General natural Einstein Kähler structures on tangent bundles

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\begin{abstract}
We study the conditions under which a Kählerian structure \((G, J)\), of general natural lift type on the tangent bundle \(TM\) of a Riemannian manifold \((M, g)\), studied in [S. Drută, V. Oproiu, General natural Kähler structures of constant holomorphic sectional curvature on tangent bundles, An. St. Univ. "Al.I. Cuza" Mat. 53 (2007) 149–166], is Einstein. We found three cases. In the first case the first proportionality factor \(\lambda\) is expressed as a rational function of the first two essential parameters involved in the definition of \(J\) and the value of the constant sectional curvature \(c\) of the base manifold \((M, g)\). If \(c\) follows that \((TM, G, J)\) has constant holomorphic sectional curvature (Theorem 8). In the second case a certain second degree homogeneous equation in the proportionality factor \(\lambda\) and its first order derivative \(\lambda'\) must be fulfilled. After some quite long computations done by using the Mathematica package RICCI for doing tensor computations, we obtain an Einstein Kähler structure only on \((T_0M, G, J)\subset(TM, G, J)\), where \(T_0M\) denotes the subset of nonzero tangent vectors to \(M\) (Theorem 9). In the last case we obtain that the Kählerian manifold \((TM, G, J)\) cannot be an Einstein manifold.
\end{abstract}

\section{1. Introduction}

The tangent bundle \(TM\) of a Riemannian manifold \((M, g)\) has many nice geometric properties, and furnishes important examples arising in various geometric classifications.

It is well known (see [12,14,15]) that the splitting of the tangent bundle to \(TM\) into the vertical and horizontal distributions, defined by the Levi-Civita connection of \(g\) on \(M\), and the corresponding Sasaki metric lead to an almost Kähler structure on \(TM\). Moreover, the possibility to consider vertical, complete and horizontal lifts, and more generally, the lifts of natural type (see [2–4]) on \(TM\) leads to interesting geometric structures, studied in the last years, and to interesting relations with some problems in Lagrangian and Hamiltonian mechanics.

The first author has studied some properties of a natural lift \(G\), of diagonal type, of the Riemannian metric \(g\) and a natural almost complex structure \(J\) of diagonal type on \(TM\) (see [7–9], and see also [10,11]). The condition for \((TM, G, J)\) to be an Einstein Kähler manifold leads to the conditions for \((M, g)\) to have constant sectional curvature, and for \((TM, G, J)\) to have constant sectional holomorphic curvature or to be a locally symmetric space.

In the paper [6], the first author has presented a general expression of the natural almost complex structures on \(TM\). In the definition of the natural almost complex structure \(J\) of general type there are involved eight parameters (smooth functions of the density energy on \(TM\)). However, from the condition for \(J\) to define an almost complex structure, four of

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the above parameters can be expressed as (rational) functions of the other four parameters. From the integrability condition for \( J \), we get (beside the condition for the base manifold to be of constant sectional curvature) that two other parameters involved in the definition of \( J \) can be expressed as functions of other two essential parameters and their first order derivatives. A Riemannian metric \( G \) which is a natural lift of general type of the metric \( g \) depends on six parameters. From the conditions for \( G \) to be Hermitian with respect to \( J \), one gets that these six parameters can be expressed with the help of the first eight (four) parameters involved in definition of \( J \) and two proportionality factors. Thus a natural Hermitian structure \((G, J)\) of general type depends on four essential parameters (two essential parameters involved in the definition of the integrable almost complex structure \( J \) and two proportionality factors). From the condition for \((G, J)\) to be almost Kählerian, we get that the second proportionality factor is the derivative of the first one. The family of natural Kählerian structures \((G, J)\) of general type on \( TM \) depends on three essential coefficients (two are involved in the expression of \( J \), and the third one is the first proportionality coefficient).

In the present paper we study the conditions under which the Kählerian manifold \((TM, G, J)\) is Einstein. After some quite involved computations we found three cases. In the first case the first proportionality factor is expressed as a rational function of the first two essential parameters, and the value of the constant sectional curvature of the base manifold is obtained by using local charts on \( TM \) (in fact it is always positive), so this case lead to no Einstein Kähler structure. Some quite long computations have been done by using the Mathematica package RICCI for doing tensor calculations.

The manifolds, tensor fields and other geometric objects we consider in this paper are assumed to be differentiable of class \( C^\infty \) (i.e. smooth). We use the computations in local coordinates in a fixed local chart but many results may be expressed in an invariant form by using the vertical and horizontal lifts. The well-known summation convention is used throughout this paper, the range of the indices \( h, i, j, k, l, r \) being always \([1, \ldots, n]\).

2. Preliminary results

Let \((M, g)\) be a smooth \(n\)-dimensional Riemannian manifold and denote its tangent bundle by \( TM \longrightarrow M \). Recall that \( TM \) has a structure of a \( 2n\)-dimensional smooth manifold, induced from the smooth manifold structure of \( M \). This structure is obtained by using local charts on \( TM \) induced from usual local charts on \( M \). If \((U, \varphi) = (U, x^1, \ldots, x^n)\) is a local chart on \( M \), then the corresponding induced local chart on \( TM \) is \((\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \ldots, x^n, y^1, \ldots, y^n)\), where the local coordinates \( x^1, y^1, \ldots, x^n, y^n \), \( i, j = 1, \ldots, n \), are defined as follows. The first \( n \) local coordinates of a tangent vector \( y \in \tau^{-1}(U) \) are the local coordinates in the local chart \((U, \varphi)\) of its base point, i.e. \( x^i = x^i \circ \tau \), by an abuse of notation. The last \( n \) local coordinates \( y^j \), \( j = 1, \ldots, n \), of \( y \in \tau^{-1}(U) \) are the vector space coordinates of \( y \) with respect to the natural basis in \( T_{\tau(y)}M \) defined by the local chart \((U, \varphi)\).

Due to this special structure of differentiable manifold for \( TM \), it is possible to introduce the \( T \)-tensor field on it (see [5]).

Denote by \( \bar{\nabla} \) the Levi-Civita connection of the Riemannian metric \( g \) on \( M \). Then we have the direct sum decomposition

\[
TM = \mathcal{V}TM \oplus \mathcal{H}TM
\]

of the tangent bundle to \( TM \) into the vertical distribution \( \mathcal{V}TM = \ker \tau_* \) and the horizontal distribution \( \mathcal{H}TM \) defined by \( \bar{\nabla} \). The set of vector fields \( \left( \frac{\partial}{\partial x^i}, \ldots, \frac{\partial}{\partial x^n} \right) \) on \( \tau^{-1}(U) \) defines a local frame field for \( \mathcal{V}TM \) and for \( \mathcal{H}TM \) we have the local frame field \( \left( \frac{\partial}{\partial y^i}, \ldots, \frac{\partial}{\partial y^n} \right) \), where

\[
\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i} \frac{\partial}{\delta y^h}, \quad \Gamma^h_{0i} = y^k \Gamma^h_{ki},
\]

and \( \Gamma^h_{ki}(x) \) are the Christoffel symbols of \( g \).

The set \( \left( \frac{\partial}{\partial y^i}, \ldots, \frac{\partial}{\partial y^n} \right) \) defines a local frame on \( TM \), adapted to the direct sum decomposition (1). Remark that

\[
\frac{\partial}{\partial y^i} = \left( \frac{\partial}{\partial x^i} \right)^V, \quad \frac{\delta}{\delta x^i} = \left( \frac{\partial}{\partial x^i} \right)^H,
\]

where \( X^V \) and \( X^H \) denote the vertical and horizontal lift of the vector field \( X \) on \( M \) respectively. We can use the vertical and horizontal lifts in order to obtain invariant expressions for almost all results in this paper. However, we should prefer to work in local coordinates since the formulas are obtained easier and, in a certain sense, they are more natural.

Consider the energy density of the tangent vector \( y \) with respect to the Riemannian metric \( g \)

\[
t = \frac{1}{2} \| y \|^2 = \frac{1}{2} G_{\tau(y)}(y, y) = \frac{1}{2} G_{\tau(x)}(x^i y^i y^j), \quad y \in \tau^{-1}(U).
\]

Obviously, we have \( t \in [0, \infty) \) for all \( y \in TM \).
Lemma 1. If $n > 1$ and $u, v$ are smooth functions on $T^*M$ such that $ug_{ij} + vp_ip_j = 0$, $p \in \pi^{-1}(U)$, on the domain of any induced local chart on $T^*M$, where $\xi_0 = \xi_0^y y^j$, then $u = 0, v = 0$.

Denote by $C = y^j \frac{\partial}{\partial x^j}$ the Liouville vector field on $TM$ and by $\tilde{C} = y^j \frac{\partial}{\partial x^j}$ the similar horizontal vector field on $TM$. Consider the real valued smooth functions $a_1, a_2, a_3, b_1, b_2, b_3, b_4$ defined on $[0, \infty) \subset \mathbb{R}$. A natural almost complex structure of general type on $TM$, defined by the Riemannian metric $g$, is obtained just like the natural 1-st order lifts of $g$ to $TM$ are obtained in [3].

Theorem 2. The natural tensor field $J$ of type $(1, 1)$ on $TM$, given by

\[
\begin{align*}
J_{\frac{\partial}{\partial y^i}} &= a_1(t) \frac{\partial}{\partial y^j} + b_1(t) g_0 C + a_4(t) \frac{\delta}{\delta x^i} + b_4(t) \delta g_0 \tilde{C}, \\
J_{\frac{\partial}{\partial y^j}} &= a_3(t) \frac{\partial}{\partial y^i} + b_3(t) g_0 C - a_2(t) \frac{\delta}{\delta x^j} - b_2(t) \delta g_0 \tilde{C},
\end{align*}
\]

defines an almost complex structure on $TM$, if and only if $a_4 = -a_3, b_4 = -b_3$ and the coefficients $a_1, a_2, a_3, b_1, b_2$ and $b_3$ are related by

\[a_1, a_2 = 1 + a_2^2, \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1 + (a_3 + 2tb_3)^2.\]

Remark. From the conditions (4) we have that the coefficients $a_1, a_2, a_1 + 2tb_1, a_2 + 2tb_2$ cannot vanish and have the same sign. We assume that $a_1 > 0, a_2 > 0, a_1 + 2tb_1 > 0, a_2 + 2tb_2 > 0$ for all $t \geq 0$.

Remark. The relations (4) allow us to express two of the coefficients $a_1, a_2, a_3, b_1, b_2, b_3$ as functions of the other four; e.g.

\[a_2 = \frac{1 + a_2^2}{a_1}, \quad b_2 = \frac{2a_3b_1 - a_2b_1 + 2tb_3}{a_1 + 2tb_1}.
\]

In the case of $TM$, the integrability conditions for the almost complex structure defined by $J$, obtained in [6] are given in the theorem:

Theorem 3. Let $(M, g)$ be an $n(>2)$-dimensional connected Riemannian manifold. The almost complex structure defined by $J$ on $TM$ is integrable if and only if $(M, g)$ has constant sectional curvature $c$ and the coefficients $b_1, b_2, b_3$ are given by:

\[
\begin{align*}
b_1 &= \frac{2c^2a_1^2 + 2cta_1a_2^2 + a_1a_2^2 - c + 3ca_3}{a_1 - 2ta_1 - 2cta_2 - 4ct^2a_2^2}, \\
b_2 &= \frac{2ta_1^2 - 2ta_1a_2^2 + ca_2^2 + 2cta_2a_3 + a_1a_3}{a_1 - 2ta_1 - 2cta_2 - 4ct^2a_2^2}, \\
b_3 &= \frac{a_1a_2^2 + 2ca_2a_3 + 4cta_2a_3 - 2cta_2a_3^2}{a_1 - 2ta_1 - 2cta_2 - 4ct^2a_2^2}.
\end{align*}
\]

Remark. Equivalently, the derivatives of the coefficients $a_1, a_2, a_3$ can be expressed as functions of $a_1, a_2, a_3$ and $b_1, b_2, b_3$.

\[
\begin{align*}
a_1' &= \frac{1}{a_1 + 2tb_1}(a_1b_1 + c - 3ca_2^2 - 4cta_3b_3), \\
a_2' &= \frac{1}{a_1 + 2tb_1}(2a_3b_3 - a_2b_1 - ca_2^2), \\
a_3' &= \frac{1}{a_1 + 2tb_1}(a_1b_3 - 2ca_2a_3 - 2cta_2b_3).
\end{align*}
\]

Remark. The second relation in (5) (or in (4)) is identically fulfilled by the expressions $b_1, b_2, b_3$ in (6).

Remark. In the case where $a_3 = 0$ it follows $b_3 = 0$ too, and we have:

\[a_2 = \frac{1}{a_1}, \quad b_1 = \frac{a_1a_1' - c}{a_1 - 2ta_1'}, \quad b_2 = \frac{c - a_1a_1'}{a_1(a_1'^2 - 2ct)}
\]

(compare with the corresponding expressions from [8] and [13]).
In the paper [6], the first author studied the conditions under which a Riemannian metric $G$ of natural type on $TM$, defined by

$$
\begin{align*}
G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) &= c_1 g_{ij} + d_1 g_{0i}g_{0j} = G_{ij}^{(1)}, \\
G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) &= c_2 g_{ij} + d_2 g_{0i}g_{0j} = G_{ij}^{(2)}, \\
G\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) &= G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = c_3 g_{ij} + d_3 g_{0i}g_{0j} = G_{ij}^{(3)},
\end{align*}
$$

is almost Hermitian with respect to the general almost complex structure $J$, i.e.

$$G(JX, JY) = G(X, Y),$$

for all vector fields $X, Y$ on $TM$. He proved the following result:

**Theorem 4.** The family of natural, Riemannian metrics $G$ on $TM$ such that $(TM, G, J)$ is an almost Hermitian manifold, is given by (8), provided that the coefficients $c_1, c_2, c_3, d_1, d_2, d_3$ are related to the coefficients $a_1, a_2, a_3, b_1, b_2, b_3$ by the following proportionality relations

$$
\begin{align*}
\frac{c_1}{a_1} &= \frac{c_2}{a_2} = \frac{c_3}{a_3} = \lambda, \\
\frac{c_1 + 2td_1}{a_1 + 2tb_1} &= \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \frac{c_3 + 2td_3}{a_3 + 2tb_3} = \lambda + 2t \mu,
\end{align*}
$$

where the proportionality coefficients $\lambda > 0$ and $\lambda + 2t \mu > 0$ are functions depending on $t$.

**Remark.** In the case where $a_3 = 0$, it follows that $c_3 = d_3 = 0$ and we obtain the almost Hermitian structure considered in [13]. Remark that the functions used in [13] are slightly different of the functions used in the present paper. Moreover, if $\lambda = 1$ and $\mu = 0$, we obtain the almost Kählerian structure considered in [9].

Considering the two-form $\Omega$ defined by the almost Hermitian structure $(G, J)$ on $TM$

$$\Omega(X, Y) = G(X, JY),$$

for all vector fields $X, Y$ on $TM$, the second author obtained the following result:

**Proposition 5.** The expression of the 2-form $\Omega$ in the local adapted frame

$$
\begin{align*}
\left(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n}\right)
\end{align*}
$$
on $TM$, is given by

$$\Omega\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0, \quad \Omega\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 0, \quad \Omega\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = \lambda g_{ij} + \mu g_{0i}g_{0j}
$$
or, equivalently

$$\Omega = (\lambda g_{ij} + \mu g_{0i}g_{0j}) \tilde{\nabla} y^i \wedge dx^j,$$

where $\tilde{\nabla} y^i = dy^i + \Gamma_{0h}^{i} dx^h$ is the absolute differential of $y^i$.

Next, by calculating the exterior differential of $\Omega$, he obtained:

**Theorem 6.** The almost Hermitian structure $(TM, G, J)$ is almost Kählerian if and only if

$$\mu = \lambda'.$$

Thus the family of general almost Kählerian structures on $TM$ depends on five essential coefficients $a_1, a_3, b_1, b_3, \lambda$. Combining the results from Theorems 3, 4 and 6, we obtain that a general natural Kählerian structure $(G, J)$ on $TM$ is defined by three essential coefficients $a_1, a_3, \lambda$. However, these coefficients must satisfy the supplementary conditions $a_1 > 0, a_1 + 2tb_1 > 0, \lambda > 0, \lambda + 2t \mu > 0$. Examples of such structures can be found in [13] (see also [9]).
3. General natural Einstein Kähler structures on tangent bundles

The Levi-Civita connection $\nabla$ of the Riemannian manifold $(M, G)$ is obtained from the well-known formula

$$2G(\nabla_X Y, Z) = X(G(Y, Z)) + Y(G(X, Z)) - Z(G(X, Y)) + G([X, Y], Z) - G([X, Z], Y) - G([Y, Z], X); \quad \forall X, Y, Z \in \mathfrak{X}(M)$$

and is characterized by the conditions

$$\nabla G = 0, \quad T = 0,$$

where $T$ is the torsion tensor of $\nabla$.

In the case of the tangent bundle $TM$ we can obtain the explicit expression of $\nabla$. The symmetric $2n \times 2n$ matrix

$$\left( \begin{array}{cc} C^{(1)}_{ij} & C^{(3)}_{ij} \\ C^{(3)}_{ij} & C^{(2)}_{ij} \end{array} \right)$$

associated to the metric $G$ in the base $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n})$ has the inverse

$$\left( \begin{array}{cc} H^{(1)}_{ij} & H^{(3)}_{ij} \\ H^{(3)}_{ij} & H^{(2)}_{ij} \end{array} \right)$$

where the entries are the blocks

$$H^{(1)}_{ij} = p_1 g^{kl} + q_1 y^k y^l, \quad H^{(2)}_{ij} = p_2 g^{kl} + q_2 y^k y^l, \quad H^{(3)}_{ij} = p_3 g^{kl} + q_3 y^k y^l. \quad (10)$$

Here $g^{kl}$ are the components of the inverse of the matrix $(g_{ij})$ and $p_1, p_2, q_1, q_2, p_3, q_3 : [0, \infty) \to \mathbb{R}$, some real smooth functions. Their expressions are obtained from the property of the matrix $H$ to be the inverse of the matrix $G$. By using Lemma 1, we get $p_1, p_2, p_3$ as functions of $c_1, c_2, c_3$

$$p_1 = \frac{c_2}{c_1 c_2 - c_3}, \quad p_2 = \frac{c_1}{c_1 c_2 - c_3}, \quad p_3 = -\frac{c_3}{c_1 c_2 - c_3}, \quad (11)$$

and $q_1, q_2, q_3$ as functions of $c_1, c_2, c_3, d_1, d_2, d_3, p_1, p_2, p_3$

$$q_1 = \frac{c_2 d_1 p_1 - c_3 d_2 p_1 - c_3 d_2 p_3 + c_2 d_3 p_1 + 2 c_1 d_2 p_1 t - 2 d_3 p_1 t}{c_1 c_2 - c_3} + \frac{2 c_2 d_1 t + 2 c_1 d_2 t - 4 c_3 d_2 t}{c_2 c_1 - c_3} + \frac{4 d_1 t^2 - 4 d_3 t^2}{c_1 c_2 - c_3},$$

$$q_2 = \frac{-(d_3 p_1 + d_2 p_3)(c_1 + 2 d_1 t) - (d_1 p_1 + d_3 p_3)(c_3 + 2 d_3 t)}{c_1 c_2 - c_3} + \frac{(d_2 p_1 + d_3 p_3)(c_1 + 2 d_1 t) - (d_3 p_1 + d_2 p_3)(c_3 + 2 d_3 t)}{c_1 c_2 - c_3},$$

$$q_3 = \frac{(c_1 + 2 d_3 t)(c_2 + 2 d_3 t) - (c_3 + 2 d_3 t)^2}{c_1 c_2 - c_3}, \quad (12)$$

Next we can obtain the expression of the Levi-Civita connection of the Riemannian metric $G$ on $TM$.

**Theorem 7.** The Levi-Civita connection $\nabla$ of $G$ has the following expression in the local adapted frame $(\frac{\partial}{\partial \varphi^1}, \ldots, \frac{\partial}{\partial \varphi^n}, \frac{\partial}{\partial \varphi^{n+1}}, \ldots, \frac{\partial}{\partial \varphi^{2n}})$

$$\nabla \frac{\partial}{\partial \varphi^i} = Q^h_{ij} \frac{\partial}{\partial \varphi^j} + \tilde{Q}^h_{ij} \frac{\partial}{\partial \varphi^h}, \quad \nabla \frac{\partial}{\partial \varphi^j} = (I_{ij}^h + \tilde{I}_{ij}^h) \frac{\partial}{\partial \varphi^j} + I_{ij}^h \frac{\partial}{\partial \varphi^h},$$

$$\nabla \frac{\partial}{\partial \varphi^i} = P^h_{ij} \frac{\partial}{\partial \varphi^j} + \tilde{P}^h_{ij} \frac{\partial}{\partial \varphi^h}, \quad \nabla \frac{\partial}{\partial \varphi^j} = (S^h_{ij} + \tilde{S}^h_{ij}) \frac{\partial}{\partial \varphi^j} + S^h_{ij} \frac{\partial}{\partial \varphi^h},$$

where $I_{ij}^h$ are the Christoffel symbols of the connection $\nabla$ and $P^h_{ij}, Q^h_{ij}, S^h_{ij}, \tilde{P}^h_{ij}, \tilde{Q}^h_{ij}, \tilde{S}^h_{ij}$ are $M$-tensor fields which can be obtained from the formula of $\nabla$, given at the beginning of this section.

Their detailed expressions can be obtained by using RICCI.

The curvature tensor field $K$ of the connection $\nabla$ is defined by the well-known formula

$$K(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad X, Y, Z \in \Gamma(TM).$$

By using the local adapted frame $(\frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j})$, $i, j = 1, \ldots, n$, we obtain, after a standard straightforward computation.
where the $M$-tensor fields appearing as coefficients are obtained in a standard way. In order to obtain the final expressions of the above $M$-tensor fields, we have to compute the first and second order partial derivatives with respect to the tangential coordinates $y^l$ of the usual tensor fields involved in the definition of the Riemannian metric $G$ (including the $M$-tensor fields $H$). Next, the first order partial derivatives with respect to the tangential coordinates $y^l$ of the $M$-tensor fields $p^h_{ij}$, $Q^h_{ij}$, $S^h_{ij}$, $\tilde{p}^h_{ij}$, $\tilde{Q}^h_{ij}$, $\tilde{S}^h_{ij}$ can be obtained easily.

Replacing these derivatives and the explicit expressions of the $M$-tensor fields $p^h_{ij}$, $Q^h_{ij}$, $S^h_{ij}$, $\tilde{p}^h_{ij}$, $\tilde{Q}^h_{ij}$, $\tilde{S}^h_{ij}$, the expressions (11), (12) of the functions $p_1$, $p_2$, $p_3$, $q_1$, $q_2$, $q_3$ and of their derivatives we can obtain the components of the curvature tensor as functions of $a_1, a_2, a_3$ and their derivatives of first, second and third order only. The respective expressions have been obtained by using the Mathematica package RICCI. It was not convenient to think $a_1, a_2, a_3, b_1, b_2, b_3$ as well as $c_1, c_2, c_3, d_1, d_2, d_3$ and $p_1, p_2, p_3, q_1, q_2, q_3$ as functions of $t$ since RICCI did not make some useful factorizations after the command TensorSimplify. We decided to consider these functions as well as their derivatives of first, second and third order, as constants, the tangent vector $y$ as a first order tensor, the components $G_{ij}^{(1)}$, $G_{ij}^{(2)}$, $G_{ij}^{(3)}$, $H_{ij}^{(1)}$, $H_{ij}^{(2)}$, $H_{ij}^{(3)}$ as second order tensors and so on, on the Riemannian manifold $M$, the associated indices being $h, i, j, k, l, r, s$.

Now we shall describe some technical aspects leading to the property of $(TM, G, J)$ to be Einstein Kähler. The components of Ricci tensor $\text{Ric}(Y, Z) = \text{trace}(X \longrightarrow K(X, Y)Z)$ of the Kählerian manifold $(TM, G, J)$ are given by the formulas:

$$
\text{Ric} \ XY_{jk} = \text{Ric} \left( \frac{\delta}{\delta x^j} \frac{\partial}{\partial y^k} \right) = \text{Ric} \ XY_{jk} = \text{Ric} \left( \frac{\partial}{\partial y^j} \frac{\delta}{\delta x^k} \right) = YYY^h_{khj} + XXY^h_{khj},
$$

$$
\text{Ric} \ XX_{jk} = \text{Ric} \left( \frac{\delta}{\delta x^j} \frac{\delta}{\delta x^k} \right) = XXX^h_{khj} + YXY^h_{khj},
$$

$$
\text{Ric} \ YY_{jk} = \text{Ric} \left( \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^k} \right) = YYY^h_{khj} + XXY^h_{khj}.
$$

From the condition

$$
\text{Ric} \ YY_{jk} - LG^{(3)}_{jk} = 0,
$$

obtained from the Einstein condition $\text{Ric} - LG = 0$, we get two values for $L$; the first one $L = LYX$ is obtained from the vanishing of the coefficient of $S_{jk}$, the second one $L = LYX2$ is obtained from the vanishing of the coefficient of $S_{0j}S_{0k}$. Similarly, from the condition

$$
\text{Ric} \ XX_{jk} - LG^{(1)}_{jk} = 0,
$$

we get $L = LXX$ and $L = LXX2$. Finally from the condition

$$
\text{Ric} \ YY_{jk} - LG^{(2)}_{jk} = 0,
$$

we get $L = LYY$ and $L = LYY2$. By a direct but quite long computation we obtain

$$
L = LYX = LXX = LYY,
$$

and the common value is given by a quite long expression.

Next from the condition $LYX2 - L = 0$ we obtain an equation of the form $\alpha n + \beta 0 = 0$, where $\alpha$ and $\beta$ are quite complicate expressions depending on $a_1, a_3, \lambda$ and their derivatives. The independence of the dimension $n$ of manifold $M$, obtained from equation $\alpha n + \beta = 0$ implies $\alpha = 0, \beta = 0$. From the condition $\alpha = 0$ we get a quite long expression of $\lambda''$. From the condition $\beta = 0$ we get another long expression of $\lambda'''$. Differentiate the above expression of $\lambda''$, and consider the equation
\[(\lambda^\prime)'' - \lambda''' = 0.\]

It follows the following relation (without the denominator)

\[
\left( a_3^2 a_1^2 \lambda + 2a_1 c_5 + 2a_1 a_2^2 c_\lambda + a_3^\prime \lambda' - 2a_1 c_\lambda t - 2a_1^\prime a_2^\prime c_\lambda t \\
+ 4a_1 a_2^3 c_\lambda t + 2a_1 c_\lambda t + 2a_1 a_2^3 c_\lambda t \right) \left( a_3^2 a_1^2 \lambda^2 + 2a_1^3 a_2^2 c_\lambda^2 + a_5^\prime \lambda \lambda' \\
- a_3^2 a_1^2 \lambda^2 t - 4a_1^3 a_2^2 c_\lambda^2 t + 4a_1 a_2^3 a_3 c_\lambda^2 t \\
- 4a_1^3 a_2^2 c_\lambda^2 \lambda' + 4a_1^3 a_2^3 c_\lambda^2 \lambda t + 4a_1^3 a_2^3 c_\lambda^2 \lambda t^2 + 4a_1^3 a_2^3 a_3^2 c_\lambda^2 t^2 \\
- 8a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 + 8a_1^3 a_2^3 c_\lambda^2 \lambda^2 t^2 + 8a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 t^2 \\
- 8a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 t^2 - 8a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 t^2 + 8a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 t^2 \\
+ 4a_1^3 a_2^3 \lambda^2 \lambda t^2 - 4a_1^3 c_\lambda^2 \lambda^2 t^2 + 4a_1^3 a_3^2 c_\lambda^2 \lambda t^2 - 4a_1^3 a_3^2 c_\lambda^2 \lambda t^2 t^2 \\
- 4a_1^3 a_2^3 c_\lambda^2 \lambda^2 t^3 + 16a_1 a_2^3 a_3 c_\lambda^2 \lambda^2 t^3 + 16a_1 a_2^3 c_\lambda^2 \lambda^2 t^3 \\
- 16a_1^3 a_2^3 c_\lambda^2 \lambda^2 t^3 + 4a_1^3 a_3^2 c_\lambda^2 \lambda^2 t^3 + 8a_1^3 a_3^2 c_\lambda^2 \lambda^2 t^3 + 4a_1^3 a_3^2 c_\lambda^2 \lambda^2 t^3 \\
\right) \times \left( a_3^2 - 2a_1 a_1^\prime t - 2a_1 c_\lambda t - 2a_1 a_2^3 c_\lambda t + 4a_1 c_\lambda t^2 + 4a_1 a_2^3 c_\lambda t^2 - 8a_1 a_3 a_3^2 c_\lambda t^2 \right) = 0
\]

and we have the following three cases, according to the vanishing of each of the three factors from this equation:

I) \( a_3^2 a_1^2 \lambda + 2a_1 c_\lambda + 2a_1 a_2^3 c_\lambda + a_3^\prime \lambda' - 2a_1 c_\lambda t - 2a_1^\prime a_2^\prime c_\lambda t + 2a_1 c_\lambda t + 2a_1 a_2^3 c_\lambda t = 0. \)

II) \( a_3^2 a_1^2 \lambda^2 + 2a_1 a_2^3 c_\lambda^2 + a_3^\prime a_\lambda' - 4a_1^2 a_2^2 c_\lambda^2 t - 4a_1^3 a_2^2 c_\lambda^2 t - 4a_1^3 a_2^3 c_\lambda^2 t^2 + 4a_1^3 a_2^3 c_\lambda^2 t^2 \\
+ 4a_1^2 a_2^3 c_\lambda^2 t^2 - 8a_1^3 a_2^3 a_3 c_\lambda^2 t^2 + 4a_1^3 a_2^3 c_\lambda^2 t^2 + 8a_1^3 a_2^3 a_3^2 c_\lambda^2 t^2 \\
+ 4a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda t^2 + 8a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda t^2 - 4a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda t^2 \\
- 8a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 t^2 + 8a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 t^2 + 8a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 t^2 \\
- 16a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 t^2 + 4a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 t^2 - 4a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 t^2 \\
- 16a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 t^2 + 4a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 t^2 + 8a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 t^2 + 4a_1^3 a_2^3 a_3^2 c_\lambda^2 \lambda^2 t^2 = 0. \)

III) \( a_3^2 - 2a_1 a_1^\prime t - 2a_1 c_\lambda t - 2a_1 a_2^3 c_\lambda t + 4a_1 c_\lambda t^2 + 4a_1 a_2^3 c_\lambda t^2 - 8a_1 a_3 a_3^2 c_\lambda t^2 = 0. \)

In the first case (case I) we get easily the following expression of \( \lambda' \)

\[
\lambda' = -\frac{a_3^2 a_1^2 + 2a_1 c_\lambda + 2a_1 a_2^3 c - 2a_1^\prime a_2^\prime c_\lambda t + 2a_1 a_2^3 c_\lambda t}{a_3^2 + 2a_1 c_\lambda t + 2a_1 a_2^3 c_\lambda t}.
\]

Replacing this expression of \( \lambda' \) in the obtained expression of \( L \) we get

\[
L = \frac{2a_1 c(n + 1)}{\lambda(a_3^2 + 2c t + 2a_2^3 c t)}.
\]

from which we can obtain the value of \( \lambda \)

\[
\lambda = \frac{2a_1 c(n + 1)}{L(a_3^2 + 2c t + 2a_2^3 c t)}.
\]

This expression of \( \lambda \) is (up to some change of constants) the same as the expression obtained in [1], in the case of the natural Kählerian structures of general type on the tangent bundles of constant holomorphic sectional curvature. Such manifolds are automatically Einstein Kähler. Thus we can state the following result:

**Theorem 8.** Let \((T M, G, J)\) be the Kählerian manifold, with \( G \) and \( J \) obtained as natural lifts of general type of the Riemannian metric \( g \) on the base manifold \( M \). Assume that the parameter \( \lambda \) is expressed by (13), where \( L \) is a nonzero real constant. Then \((T M, G, J)\) is an Einstein Kähler manifold, i.e. \( \text{Ric} = L G \).

**Remark.** Taking into account of Theorem 3.3 in [1] we remark that the expression (13) of \( \lambda \) implies that the Kählerian manifold \((T M, G, J)\) has constant holomorphic sectional curvature \( k = \frac{2L}{n+1} \).

**Remark.** In the particular subcase where \( a_3 = 0 \) and \( \lambda = 1 \) we obtain Theorems 4.2 and 5.1 from [8].
Case II). The relation obtained in the case II can be thought of as an equation of second order in $\lambda'$. This equation can be solved easily and we get some quite interesting results related to the property of $(TM, G, J)$ to be Einstein Kähler. First of all we shall use the following notation

$$\lambda = \sqrt{a_1^4 - 4a_1^2ct + 4a_1^2a_3^2ct + 4c^2t^2 + 8a_3^2c^2t^2 + 4a_3^4c^2t^2}.$$  

In the case $c > 0$, the expression under square root can be written in the form

$$(a_1^2 - 2ct)^2 + 4a_1^2a_3^2ct + 8a_3^2c^2t^2 + 4a_3^4c^2t^2 > 0.$$  

In the case $c < 0$, the same expression can be written in the form

$$(a_1^2 + 2a_3^2ct)^2 - 4a_1^2ct + 8a_3^2c^2t^2 + 4c^2t^2 > 0$$  

and we see that $A$ is real.

Then the solutions of the equation from the case II) are

$$\lambda' = \lambda \left( -\frac{1}{2t} \pm \frac{a_1 - 2a_1a_1t - 2a_1a_3^2ct + 4a_1^2t + 4a_3^2ct^2 - 8a_1a_3^2ct^2}{2a_1't} \right).$$  

From now on we shall study only the case of the solution with +. Replacing the above expression of $\lambda'$ as well as its derivative $\lambda''$ in the expression of $L$ we get

$$L = \frac{n(a_1^2 + 2ct + 2a_3^2ct - A)}{4a_1'tt}.$$  

Remark that $L$ is defined on the set $T_0M \subset TM$ of the nonzero tangent vectors. Then, the expression of $\lambda$ is given by

$$\lambda = \frac{n(a_1^2 + 2ct + 2a_3^2ct - A)}{4a_1'tt}. \quad (14)$$  

By using this expression of $\lambda$ in the condition for $(T_0M, G, J)$ to be an Einstein manifold, we obtain, after quite long computations the property $\text{Ric} - LG = 0$. Hence we can state

**Theorem 9.** Let $(T_0M, G, J)$ be the Kählerian manifold, with $G$ and $J$ obtained as natural lifts of general type of the Riemannian metric $g$ on the base manifold $M$. Assume that the parameter $\lambda$ is expressed by $(14)$, where $L$ is a nonzero real constant. Then $(T_0M, G, J)$ is an Einstein Kähler manifold.

**Remark.** In the particular case where $a_1 = 1$, $a_3 = 0$ and assuming that $\lambda = 1$, we obtain the main result obtained by the first author in [7] (Theorem 5).

In the last case (case III), replacing $a_1', a_3'$ from (7) we obtain that the first member becomes

$$\frac{a_1^4 - 4a_1^2ct + 4a_1^2a_3^2ct + 4c^2t^2 + 8a_3^2c^2t^2 + 4a_3^4c^2t^2}{a_1 + 2tb_1} = \frac{A^2}{a_1 + 2tb_1} > 0.$$  

Therefore, in this case, the first member of the considered equation is always positive, i.e. the Kählerian manifold $(TM, G, J)$ cannot be Einstein.

**Remark.** In the particular subcase, where $a_3 = 0$ and $c > 0$, it follows that $a_1 = \sqrt{2ct}$ and the obtained Kählerian structure is that defined by the Lagrangian $L$ on $T_0M$ studied by the present authors in [10]. Therefore we obtain Proposition 7 in [10], i.e., $(T_0M, G, J)$ cannot be an Einstein manifold.

**General remark.** From the above results it follows that $TM$ endowed with a Kählerian structure $(G, J)$ of general natural lift type is Einstein if and only if it has constant holomorphic sectional curvature. The subset $T_0M \subset TM$ becomes Einstein if and only if the parameter $\lambda$ is given by $(14)$.

**References**


