfies the following conditions:

(1) ${}_{\Lambda}T \in \operatorname{mod}\Lambda$ and $T_{\Gamma} \in \operatorname{mod}\Gamma^{\mathrm{op}}$.
(2) The natural maps $\Lambda \to \operatorname{End}(T_{\Gamma})$ and $\Gamma \to \operatorname{End}({}_{\Lambda}T)^{\mathrm{op}}$ are isomorphisms.
(3) $\operatorname{Ext}^i_{\Lambda}({}_{\Lambda}T, {}_{\Lambda}T) = 0$ and $\operatorname{Ext}^i_{\Gamma}(T_{\Gamma}, T_{\Gamma}) = 0$ for any $i \geqslant 1$.

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Definition 1.2. Let U be in mod Λ (respectively mod Γ^{op}) and n a non-negative integer. For a module M in mod Λ (respectively mod Γ^{op}),

(1) M is said to have U-dominant dimension greater than or equal to n, written U-dom.dim$({}_{\Lambda}M)$ (respectively U-dom.dim$(M_{\Gamma})) \geqslant n$, if each of the first n terms in a minimal injective resolution of M is cogenerated by ${}_{\Lambda}U$ (respectively U_{Γ}), that is, each of these terms can be embedded into a direct product of copies of ${}_{\Lambda}U$ (respectively U_{Γ}) [10].

(2) M is said to have dominant dimension greater than or equal to n, written dom.dim$({}_{\Lambda}M)$ (respectively dom.dim$(M_{\Gamma})) \geqslant n$, if each of the first n terms in a minimal injective resolution of M is Λ-projective (respectively Γ^{op}-projective) [12].

Assume that Λ is an artin algebra. By [4, Theorem 3.3], Λ^I and each of its direct summands are projective for any index set I. So, when ${}_{\Lambda}U = {}_{\Lambda}\Lambda$ (respectively $U_{\Gamma} = \Lambda_{\Lambda}$), the notion of U-dominant dimension coincides with that of (ordinary) dominant dimension. Tachikawa in [12] showed that if Λ is a left and right artinian ring then the dominant dimensions of ${}_{\Lambda}\Lambda$ and Λ_{Λ} are identical. Hoshino then in [6] generalized this result to left and right noetherian rings. Kato in [10] characterized the modules with U-dominant dimension greater than or equal to one. Colby and Fuller in [5] gave some equivalent conditions of dom.dim$({}_{\Lambda}\Lambda) \geqslant 1$ (or 2) in terms of the properties of the double dual functors (with respect to ${}_{\Lambda}\Lambda_{\Lambda}$).

The results mentioned above motivate our interests in establishing the identity of U-dominant dimensions of ${}_{\Lambda}U$ and U_{Γ} and characterizing the properties of modules with a given U-dominant dimension. Our characterizations will lead a better comprehension about U-dominant dimension and the theory of selforthogonal bimodules.

Throughout this paper, Λ and Γ are artin algebras and ${}_{\Lambda}U_{\Gamma}$ is a faithfully balanced selforthogonal bimodule. The main result in this paper is the following

Theorem 1.3. U-dom.dim$({}_{\Lambda}U) = U$-dom.dim(U_{Γ}).

Put ${}_{\Lambda}U_{\Gamma} = {}_{\Lambda}\Lambda_{\Lambda}$, we immediately get the following result, which is due to Tachikawa (see [12]).

Corollary 1.4. dom.dim$({}_{\Lambda}\Lambda) =$ dom.dim(Λ_{Λ}).

Let M be in mod Λ (respectively mod Γ^{op}) and $G(M)$ the subcategory of mod Λ (respectively mod Γ^{op}) consisting of all submodules of the modules generated by M. M is called a QF-3 module if $G(M)$ has a cogenerator which is a direct summand of every other cogenerator [13]. By [13] Proposition 2.2 we have that a finitely cogenerated Λ-module (respectively Γ^{op}-module) M is a QF-3 module if and only if M cogenerates its injective envelope. So by Theorem 1.3 we have

Corollary 1.5. ${}_{\Lambda}U$ *is* QF-3 *if and only if* U_{Γ} *is* QF-3.

We shall prove our main result in Section 2. We study the case that the double U-dual functors $(-)^{**}$ preserves monomorphisms by the language of Lambek torsion theory, show the left–right symmetry of the fact that $(-)^{**}$ preserves monomorphisms, and then prove the main result. It should be pointed out that this strategy is similar to that of Hoshino [7]. As applications of the results obtained in Section 2, we give in Section 3 some characterizations of the double U-dual functors $(-)^{**}$ preserving monomorphisms and being left exact respectively. The results of this paper are natural generalizations of (ordinary) dominant dimension and of several author's approach to dominant dimension (see Tachikawa [12], Colby–Fuller [5] and Hoshino [6,7]). In fact, most of the results here are the U-dual versions of the results in [6,7].

2. The proof of main result

Let E_0 be the injective envelope of ${}_{\Lambda}U$. Then E_0 defines a torsion theory in $\operatorname{mod}\Lambda$. The torsion class $\mathcal{T}$ is the subcategory of $\operatorname{mod}\Lambda$ consisting of the modules X satisfying $\operatorname{Hom}_{\Lambda}(X, E_0) = 0$, and the torsionfree class $\mathcal{F}$ is the subcategory of $\operatorname{mod}\Lambda$ consisting of the modules Y cogenerated by E_0 (equivalently, Y can be embedded in E_0^I for some index set I). A module in $\operatorname{mod}\Lambda$ is called torsion (respectively torsionfree) if it is in $\mathcal{T}$ (respectively $\mathcal{F}$). The injective envelope E_0' of U_Γ also defines a torsion theory in $\operatorname{mod}\Gamma^{\mathrm{op}}$ and we may give in $\operatorname{mod}\Gamma^{\mathrm{op}}$ the corresponding notions as above. Let X be in $\operatorname{mod}\Lambda$ (respectively $\operatorname{mod}\Gamma^{\mathrm{op}}$) and $t(X)$ the torsion submodule, that is, $t(X)$ is the submodule X such that $\operatorname{Hom}_{\Lambda}(t(X), E_0) = 0$ (respectively $\operatorname{Hom}_{\Gamma}(t(X), E_0') = 0$) and E_0 (respectively E_0') cogenerates $X/t(X)$ (cf. [9]).

Let A be in $\operatorname{mod}\Lambda$ (respectively $\operatorname{mod}\Gamma^{\mathrm{op}}$). We call $\operatorname{Hom}_{\Lambda}({}_{\Lambda}A, {}_{\Lambda}U_{\Gamma})$ (respectively $\operatorname{Hom}_{\Gamma}(A_{\Gamma}, {}_{\Lambda}U_{\Gamma})$) the dual module of A with respect to ${}_{\Lambda}U_{\Gamma}$, and denote either of these modules by A^*. For a homomorphism f between Λ-modules (respectively Γ^{op}-modules), we put $f^* = \operatorname{Hom}(f, {}_{\Lambda}U_{\Gamma})$. Let $\sigma_A : A \to A^{**}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^*$ be the canonical evaluation homomorphism. A is called U-torsionless (respectively U-reflexive) if σ_A is a monomorphism (respectively an isomorphism).

The following result is analogous to [7, Lemma 4].

Lemma 2.1. *For a module X in* $\operatorname{mod}\Lambda$ (*respectively* $\operatorname{mod}\Gamma^{\mathrm{op}}$), $t(X) = \operatorname{Ker}\sigma_X$ *if and only if* $\operatorname{Hom}_{\Lambda}(\operatorname{Ker}\sigma_X, E_0) = 0$ (*respectively* $\operatorname{Hom}_{\Gamma}(\operatorname{Ker}\sigma_X, E_0') = 0$).

Proof. The necessity is trivial. Now we prove the sufficiency.

We have the following commutative diagram with the upper row exact:

$$\begin{tikzcd}
0 \arrow[r] & t(X) \arrow[r] & X \arrow[r, "\pi"] \arrow[d, "\sigma_X"] & X/t(X) \arrow[r] \arrow[d, "\sigma_{X/t(X)}"] & 0 \\
 & & X^{**} \arrow[r, "\pi^{**}"] & {[X/t(X)]^{**}} &
\end{tikzcd}$$

Since $\operatorname{Hom}_{\Lambda}(t(X), E_0) = 0$, $[t(X)]^* = 0$ and π^* is an isomorphism. So π^{**} is also an isomorphism and hence $t(X) \subset \operatorname{Ker}\sigma_X$. On the other hand, $\operatorname{Hom}_{\Lambda}(\operatorname{Ker}\sigma_X, E_0) = 0$ by assumption, which implies that $\operatorname{Ker}\sigma_X$ is a torsion module and contained in X. So we conclude that $\operatorname{Ker}\sigma_X \subset t(X)$ and $\operatorname{Ker}\sigma_X = t(X)$. □

Remark. From the above proof we always have $t(X) \subset \operatorname{Ker}\sigma_X$.

Suppose that $A \in \operatorname{mod}\Lambda$ (respectively $\operatorname{mod}\Gamma^{\mathrm{op}}$) and $P_1 \xrightarrow{f} P_0 \to A \to 0$ is a (minimal) projective resolution of A. Then we have an exact sequence

$$0 \to A^* \to P_0^* \xrightarrow{f^*} P_1^* \to \operatorname{Coker} f^* \to 0.$$

We call $\operatorname{Coker} f^*$ the transpose (with respect to ${}_\Lambda U_\Gamma$) of A, and denote it by $\operatorname{Tr}_U A$.

The following result is the U-dual version of [7, Theorem A].

Proposition 2.2. *The following statements are equivalent.*

(1) $t(X) = \operatorname{Ker}\sigma_X$ *for every* $X \in \operatorname{mod}\Lambda$.
(2) f^{**} *is monic for every monomorphism* $f : A \to B$ *in* $\operatorname{mod}\Lambda$.
$(1)^{\mathrm{op}}$ $t(Y) = \operatorname{Ker}\sigma_Y$ *for every* $Y \in \operatorname{mod}\Gamma^{\mathrm{op}}$.
$(2)^{\mathrm{op}}$ g^{**} *is monic for every monomorphism* $g : C \to D$ *in* $\operatorname{mod}\Gamma^{\mathrm{op}}$.

Proof. By symmetry, it suffices to prove the implications of $(1) \Rightarrow (2)^{\mathrm{op}} \Rightarrow (1)^{\mathrm{op}}$.

$(1) \Rightarrow (2)^{\mathrm{op}}$. Let $g : C \to D$ be monic in $\operatorname{mod}\Gamma^{\mathrm{op}}$. Set $X = \operatorname{Coker} g$. We have that $\operatorname{Ker}\sigma_{\operatorname{Tr}_U X} \cong \operatorname{Ext}^1_\Gamma(X, U)$ and $\operatorname{Tr}_U X \in \operatorname{mod}\Lambda$ by [8, Lemma 2.1]. By (1) and Lemma 2.1, $\operatorname{Hom}_\Lambda(\operatorname{Ext}^1_\Gamma(X, U), E_0) = 0$. Since $\operatorname{Coker} g^*$ can be imbedded in $\operatorname{Ext}^1_\Gamma(X, U)$, $\operatorname{Hom}_\Lambda(\operatorname{Coker} g^*, E_0) = 0$. But $(\operatorname{Coker} g^*)^* \subset \operatorname{Hom}_\Lambda(\operatorname{Coker} g^*, E_0)$, so $(\operatorname{Coker} g^*)^* = 0$ and hence $\operatorname{Ker} g^{**} \cong (\operatorname{Coker} g^*)^* = 0$, which implies that g^{**} is monic.

$(2)^{\mathrm{op}} \Rightarrow (1)^{\mathrm{op}}$. Let Y be in $\operatorname{mod}\Gamma^{\mathrm{op}}$ and X any submodule of $\operatorname{Ker}\sigma_Y$ and $f_1 : X \to \operatorname{Ker}\sigma_Y$ the inclusion. Assume that f is the composition:

$$X \xrightarrow{f_1} \operatorname{Ker}\sigma_Y \to Y.$$

Then $\sigma_Y f = 0$ and $f^*\sigma_Y^* = (\sigma_Y f)^* = 0$. But σ_Y^* is epic by [1, Proposition 20.14], so $f^* = 0$ and $f^{**} = 0$. By $(2)^{\mathrm{op}}$, f^{**} is monic, so $X^{**} = 0$ and $X^{***} = 0$. Since X^* is isomorphic to a submodule of X^{***} by [1, Proposition 20.14], $X^* = 0$.

We claim: $\operatorname{Hom}_\Gamma(\operatorname{Ker}\sigma_Y, E_0') = 0$. Otherwise, there exists $0 \neq \alpha \in \operatorname{Hom}_\Gamma(\operatorname{Ker}\sigma_Y, E_0')$. Then $\operatorname{Im}\alpha \cap U_\Gamma \neq 0$ since U_Γ is an essential submodule of E_0'. So $\alpha^{-1}(\operatorname{Im}\alpha \cap U_\Gamma)$ is a non-zero submodule of $\operatorname{Ker}\sigma_Y$ and there exists a non-zero map $\alpha^{-1}(\operatorname{Im}\alpha \cap U_\Gamma) \to U_\Gamma$, which implies that $(\alpha^{-1}(\operatorname{Im}\alpha \cap U_\Gamma))^* \neq 0$, a contradiction with the former argument. Hence we conclude that $t(Y) = \operatorname{Ker}\sigma_Y$ by Lemma 2.1. □

Let A be a Λ-module (respectively a Γ^{op}-module). Denote either of $\operatorname{Hom}_\Lambda({}_\Lambda U_\Gamma, {}_\Lambda A)$ and $\operatorname{Hom}_\Gamma({}_\Lambda U_\Gamma, A_\Gamma)$ by ${}^*A$, and the left (respectively right) flat dimension of A by $\operatorname{l.fd}_\Lambda(A)$ (respectively $\operatorname{r.fd}_\Gamma(A)$). We give a remark as follows. For an artin algebra R and a left (respectively right) R-module A, we have that the left (respectively right) flat dimension of A and its left (respectively right) projective dimension are identical; especially, A is left (respectively right) flat if and only if it is left (respectively right) projective.

Lemma 2.3. *Let ${}_{\Lambda}E$ (respectively E_{Γ}) be injective and n a non-negative integer. Then* $\mathrm{l.fd}_{\Gamma}({}^*E)$ (*respectively* $\mathrm{r.fd}_{\Lambda}({}^*E) \leqslant n$) *if and only if* $\mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Gamma}^{n+1}(A,U),E)$ (*respectively* $\mathrm{Hom}_{\Gamma}(\mathrm{Ext}_{\Lambda}^{n+1}(A,U),E)=0$) *for any* $A \in \mathrm{mod}\,\Gamma^{\mathrm{op}}$ (*respectively* $\mathrm{mod}\,\Lambda$).

Proof. It is trivial by [3, Chapter VI, Proposition 5.3]. □

The following result is similar to [7, Proposition B]. In fact, we obtain the first two statements of this result by replacing "$E({}_R R)$ is flat" and "E is flat" of [7, Proposition B] by "${}^*E_0$ is flat" and "${}^*E$ is flat" respectively. The third statement is analogous to the corresponding one of [7, Proposition B].

Proposition 2.4. *The following statements are equivalent.*

(1) ${}^*E_0$ *is flat.*
(2) *There is an injective Λ-module E such that ${}^*E$ is flat and E cogenerates E_0.*
(3) $t(X) = \mathrm{Ker}\,\sigma_X$ *for any* $X \in \mathrm{mod}\,\Lambda$.

Proof. (1) $\Rightarrow$ (2). It is trivial.

(2) $\Rightarrow$ (3). Let $X \in \mathrm{mod}\,\Lambda$. Since $\mathrm{Ker}\,\sigma_X \cong \mathrm{Ext}_{\Gamma}^1(\mathrm{Tr}_U X, U)$ with $\mathrm{Tr}_U X \in \mathrm{mod}\,\Gamma^{\mathrm{op}}$ by [8, Lemma 2.1]. By (2) and Lemma 2.3, $\mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Gamma}^1(\mathrm{Tr}_U X, U), E) = 0$.

Since E cogenerates E_0, there is an exact sequence $0 \to E_0 \to E^I$ for some index set I. So

$$\mathrm{Hom}_{\Lambda}\big(\mathrm{Ext}_{\Gamma}^1(\mathrm{Tr}_U X, U), E_0\big) \subset \mathrm{Hom}_{\Lambda}\big(\mathrm{Ext}_{\Gamma}^1(\mathrm{Tr}_U X, U), E^I\big)$$
$$\cong \big[\mathrm{Hom}_{\Lambda}\big(\mathrm{Ext}_{\Gamma}^1(\mathrm{Tr}_U X, U), E\big)\big]^I = 0 \quad \text{and}$$

$$\mathrm{Hom}_{\Lambda}\big(\mathrm{Ext}_{\Gamma}^1(\mathrm{Tr}_U X, U), E_0\big) = 0.$$

By Lemma 2.1, $t(X) = \mathrm{Ker}\,\sigma_X$.

(3) $\Rightarrow$ (1). Let $N \in \mathrm{mod}\,\Gamma^{\mathrm{op}}$. Since $\mathrm{Ker}\,\sigma_{\mathrm{Tr}_U N} \cong \mathrm{Ext}_{\Gamma}^1(N, U)$ with $\mathrm{Tr}_U N \in \mathrm{mod}\,\Lambda$ by [8, Lemma 2.1], By (3) and Lemma 2.1 we have $\mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Gamma}^1(N, U), E_0) \cong \mathrm{Hom}_{\Lambda}(\mathrm{Ker}\,\sigma_{\mathrm{Tr}_U N}, E_0) = 0$, and so ${}^*E_0$ is flat by Lemma 2.3. □

Dually, we have the following

Proposition 2.4′. *The following statements are equivalent.*

(1) ${}^*E_0'$ *is flat.*
(2) *There is an injective Γ^{op}-module E' such that ${}^*E'$ is flat and E' cogenerates E_0'.*
(3) $t(Y) = \mathrm{Ker}\,\sigma_Y$ *for any* $Y \in \mathrm{mod}\,\Gamma^{\mathrm{op}}$.

Corollary 2.5. ${}^*E_0$ *is flat if and only if* ${}^*E_0'$ *is flat.*

Proof. By Propositions 2.2, 2.4 and 2.4′. □

Let $A \in \operatorname{mod}\Lambda$ (respectively $\operatorname{mod}\Gamma^{\mathrm{op}}$) and i a non-negative integer. We say that the grade of A with respect to ${}_\Lambda U_\Gamma$, written $\operatorname{grade}_U A$, is greater than or equal to i if $\operatorname{Ext}^j_\Lambda(A, U) = 0$ (respectively $\operatorname{Ext}^j_\Gamma(A, U) = 0$) for any $0 \leqslant j < i$.

Lemma 2.6. *Let X be in $\operatorname{mod}\Gamma^{\mathrm{op}}$ and n a non-negative integer. If $\operatorname{grade}_U X \geqslant n$ and $\operatorname{grade}_U \operatorname{Ext}^n_\Gamma(X, U) \geqslant n + 1$, then $\operatorname{Ext}^n_\Gamma(X, U) = 0$.*

Proof. Since X^* is U-torsionless, $X^{**} = 0$ if and only if $X^* = 0$. Then the case $n = 0$ follows.

Now let $n \geqslant 1$ and

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to X \to 0$$

be a projective resolution of X in $\operatorname{mod}\Gamma^{\mathrm{op}}$. Put $X_n = \operatorname{Coker}(P_{n+1} \to P_n)$. Then we have an exact sequence

$$0 \to P_0^* \to \cdots \to P_{n-1}^* \xrightarrow{f} X_n^* \to \operatorname{Ext}^n_\Gamma(X, U) \to 0$$

in $\operatorname{mod}\Lambda$ with each $P_i^* \in \operatorname{add} {}_\Lambda U$. Since $\operatorname{grade}_U \operatorname{Ext}^n_\Gamma(X, U) \geqslant n + 1$,

$$\operatorname{Ext}^i_\Lambda\bigl(\operatorname{Ext}^n_\Gamma(X, U), U\bigr) = 0 \quad \text{for any } 0 \leqslant i \leqslant n.$$

So $\operatorname{Ext}^i_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), P_j^*) = 0$ for any $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n - 1$, and hence $\operatorname{Ext}^1_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), \operatorname{Im} f) \cong \operatorname{Ext}^n_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), P_0^*) = 0$, which implies that we have an exact sequence $\operatorname{Hom}_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), X_n^*) \to \operatorname{Hom}_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), \operatorname{Ext}^n_\Gamma(X, U)) \to 0$. Notice that X_n^* is U-torsionless and $\operatorname{Hom}_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), U) = 0$. So $\operatorname{Hom}_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), X_n^*) = 0$ and $\operatorname{Hom}_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), \operatorname{Ext}^n_\Gamma(X, U)) = 0$, which implies that $\operatorname{Ext}^n_\Gamma(X, U) = 0$. □

Remark. We point out that all of the above results (from 2.1 to 2.6) in this section also hold in the case Λ and Γ are left and right noetherian rings.

For a module T in $\operatorname{mod}\Lambda$ (respectively $\operatorname{mod}\Gamma^{\mathrm{op}}$), we use $\operatorname{add} {}_\Lambda T$ (respectively $\operatorname{add} T_\Gamma$) to denote the subcategory of $\operatorname{mod}\Lambda$ (respectively $\operatorname{mod}\Gamma^{\mathrm{op}}$) consisting of all modules isomorphic to direct summands of finite direct sums of copies of ${}_\Lambda T$ (respectively T_Γ). Let A be in $\operatorname{mod}\Lambda$. If there is an exact sequence $\cdots \to U_n \to \cdots \to U_1 \to U_0 \to A \to 0$ in $\operatorname{mod}\Lambda$ with each $U_i \in \operatorname{add} {}_\Lambda U$ for any $i \geqslant 0$, then we define $U\text{-resol.dim}_\Lambda(A) = \inf\{n \mid$ there is an exact sequence $0 \to U_n \to \cdots \to U_1 \to U_0 \to A \to 0$ in $\operatorname{mod}\Lambda$ with each $U_i \in \operatorname{add} {}_\Lambda U$ for any $0 \leqslant i \leqslant n\}$. We set $U\text{-resol.dim}_\Lambda(A)$ infinity if no such an integer exists. Dually, for a module B in $\operatorname{mod}\Gamma^{\mathrm{op}}$, we may define $U\text{-resol.dim}_\Gamma(B)$ (see [2]).

Lemma 2.7. *Let E be injective in $\operatorname{mod}\Lambda$ (respectively $\operatorname{mod}\Gamma^{\mathrm{op}}$). Then $\mathrm{l.fd}_\Gamma({}^*E)$ (respectively $\mathrm{r.fd}_\Lambda({}^*E) \leqslant n$) if and only if $U\text{-resol.dim}_\Lambda(E)$ (respectively $U\text{-resol.dim}_\Gamma(E) \leqslant n$).*

Proof. Assume that E is injective in $\operatorname{mod}\Lambda$ and $\text{l.fd}_{\Gamma}({}^*E) \leqslant n$. Then there is an exact sequence $0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to^* E \to 0$ with each Q_i flat (and hence projective) in $\operatorname{mod}\Gamma$ for any $0 \leqslant i \leqslant n$. By [3, Chapter VI, Proposition 5.3] $\operatorname{Tor}_j^{\Gamma}(U, {}^*E) \cong \operatorname{Hom}_{\Lambda}(\operatorname{Ext}_{\Gamma}^j(U, U), E) = 0$ for any $j \geqslant 1$. Then we easily have an exact sequence:

$$0 \to U \otimes_{\Gamma} Q_n \to \cdots \to U \otimes_{\Gamma} Q_1 \to U \otimes_{\Gamma} Q_0 \to U \otimes_{\Gamma}^* E \to 0.$$

It is clear that $U \otimes_{\Gamma} Q_i \in \operatorname{add}{}_{\Lambda}U$ for any $0 \leqslant i \leqslant n$. By [11, p. 47], $U \otimes_{\Gamma}^* E \cong \operatorname{Hom}_{\Lambda}(\operatorname{Hom}_{\Gamma}(U, U), E) \cong E$. Hence we conclude that $U\text{-resol.dim}_{\Lambda}(E) \leqslant n$.

Conversely, if $U\text{-resol.dim}_{\Lambda}(E) \leqslant n$ then there is an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to E \to 0$ with each X_i in $\operatorname{add}{}_{\Lambda}U$ for any $0 \leqslant i \leqslant n$. Since $\operatorname{Ext}_{\Lambda}^j(U, X_i) = 0$ for any $j \geqslant 1$ and $0 \leqslant i \leqslant n$, $0 \to {}^*X_n \to \cdots \to {}^*X_1 \to {}^*X_0 \to^* E \to 0$ is exact with each ${}^*X_i$ $(0 \leqslant i \leqslant n)$ Γ-projective. Hence we are done. $\square$

Corollary 2.8. *Let E be injective in* $\operatorname{mod}\Lambda$ (*respectively* $\operatorname{mod}\Gamma^{\text{op}}$). *Then ${}^*E$ is flat in* $\operatorname{mod}\Gamma$ (*respectively* $\operatorname{mod}\Lambda^{\text{op}}$) *if and only if* ${}_{\Lambda}E \in \operatorname{add}{}_{\Lambda}U$ (*respectively* $E_{\Gamma} \in \operatorname{add}U_{\Gamma}$).

From now on, assume that

$$0 \to {}_{\Lambda}U \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} E_i \xrightarrow{f_{i+1}} \cdots$$

is a minimal injective resolution of ${}_{\Lambda}U$.

The following result is the U-dual version of [6, Lemma 2.2].

Lemma 2.9. *Suppose $U\text{-dom.dim}({}_{\Lambda}U) \geqslant 1$. Then, for any $n \geqslant 2$, $U\text{-dom.dim}({}_{\Lambda}U) \geqslant n$ if and only if $\operatorname{grade}_U M \geqslant n$ for any $M \in \operatorname{mod}\Lambda$ with $M^* = 0$.*

Proof. For any $M \in \operatorname{mod}\Lambda$ and $i \geqslant 1$, we have an exact sequence

$$\operatorname{Hom}_{\Lambda}(M, E_{i-1}) \to \operatorname{Hom}_{\Lambda}(M, \operatorname{Im} f_i) \to \operatorname{Ext}_{\Lambda}^i(M, U) \to 0. \tag{†}$$

Suppose $U\text{-dom.dim}({}_{\Lambda}U) \geqslant n$. Then E_i is cogenerated by ${}_{\Lambda}U$ for any $0 \leqslant i \leqslant n-1$. So, for a given $M \in \operatorname{mod}\Lambda$ with $M^* = 0$ we have that $\operatorname{Hom}_{\Lambda}(M, E_i) = 0$ and $\operatorname{Hom}_{\Lambda}(M, \operatorname{Im} f_i) = 0$ for any $0 \leqslant i \leqslant n-1$. Then by the exactness of (†), $\operatorname{Ext}_{\Lambda}^i(M, U) = 0$ for any $1 \leqslant i \leqslant n-1$, and so $\operatorname{grade}_U M \geqslant n$.

Now we prove the converse, that is, we will prove that $E_i \in \operatorname{add}{}_{\Lambda}U$ for any $0 \leqslant i \leqslant n-1$.

First, $E_0 \in \operatorname{add}{}_{\Lambda}U$ by assumption. We next prove $E_1 \in \operatorname{add}{}_{\Lambda}U$. For any $0 \neq x \in \operatorname{Im} f_1$, we claim that $M^* = \operatorname{Hom}_{\Lambda}(M, U) \neq 0$, where $M = \Lambda x$. Otherwise, we have $\operatorname{Ext}_{\Lambda}^i(M, U) = 0$ for any $0 \leqslant i \leqslant n-1$ by assumption. Since $E_0 \in \operatorname{add}{}_{\Lambda}U$, $\operatorname{Hom}_{\Lambda}(M, E_0) = 0$. So from the exactness of (†) we know that $\operatorname{Hom}_{\Lambda}(M, \operatorname{Im} f_1) = 0$, which is a contradiction. Then we conclude that $\operatorname{Im} f_1$, and hence E_1, is cogenerated by ${}_{\Lambda}U$. Notice that E_1 is finitely cogenerated, so $E_1 \in \operatorname{add}{}_{\Lambda}U$. Finally, suppose that $n \geqslant 3$ and $E_i \in \operatorname{add}{}_{\Lambda}U$ for any $0 \leqslant i \leqslant n-2$. Then by using a similar argument to that above we have $E_{n-1} \in \operatorname{add}{}_{\Lambda}U$. The proof is finished. $\square$

Dually, we have the following

Lemma 2.9′. *Suppose* U-dom.dim$(U_\Gamma) \geqslant 1$. *Then, for any* $n \geqslant 2$, U-dom.dim$(U_\Gamma) \geqslant n$ *if and only if* $\mathrm{grade}_U N \geqslant n$ *for any* $N \in \mathrm{mod}\, \Gamma^{\mathrm{op}}$ *with* $N^* = 0$.

We now are in a position to prove the main result in this paper.

Proof of Theorem 1.3. We only need to prove U-dom.dim$({}_\Lambda U) \leqslant U$-dom.dim$(U_\Gamma)$. Without loss of generality, suppose U-dom.dim$({}_\Lambda U) = n$.

The case $n = 1$ follows from Corollaries 2.5 and 2.8. Let $n \geqslant 2$. Notice that U-dom.dim$({}_\Lambda U) \geqslant 1$ and U-dom.dim$(U_\Gamma) \geqslant 1$. By Lemma 2.9′ it suffices to show that $\mathrm{grade}_U N \geqslant n$ for any $N \in \mathrm{mod}\, \Gamma^{\mathrm{op}}$ with $N^* = 0$. By Lemmas 2.3 and 2.7, for any $i \geqslant 1$, $\mathrm{Hom}_\Lambda(\mathrm{Ext}^i_\Gamma(N, U), E_0) \cong \mathrm{Tor}_i^\Gamma(N, {}^*E_0) = 0$, so $[\mathrm{Ext}^i_\Gamma(N, U)]^* = 0$. Then by assumption and Lemma 2.9, $\mathrm{grade}_U \mathrm{Ext}^i_\Gamma(N, U) \geqslant n$ for any $i \geqslant 1$. It follows from Lemma 2.6 that $\mathrm{grade}_U N \geqslant n$. $\square$

3. Some applications

As applications of the results in above section, we give in this section some characterizations of $(-)^{**}$ preserving monomorphisms and being left exact respectively.

Assume that

$$0 \to U_\Gamma \xrightarrow{f'_0} E'_0 \xrightarrow{f'_1} E'_1 \xrightarrow{f'_2} \cdots \xrightarrow{f'_i} E'_i \xrightarrow{f'_{i+1}} \cdots$$

is a minimal injective resolution of U_Γ. We first have the following

Proposition 3.1. *The following statements are equivalent for any positive integer* k.

(1) U-dom.dim$({}_\Lambda U) \geqslant k$.
(2) $0 \to ({}_\Lambda U)^{**} \xrightarrow{f_0^{**}} E_0^{**} \xrightarrow{f_1^{**}} E_1^{**} \xrightarrow{f_2^{**}} \cdots \xrightarrow{f_{k-1}^{**}} E_{k-1}^{**}$ *is exact.*
$(1)^{\mathrm{op}}$ U-dom.dim$(U_\Gamma) \geqslant k$.
$(2)^{\mathrm{op}}$ $0 \to (U_\Gamma)^{**} \xrightarrow{(f'_0)^{**}} (E'_0)^{**} \xrightarrow{(f'_1)^{**}} (E'_1)^{**} \xrightarrow{(f'_2)^{**}} \cdots \xrightarrow{(f'_{k-1})^{**}} (E'_{k-1})^{**}$ *is exact.*

Proof. By Theorem 1.3 we have $(1) \Leftrightarrow (1)^{\mathrm{op}}$. By symmetry, we only need to prove $(1) \Leftrightarrow (2)$.

If U-dom.dim$({}_\Lambda U) \geqslant k$, then E_i is in $\mathrm{add}_\Lambda U$ for any $1 \leqslant i \leqslant k-1$. Notice that ${}_\Lambda U$ and each E_i $(0 \leqslant i \leqslant k-1)$ are U-reflexive and hence we have that

$$0 \to ({}_\Lambda U)^{**} \xrightarrow{f_0^{**}} E_0^{**} \xrightarrow{f_1^{**}} E_1^{**} \xrightarrow{f_2^{**}} \cdots \xrightarrow{f_{k-1}^{**}} E_{k-1}^{**}$$

is exact. Assume that (2) holds. We proceed by induction on k. By assumption we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & {}_{\Lambda}U \arrow[r, "f_0"] \arrow[d, "\sigma_U"] & E_0 \arrow[r, "f_1"] \arrow[d, "\sigma_{E_0}"] & E_1 \arrow[r, "f_2"] \arrow[d, "\sigma_{E_1}"] & \cdots \arrow[r, "f_{k-1}"] & E_{k-1} \arrow[d, "\sigma_{E_{k-1}}"] \\
0 \arrow[r] & ({}_{\Lambda}U)^{**} \arrow[r, "f_0^{**}"] & E_0^{**} \arrow[r, "f_1^{**}"] & E_1^{**} \arrow[r, "f_2^{**}"] & \cdots \arrow[r, "f_{k-1}^{**}"] & E_{k-1}^{**}
\end{tikzcd}$$

Since σ_U is an isomorphism, $\sigma_{E_0} f_0 = f_0^{**}\sigma_U$ is a monomorphism. But f_0 is essential, so σ_{E_0} is monic, that is, E_0 is U-torsionless and E_0 is cogenerated by ${}_{\Lambda}U$. Moreover, E_0 is finitely cogenerated, so we have that $E_0 \in \operatorname{add} {}_{\Lambda}U$ (and hence σ_{E_0} is an isomorphism). The case $k = 1$ is proved. Now suppose that $k \geqslant 2$ and $E_i \in \operatorname{add} {}_{\Lambda}U$ (and then σ_{E_i} is an isomorphism) for any $0 \leqslant i \leqslant k-2$. Put $A_0 = {}_{\Lambda}U$, $B_0 = ({}_{\Lambda}U)^{**}$, $g_0 = f_0$, $g_0' = f_0^{**}$ and $h_0 = \sigma_U$. Then, for any $0 \leqslant i \leqslant k-2$, we get the following commutative diagrams with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A_i \arrow[r, "g_i"] \arrow[d, "h_i"] & E_i \arrow[r] \arrow[d, "\sigma_{E_i}"] & A_{i+1} \arrow[r] \arrow[d, "h_{i+1}"] & 0 \\
0 \arrow[r] & B_i \arrow[r, "g_i'"] & E_i^{**} \arrow[r] & B_{i+1} \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & A_{i+1} \arrow[r, "g_{i+1}"] \arrow[d, "h_{i+1}"] & E_{i+1} \arrow[d, "\sigma_{E_{i+1}}"] \\
0 \arrow[r] & B_{i+1} \arrow[r, "g_{i+1}'"] & E_{i+1}^{**}
\end{tikzcd}$$

where $A_i = \operatorname{Im} f_i$ and $A_{i+1} = \operatorname{Im} f_{i+1}$, $B_i = \operatorname{Im} f_i^{**}$ and $B_{i+1} = \operatorname{Im} f_{i+1}^{**}$, g_i and g_{i+1} are essential monomorphisms, h_i and h_{i+1} are induced homomorphisms. We may get inductively that each h_j is an isomorphism for any $0 \leqslant j \leqslant k-1$. Because $\sigma_{E_{k-1}} g_{k-1} = g_{k-1}' h_{k-1}$ is a monomorphism, by using a similar argument to that above we have $E_{k-1} \in \operatorname{add} {}_{\Lambda}U$. Hence we conclude that U-dom.dim$({}_{\Lambda}U) \geqslant k$. $\square$

The following result develops [5, Theorem 1] and [6, Proposition 3.1].

Proposition 3.2. *The following statements are equivalent.*

(1) U-dom.dim$({}_{\Lambda}U) \geqslant 1$.

(2) $(-)^{**} : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda$ *preserves monomorphisms.*

(3) $0 \to ({}_\Lambda U)^{**} \xrightarrow{f_0^{**}} E_0^{**}$ *is exact.*

$(1)^{\mathrm{op}}$ U-dom.dim$(U_\Gamma) \geqslant 1$.

$(2)^{\mathrm{op}}$ $(-)^{**}: \mathrm{mod}\,\Gamma^{\mathrm{op}} \to \mathrm{mod}\,\Gamma^{\mathrm{op}}$ *preserves monomorphisms.*

$(3)^{\mathrm{op}}$ $0 \to (U_\Gamma)^{**} \xrightarrow{(f_0')^{**}} (E_0')^{**}$ *is exact.*

Proof. By Theorem 1.3 we have (1) $\Leftrightarrow$ $(1)^{\mathrm{op}}$. By symmetry, we only need to prove that the conditions of (1), (2) and (3) are equivalent.

(1) $\Rightarrow$ (2). If U-dom.dim$({}_\Lambda U) \geqslant 1$ then $t(X) = \mathrm{Ker}\,\sigma_X$ for any $X \in \mathrm{mod}\,\Lambda$ by Corollary 2.8 and Proposition 2.4. So $(-)^{**}$ preserves monomorphisms by Proposition 2.2.

(2) $\Rightarrow$ (3) is trivial and (3) $\Rightarrow$ (1) follows from Proposition 3.1. $\square$

The following result except (3) and $(3)^{\mathrm{op}}$ is the U-dual version of [7, Proposition E], which develops [5, Theorem 2].

Proposition 3.3. *The following statements are equivalent.*

(1) U-dom.dim$({}_\Lambda U) \geqslant 2$.

(2) $(-)^{**}: \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$ *is left exact.*

(3) $0 \to ({}_\Lambda U)^{**} \xrightarrow{f_0^{**}} E_0^{**} \xrightarrow{f_1^{**}} E_1^{**}$ *is exact.*

(4) $(-)^{**}: \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$ *preserves monomorphisms and* $\mathrm{Ext}^1_\Gamma(\mathrm{Ext}^1_\Lambda(X, U), U) = 0$ *for any* $X \in \mathrm{mod}\,\Lambda$.

$(1)^{\mathrm{op}}$ U-dom.dim$(U_\Gamma) \geqslant 2$.

$(2)^{\mathrm{op}}$ $(-)^{**}: \mathrm{mod}\,\Gamma^{\mathrm{op}} \to \mathrm{mod}\,\Gamma^{\mathrm{op}}$ *is left exact.*

$(3)^{\mathrm{op}}$ $0 \to (U_\Gamma)^{**} \xrightarrow{(f_0')^{**}} (E_0')^{**} \xrightarrow{(f_1')^{**}} (E_1')^{**}$ *is exact.*

$(4)^{\mathrm{op}}$ $(-)^{**}: \mathrm{mod}\,\Gamma^{\mathrm{op}} \to \mathrm{mod}\,\Gamma^{\mathrm{op}}$ *preserves monomorphisms and* $\mathrm{Ext}^1_\Lambda(\mathrm{Ext}^1_\Gamma(Y, U), U) = 0$ *for any* $Y \in \mathrm{mod}\,\Gamma^{\mathrm{op}}$.

Proof. By Theorem 1.3 we have (1) $\Leftrightarrow$ $(1)^{\mathrm{op}}$ and by Proposition 3.1 we have (1) $\Leftrightarrow$ (3). So, by symmetry we only need to prove that (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (4) $\Rightarrow$ $(1)^{\mathrm{op}}$.

(1) $\Leftrightarrow$ (2). Assume that $(-)^{**}: \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$ is left exact. Then, by Proposition 3.2, we have that U-dom.dim$({}_\Lambda U) \geqslant 1$ and $E_0 \in \mathrm{add}\,{}_\Lambda U$.

Let $K = \mathrm{Im}(E_0 \to E_1)$ and $v: K \to E_1$ be the essential monomorphism. By assumption and the exactness of the sequences $0 \to U \to E_0 \to K \to 0$ and $0 \to K \xrightarrow{v} E_1$, we have the following exact commutative diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & E_0 & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_U} & & \downarrow{\scriptstyle \sigma_{E_0}} & & \downarrow{\scriptstyle \sigma_K} & & \\
0 & \longrightarrow & U^{**} & \longrightarrow & E_0^{**} & \longrightarrow & K^{**} & &
\end{array}$$

and

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{v} & E_1 \\ & & \downarrow{\scriptstyle \sigma_K} & & \downarrow{\scriptstyle \sigma_{E_1}} \\ 0 & \longrightarrow & K^{**} & \xrightarrow{v^{**}} & E_1^{**} \end{array}$$

where σ_U and σ_{E_0} are isomorphisms. By applying the snake lemma to the first diagram we have that σ_K is monic. Then we know from the second diagram that $\sigma_{E_1} v = v^{**}\sigma_K$ is a monomorphism. However, v is essential, so σ_{E_1} is monic, that is, E_1 is U-torsionless and E_1 is cogenerated by ${}_\Lambda U$. Moreover, E_1 is finitely cogenerated, so we conclude that $E_1 \in \operatorname{add} {}_\Lambda U$.

Conversely, assume that U-dom.dim$({}_\Lambda U) \geqslant 2$ and $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is an exact sequence in $\operatorname{mod} \Lambda$. By Proposition 3.2, α^{**} is monic. By assumption, Corollary 2.8 and Lemma 2.3 we have $\operatorname{Hom}_\Gamma(\operatorname{Ext}^1_\Lambda(C, U), E_0) = 0$. Since $\operatorname{Coker} \alpha^*$ is isomorphic to a submodule of $\operatorname{Ext}^1_\Lambda(C, U)$, $\operatorname{Hom}_\Gamma(\operatorname{Coker} \alpha^*, E_0) = 0$ and $\operatorname{Hom}_\Gamma(\operatorname{Coker} \alpha^*, U) = 0$. Then, by Theorem 1.3 and Lemma 2.9$'$, $\operatorname{grade}_U \operatorname{Coker} \alpha^* \geqslant 2$. It follows easily that $0 \to A^{**} \xrightarrow{\alpha^{**}} B^{**} \xrightarrow{\beta^{**}} C^{**}$ is exact.

(1) $\Rightarrow$ (4). Suppose U-dom.dim$({}_\Lambda U) \geqslant 2$. By Proposition 3.2, $(-)^{**} : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda$ preserves monomorphisms. On the other hand, we have that U-dom.dim$(U_\Gamma) \geqslant 2$ by Theorem 1.3. It follows from Corollary 2.8 and Lemma 2.3 that $\operatorname{Hom}_\Gamma(\operatorname{Ext}^1_\Lambda(X, U), E_0') = 0$ for any $X \in \operatorname{mod} \Lambda$. So $[\operatorname{Ext}^1_\Lambda(X, U)]^* = 0$ and hence $\operatorname{Ext}^1_\Gamma(\operatorname{Ext}^1_\Lambda(X, U), U) = 0$ by Lemma 2.9$'$.

(4) $\Rightarrow$ (1)$^{\mathrm{op}}$. Suppose that (4) holds. Then U-dom.dim$(U_\Gamma) \geqslant 1$ by Proposition 3.2.

Let A be in $\operatorname{mod} \Lambda$ and B any submodule of $\operatorname{Ext}^1_\Lambda(A, U)$ in $\operatorname{mod} \Gamma^{\mathrm{op}}$. Since U-dom.dim$(U_\Gamma) \geqslant 1$, $\operatorname{Hom}_\Gamma(\operatorname{Ext}^1_\Lambda(A, U), E_0') = 0$ by Corollary 2.8 and Lemma 2.3. So $\operatorname{Hom}_\Gamma(B, E_0') = 0$ and hence $\operatorname{Hom}_\Gamma(B, E_0'/U) \cong \operatorname{Ext}^1_\Gamma(B, U)$. On the other hand, $\operatorname{Hom}_\Gamma(B, E_0') = 0$ implies $B^* = 0$. Then by [8, Lemma 2.1] we have that $B \cong \operatorname{Ext}^1_\Lambda(\operatorname{Tr}_U B, U)$ with $\operatorname{Tr}_U B$ in $\operatorname{mod} \Lambda$. By (4), $\operatorname{Hom}_\Gamma(B, E_0'/U) \cong \operatorname{Ext}^1_\Gamma(B, U) \cong \operatorname{Ext}^1_\Gamma(\operatorname{Ext}^1_\Lambda(\operatorname{Tr}_U B, U), U) = 0$. Then by using a similar argument to that in the proof (2)$^{\mathrm{op}}$ $\Rightarrow$ (1)$^{\mathrm{op}}$ in Proposition 2.2, we have that $\operatorname{Hom}_\Gamma(\operatorname{Ext}^1_\Lambda(A, U), E_1') = 0$ (note: E_1' is the injective envelope of E_0'/U). Thus $E_1' \in \operatorname{add} U_\Gamma$ by Lemma 2.3 and Corollary 2.8, and therefore U-dom.dim$(U_\Gamma) \geqslant 2$. $\square$

Finally we give some equivalent characterizations of U-resol.dim$_\Lambda(E_0) \leqslant 1$ as follows, which is the U-dual version of [7, Proposition D].

Proposition 3.4. *The following statements are equivalent.*

(1) U-resol.dim$_\Lambda(E_0) \leqslant 1$.
(2) σ_X *is an essential monomorphism for any U-torsionless module X in* $\operatorname{mod} \Lambda$.
(3) f^{**} *is a monomorphism for any monomorphism $f : X \to Y$ in* $\operatorname{mod} \Lambda$ *with Y U-torsionless.*

(4) $\operatorname{grade}_U \operatorname{Ext}^1_\Lambda(X,U) \geqslant 1$ (*that is*, $[\operatorname{Ext}^1_\Lambda(X,U)]^* = 0$) *for any* X *in* $\operatorname{mod}\Lambda$.

Proof. (1) $\Rightarrow$ (2). Assume that X is U-torsionless in $\operatorname{mod}\Lambda$. Then $\operatorname{Coker}\sigma_X \cong \operatorname{Ext}^2_\Gamma(\operatorname{Tr}_U X, U)$ by [8, Lemma 2.1]. By Lemmas 2.7 and 2.3 we have

$$\operatorname{Hom}_\Lambda(\operatorname{Coker}\sigma_X, E_0) = \operatorname{Hom}_\Lambda\big(\operatorname{Ext}^2_\Gamma(\operatorname{Tr}_U X, U), E_0\big) = 0.$$

Then $\operatorname{Hom}_\Lambda(A, {}_\Lambda U) = 0$ for any submodule A of $\operatorname{Coker}\sigma_X$, which implies that any non-zero submodule of $\operatorname{Coker}\sigma_X$ is not U-torsionless.

Let B be a submodule of X^{**} with $X \cap B = 0$. Then $B \cong B/(X\cap B) \cong (X+B)/X$ is isomorphic to a submodule of $\operatorname{Coker}\sigma_X$. On the other hand, B is clearly U-torsionless. So $B = 0$ and hence σ_X is essential.

(2) $\Rightarrow$ (3). Let $f: X \to Y$ be monic in $\operatorname{mod}\Lambda$ with Y U-torsionless. Then $f^{**}\sigma_X = \sigma_Y f$ is monic. By (2), σ_X is an essential monomorphism, so f^{**} is monic.

(3) $\Rightarrow$ (4). Let X be in $\operatorname{mod}\Lambda$ and $0 \to Y \xrightarrow{g} P \to X \to 0$ an exact sequence in $\operatorname{mod}\Lambda$ with P projective. It is easy to see that $[\operatorname{Ext}^1_\Lambda(X,U)]^* \cong \operatorname{Ker} g^{**}$. On the other hand, g^{**} is monic by (3). So $\operatorname{Ker} g^{**} = 0$ and $[\operatorname{Ext}^1_\Lambda(X,U)]^* = 0$.

(4) $\Rightarrow$ (1). Let M be in $\operatorname{mod}\Gamma^{\mathrm{op}}$ and $\cdots \to P_1 \to P_0 \to M \to 0$ a projective resolution of M in $\operatorname{mod}\Gamma^{\mathrm{op}}$. Put $N = \operatorname{Coker}(P_2 \to P_1)$. By [8, Lemma 2.1], $\operatorname{Ext}^2_\Gamma(M,U) \cong \operatorname{Ext}^1_\Gamma(N,U) \cong \operatorname{Ker}\sigma_{\operatorname{Tr}_U N}$. On the other hand, since N is U-torsionless, $\operatorname{Ext}^1_\Lambda(\operatorname{Tr}_U N, U) \cong \operatorname{Ker}\sigma_N = 0$.

Let X be any finitely generated submodule of $\operatorname{Ext}^2_\Gamma(M,U)$ and $f_1: X \to \operatorname{Ext}^2_\Gamma(M,U)$ ($\cong \operatorname{Ker}\sigma_{\operatorname{Tr}_U N}$) the inclusion, and let f be the composition:

$$X \xrightarrow{f_1} \operatorname{Ext}^2_\Gamma(M,U) \xrightarrow{g} \operatorname{Tr}_U N,$$

where g is a monomorphism. By using the same argument as that in the proof of $(2)^{\mathrm{op}} \Rightarrow (1)^{\mathrm{op}}$ in Proposition 2.2, we get that $f^* = 0$. Hence, by applying $\operatorname{Hom}_\Lambda(-,U)$ to the exact sequence

$$0 \to X \xrightarrow{f} \operatorname{Tr}_U N \to \operatorname{Coker} f \to 0,$$

we have $X^* \cong \operatorname{Ext}^1_\Lambda(\operatorname{Coker} f, U)$. Then $X^{**} \cong [\operatorname{Ext}^1_\Lambda(\operatorname{Coker} f, U)]^* = 0$ by (4), which implies that $X^* = 0$ since X^* is a direct summand of $X^{***} (= 0)$ by [1, Proposition 20.24]. Also by using the same argument as that in the proof of $(2)^{\mathrm{op}} \Rightarrow (1)^{\mathrm{op}}$ in Proposition 2.2, we get that $\operatorname{Hom}_\Lambda(\operatorname{Ext}^2_\Gamma(M,U), E_0) = 0$. It follows from Lemma 2.3 that $\operatorname{l.fd}_\Gamma({}^*E_0) \leqslant 1$. Therefore $U\text{-resol.dim}_\Lambda(E_0) \leqslant 1$ by Lemma 2.7. □

Remark. By Theorem 1.3, we have that $E_0 \in \operatorname{add}{}_\Lambda U$ if and only if $E_0' \in \operatorname{add} U_\Gamma$, that is, $U\text{-resol.dim}_\Lambda(E_0) = 0$ if and only if $U\text{-resol.dim}_\Gamma(E_0') = 0$. However, in general, we don't have the fact that $U\text{-resol.dim}_\Lambda(E_0) \leqslant 1$ if and only if $U\text{-resol.dim}_\Gamma(E_0') \leqslant 1$ even when ${}_{\Lambda}U_{\Gamma} = {}_{\Lambda}\Lambda_{\Lambda}$. We use I_0 and I_0' to denote the injective envelope of ${}_{\Lambda}\Lambda$ and Λ_{Λ}, respectively. Consider the following example. Let K be a field and Δ the quiver:

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \xrightarrow{\gamma} 3.$$

(1) If $\Lambda = K\Delta/(\alpha\beta\alpha)$. Then $\mathrm{l.fd}_{\Lambda}(I_0) = 1$ and $\mathrm{r.fd}_{\Lambda}(I_0') \geqslant 2$. (2) If $\Lambda = K\Delta/(\gamma\alpha, \beta\alpha)$. Then $\mathrm{l.fd}_{\Lambda}(I_0) = 2$ and $\mathrm{r.fd}_{\Lambda}(I_0') = 1$.

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