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Behavior of the Periodic Surface for a Periodically Perturbed Autonomous System and Periodic Solutions*

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A periodically perturbed autonomous system with an isolated periodic orbit has a periodic surface as is well known. The behavior of this surface with respect to T, the period of the disturbance, is studied. The results include that the surface exists for all T when ϵ is sufficiently small, where ϵ is the perturbation parameter. Stability properties hold uniformly with respect to T. Further, growth estimates with respect to ϵ of the surface and its first T derivative are found. These estimates are extremely useful in discussing periodic solutions on the periodic surface. Incidental in the work is the formula for the invariant manifold for the quasilinear version of the system considered by Coppel and Palmer.

1. INTRODUCTION

Consider the perturbed autonomous system

$$x' = X(x) + \epsilon R(t/T, x, \epsilon). \tag{1.1}$$

We assume

$$x' = X(x) \tag{1.2}$$

has a periodic solution $u(t)$ of least period 1. The variational equation,

$$w' = X_x(u(t))w \tag{1.3}$$

(considered as a system with period 1), has 1 as a characteristic multiplier due to the solution $w = u'$. We assume that this is the only characteristic multiplier with absolute value 1. X and R are C^3 with respect to x in a neighborhood of the curve $x = u(\theta)$: $0 \leqslant \theta \leqslant 1$, and R has period 1 in its first variable.

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It is well known (cf. [10, pp. 210–215; 11, p. 729; 8, pp. 212–215] and [13, pp. 42–53]) that normal coordinates exist for u,

$$x = u(\theta) + Z(\theta)y, \tag{1.4}$$

where $Z(\theta)$ is a C^3 1-periodic $n \times (n-1)$ matrix and y is an $n-1$ vector. Equation (1.4) is one-to-one for $0 \leqslant \theta < 1$ and $\| y \|$ sufficiently small; and Eq. (1.4) used in (1.1) gives the normal system

$$\begin{aligned} \theta' &= 1 + F(\theta, y)y + \epsilon U(t/T, \theta, y, \epsilon), \\ y' &= A(\theta)y + G(\theta, y)y + \epsilon V(t/T, \theta, y, \epsilon), \end{aligned} \tag{1.5}$$

where $G(\theta, 0) = 0$ and $y' = A(t)y$ has an exponential dichotomy [10, p. 216], the right-hand side is C^2 in θ and y and 1-periodic in θ, T-periodic in t. Equation (1.5) is used to show the existence of an invariant periodic surface, $x = H(t, \theta, \epsilon)$, which is 1-periodic in θ, T-periodc in t, and $H(t, \theta, 0) \equiv u(\theta)$ (cf. [11, 8, 9], also see [7] where θ is used as the independent variable).

Previously, the variation of H with respect to T had not been considered. The periodic surface depends on T and can be written as $x = H(t/T, \theta, T, \epsilon)$, where H is 1-periodic in its first two variables. Theorem 2 will show $H(t, \theta, T, \epsilon)$ is C^1 in T as well as in t and θ. Further, S/ϵ and TS_T/ϵ are uniformly bounded in t, θ, T, ϵ for all sufficiently small ϵ where $S(t, \theta, T, \epsilon)$ is defined by $H = u + ZS$. Solutions of (1.1) on $x = H$ satisfy a differential equation on a torus

$$\theta' = k(t/T, \theta, T, \epsilon), \tag{1.6}$$

and the estimates on S will lead to an excellent qualitative description, for all small ϵ, of the points in the T–ϵ plane for which periodic solutions (on the torus) exist to (1.6), a description, which is in a large sense best possible [1].

Diliberto [9] has treated autonomous systems like (1.5) where θ is a p vector and $A(\theta)$ is block diagonal definite [t of (1.5) is one of the components of θ]. In (1.5), $A(\theta)$ is not necessarily block diagonal definite. If a real invertible 1-periodic C^2 matrix $B(t)$ exists such that

$$C(t) = B^{-1}(t)(A(t)\,B(t) - B'(t))$$

is block diagonal definite, then the transformation $y = B(\theta)z$ changes (1.5) to a similar system in (θ, z) where $A(\theta)$ is replaced by $C(\theta)$. However, there are systems $y' = A(t)y$ for which no such $B(t)$ exists.

Coppel and Palmer [7] consider a system like (1.5) where θ is a p vector but $A(\theta)$ is replaced by $A(t)$. The use of θ as the independent variable would put (1.5) in the Coppel–Palmer form. However, the author has not found useful results in this form concerning the behavior of the periodic surface with respect to T.

We shall treat a system which is a cross between the types considered by Diliberto and by Coppel and Palmer in the case where the rotation variable θ is one dimensional.

Concerning notation, R^1 will denote the real number system, and $\| \ \|$ is a vector norm, or its corresponding sup-norm or matrix norm. Variables will be systematicaly omitted when no confusion results.

2. The Normal System

Let the system

$$\begin{aligned} \theta' &= T[1 + f(t, \theta, y, T, \mu, \epsilon)], \\ y' &= T[A(\theta, u)y + g(t, \theta, y, T, \mu, \epsilon)], \end{aligned} \tag{2.1}$$

be 1-periodic in t and θ, where the right-hand side is defined for all $(t, \theta, y, T, \mu, \epsilon)$ in $K_{\bar{r}\bar{\epsilon}} = R^1 \times R^1 \times \{y \mid y \text{ in } R^{n-1}, \|y\| \leqslant \bar{r}\} \times (0, \infty) \times J \times (-\bar{\epsilon}, \bar{\epsilon})$, where it is continuous and C^2 in θ, y, T and satisfies the following inequalities:

(i) $$\|g\|, \|Tg_T\|, \|g_\theta\|, \|Tg_{T\theta}\|, \|T(Tg_T)_T\|, \|g_{\theta\theta}\| \leqslant M(\|y\|^2 + |\epsilon|);$$

(ii) $$|f|, |Tf_T|, |f_\theta|, |Tf_{T\theta}|, |T(Tf_T)_T|, |f_{\theta\theta}|, \|g_y\|, \|g_{y\theta}\|, \|Tg_{Ty}\| \leqslant M(\|y\| + |\epsilon|);$$

(iii) $$\|f_y\|, \|Tf_{Ty}\|, \|f_{y\theta}\|, \|f_{yy}\|, \|g_{yy}\|, \|A\|, \|A_\theta\|, \|A_{\theta\theta}\| \leqslant M.$$

Further, let $w' = A(t, \mu)w$ have a fundamental matrix $\Phi(t, \mu)$ such that

(iv) $$\begin{aligned} \|\Phi(t, \mu) P \Phi^{-1}(s, \mu)\| &\leqslant Me^{-\alpha(t-s)}, \qquad t \geqslant s, \\ \|\Phi(t, \mu)(I - P) \Phi^{-1}(s, \mu)\| &\leqslant Me^{-\alpha(s-t)}, \qquad t \leqslant s, \end{aligned}$$

where $\alpha > 0$ and P is a constant projection matrix (i.e., $P^2 = P$). Lastly, by possibly taking M larger we can assume

(v) $$h(t, \theta, y, T, \mu, \epsilon) = g(t, \theta, y, T, \mu, \epsilon) - f(t, \theta, y, T, \mu, \epsilon) A(\theta, \mu)y$$

satisfies the same inequalities as g.

Some comments are in order. First (2.1) is more general than (1.5) with t/T replaced by t and where the right-hand side is allowed to vary with T.

Secondly, J is a parameter set and (2.1) can be thought of as the result of a normal coordinate transformation, $x = u(\theta, \mu) + Z(\theta, \mu)y$, applied to

$$x' = X(x, \mu) + R(t/T, x, T, \mu, \epsilon) \tag{2.2}$$

valid uniformly for μ in J, where R, TR_T, TR_{Tx}, $T(TR_T)_T$ are bounded in a neighborhood of $x = u(\theta, \mu) : 0 \leqslant \theta \leqslant 1$. The model that the author has in mind is a perturbed van der Pol equation,

$$x'' + \mu(x^2 - 1)\, x' + x = \epsilon \cos(2\pi t/T^*),$$

whose autonomous $L(\mu)$-periodic limit cycle has been normalized to have period 1, $T = T^*L(\mu)$, and J is any compact subset of $(0, \infty)$. Lastly, we are not requiring (2.1) to be even C^1 in t.

3. The Functional Equation for the Periodic Surface

Consider the quasilinear system

$$\begin{aligned} \theta' &= f(t, \theta), \\ y' &= A(t)y + g(t, \theta), \end{aligned} \tag{3.1}$$

which is not necessarily periodic, where θ can be a p vector, and where $y' = A(t)y$ has an exponential dichotomy, that is, there is a fundamental matrix $\Phi(t)$, positive constants, K, L, α and β and a constant projection matrix P such that

$$\begin{aligned} \| \Phi(t)\, P\Phi^{-1}(s)\| &\leqslant Ke^{-\alpha(t-s)}, \qquad t \geqslant s, \\ \| \Phi(t)(I - P)\, \Phi^{-1}(s)\| &\leqslant Le^{-\beta(t-s)}, \qquad t \leqslant s. \end{aligned}$$

Theorem 1. *Let $\theta' = f(t, \theta)$ have unique solutions existing for all t, let $\psi(t, \tau, \theta)$ be the solution such that $\psi(\tau, \tau, \theta) = \theta$ and let g be bounded. Then the unique invariant manifold for* (3.1) *is*

$$S(t, \theta) = \left\{\int_{-\infty}^{t} \Phi(t)\, P\Phi^{-1}(v) - \int_{t}^{\infty} \Phi(t)(I - P)\, \Phi^{-1}(v)\right\} [g(v, \psi(v, t, \theta))]\, dv.$$

Further, if (3.1) *is T periodic in t and L periodic in θ, so is S.*

Remark. This theorem gives the unique invariant manifold for the quasilinear case of the Coppel–Palmer system [7].

Proof. By uniqueness, $\psi(t, \tau, \theta) = \psi(t, v, \psi(v, \tau, \theta))$. The equation

$$y' = A(t)\, y + g(t, \psi(t, \tau, \theta))$$

has the unique bounded solution

$$\begin{aligned} y(t) &= \left\{\int_{-\infty}^{t} \Phi(t)\, P\Phi^{-1}(v) - \int_{t}^{\infty} \Phi(t)(I-P)\,\Phi^{-1}(v)\right\} [g(v, \psi(v, \tau, \theta))]\, dv \\ &= \left\{\int_{-\infty}^{t} \Phi(t)\, P\Phi^{-1}(v) - \int_{t}^{\infty} \Phi(t)(I-P)\,\Phi^{-1}(v)\right\} \\ &\quad \times [g(v, \psi(v, t, \psi(t, \tau, \theta)))]\, dv \\ &= S(t, \psi(t, \tau, \theta)). \end{aligned}$$

In case (3.1) is periodic, then we have

$$\psi(t, \tau, \theta + L) = \psi(t, \tau, \theta) + L$$

and

$$\psi(t, \tau + T, \theta) = \psi(t - T, \tau, \theta).$$

Also $\Phi(t+T) = \Phi(t)C$ for some nonsingular matrix C. But $\Phi(t+T)$ satisfies the same exponential dichotomy that $\Phi(t)$ does, and hence by Lemma 4 of Coppel [6, p. 504], $CP = PC$. Thus, we have

$$\Phi(t+T)\, P\Phi^{-1}(v+T) = \Phi(t)\, P\Phi^{-1}(v)$$

together with

$$\Phi(t+T)(I-P)\,\Phi^{-1}(v+T) = \Phi(t)(I-P)\,\Phi^{-1}(v).$$

With the above facts, it is easy to show $S(t, \theta) = S(t+T, \theta) = S(t, \theta + L)$.

The functional equation for the periodic surface of (2.1) is found in a similar manner, and as a first step we write (2.1) as

$$\begin{aligned} \theta &= T[1 + f(t, \theta, y, T, \mu, \epsilon)], \\ y' &= \theta' A(\theta, \mu) y + Th(t, \theta, y, T, \mu, \epsilon), \end{aligned} \tag{3.2}$$

where (v) of Section 2 was used. Let $S(t, \theta, T, \mu, \epsilon)$ be 1-periodic in t and θ and let $\psi(t, \tau, \theta, T, \mu, \epsilon; S)$ be the solution of the θ' equation in (3.2) with y replaced by S satisfying $\psi(\tau) = \theta$. Then $\Phi(\psi(t, \mu), \mu)$ is a fundamental matrix for $y' = \psi' A(\psi, \mu) y$. It can be shown by the methods of the proof of Theorem 2, that if $\| S \|$ and $| \epsilon |$ are small enough, then $\Phi(\psi(t, \mu), \mu)$ satisfies (iv) with α replaced by $\alpha T(1 - M(\| S \| + | \epsilon |))$. Thus ,if $S(t, \theta, T, \mu, \epsilon)$ is a periodic surface which is sufficiently small, then $y(t) = S(t, \psi(t))$ satisfies

$$y' = \psi' A(\psi)\, y + Th(t, \psi, S(t, \psi))$$

and so must be given by the unique bounded solution

$$y(t) = T\left\{\int_{-\infty}^{t} \Phi(\psi(t, \tau, \theta))\, P\Phi^{-1}(\psi(v, \tau, \theta)) - \int_{t}^{\infty} \Phi(\psi(t, \tau, \theta))(I - P)\, \Phi^{-1}(\psi(v, \tau, \theta))\right\} \times [h(v, \psi(v, \tau, \theta), S(v, \psi(v, \tau, \theta)))]\, dv.$$

Recalling that $\psi(v, \tau, \theta) = \psi(v, t, \psi(t, \tau, \theta))$, we see that

$$y(t) = (\mathscr{G}(S))(t, \psi(t, \tau, \theta)),$$

where

$$\mathscr{G}(S) = T\left\{\int_{-\infty}^{t} \Phi(\theta)\, P\Phi^{-1}(\psi(v, t, \theta)) - \int_{t}^{\infty} \Phi(\theta)(I - P)\, \Phi^{-1}(\psi(v, t, \theta))\right\} \times [h(v, \psi(v, t, \theta), S(v, \psi(v, t, \theta)))]\, dv. \tag{3.3}$$

Thus S must satisfy

$$S = \mathscr{G}(S). \tag{3.4}$$

One can check that $\mathscr{G}(S)$ is 1-periodic in t and θ.

4. Principle Properties of the Surface

THEOREM 2. *Under assumptions* (i)–(iv) *there exists a* $\delta > 0$ *such that the system* (2.1) *has a periodic surface* $y = S(t, \theta, T, \mu, \epsilon)$ *where* S *is defined and continuous for all* $(t, \theta, T, \mu, \epsilon)$ *in* $K_\delta = R^1 \times R^1 \times (0, \infty) \times J \times [-\delta, \delta]$ *and is* 1*-periodic in* t *and* θ. *Further, for each fixed* μ, ϵ, S *is* C^1 *in* t, θ *and* T, S_θ *is of bounded variation in* θ *on* $\theta : 0 \leqslant \theta \leqslant 1$ *uniformly in* t, T, μ, ϵ, *and, finally, there is a constant* N *such that*

$$\| S/\epsilon \| \leqslant N, \tag{4.1}$$

$$\| TS_T/\epsilon \| \leqslant N. \tag{4.2}$$

If $g_\epsilon(t, \theta, 0, T, \mu, 0)$ *exists uniformly with respect to* t *and* θ, *then*

$$S_\epsilon(t, \theta, T, \mu, 0) = T\left\{\int_{-\infty}^{t} \Phi(\theta, \mu)\, P\Phi^{-1}(\theta + T(v - t), \mu) - \int_{t}^{\infty} \Phi(\theta, \mu)(I - P)\, \Phi^{-1}(\theta + T(v - t), \mu)\right\} \times [g_\epsilon(v, \theta + T(v - t), 0, T, \mu, 0)]\, dv.$$

Remarks. The proof will show there exists a constant $r > 0$ such that $\| S_\theta(t, \theta_1, T, \mu, \epsilon) - S_\theta(t, \theta_2, T, \mu, \epsilon)\| \leqslant r \mid \theta_1 - \theta_2 \mid$ for all $(t, \theta_1, T, \mu, \epsilon)$ and $(t, \theta_2, T, \mu, \epsilon)$ in K_δ and $\| T_1 S_T(t, \theta, T_1, \mu, \epsilon) - T_2 S_T(t, \theta, T_2, \mu, \epsilon)\| \leqslant r \mid \ln(T_1/T_2)\mid$ for all $(t, \theta, T_1, \mu, \epsilon)$ and $(t, \theta, T_2, \mu, \epsilon)$ in K_δ. The importance of the first inequality is well known while the author is not aware of any use for the second. Further, if for some constant M_1 we have

$$\| g(t, \theta, 0, T, \mu, \epsilon)/\epsilon - g_\epsilon(t, \theta, 0, T, \mu, 0)\| \leqslant M_1 \mid \epsilon \mid$$

for all $(t, \theta, T, \mu, \epsilon)$ in K_δ, then it is easily shown that a constant M_2 exists such that $\| S(t, \theta, T, \mu, \epsilon) - \epsilon S_\epsilon(t, \theta, T, \mu, 0)\| \leqslant M_2 \mid \epsilon \mid^2$ for all $(t, \theta, T, \mu, \epsilon)$ in K_δ. This is the case for (1.5).

The results above, of course, include Levinson's [11] and Diliberto and Hufford's [8]. The method of proof is much like that of Theorem 3.1 of Diliberto [9, p. 66] and employs a fixed-point theorem of the type originally used by Levinson in [11] to study (1.5). The estimation process will be quite crude.

Proof. Let

$$\beta = \alpha(1 - M(r + \delta))$$

and

$$\gamma = 2M(r + \delta).$$

Without loss of generality we assume that

$$\alpha \leqslant 1 \leqslant M.$$

Choose $r > 0$ and $\delta > 0$ so that $r \leqslant \bar{r}$, $\delta \leqslant \bar{\epsilon}$ and

$$0 < 4256 M^6(r^2 + \delta)/(\beta - 2\gamma)^3 \leqslant r. \tag{4.3}$$

Some consequences of (4.3) are that $0 < \beta - 2\gamma < \beta - \gamma < \beta < \alpha \leqslant 1$ and these inequalities will be used throughout to simplify estimates.

Further consequences are

$$r < 1, \qquad \nu \equiv 14M^4(r + \delta)/(\beta(\beta - \gamma)) < 1$$

and

$$0 < \beta - 2M^2 r.$$

Since most of the estimates below are found by similar techniques and are very tedious, most of them will not be derived, merely stated.

Let $\mathscr{S}$ consist of all $n - 1$ vector functions $S(t, \theta, T, \mu, \epsilon)$ with domain K_δ

which are 1-periodic in t and θ. Let $\mathscr{S}_r$ be the subset of $\mathscr{S}$ consisting of those functions which are C^2 in θ and T such that $\| S \|$, $\| S_\theta \|$, $\| TS_T \|$, $\| S_{\theta\theta} \|$, $\| TS_{T\theta} \|$, $\| T(TS_T)_T \| \leqslant r$, where $\| S \| = \sup\{\| S(P) \| : P \text{ in } K_\delta\}$ for continuous functions S on K_δ. *Define a mapping* $\mathscr{G} : \mathscr{S}_r \to \mathscr{S}$ *by* (3.3). r *and* δ *have been chosen so that* $\mathscr{G}(\mathscr{S}_r) \subset \mathscr{G}_r$ *and* $\mathscr{G}$ *is a contraction in the sense that* $\| S \|$, $\| Q \|$, $\| S_\theta \|$, $\| Q_\theta \| \leqslant r$ *implies* $\| \mathscr{G}(S) - \mathscr{G}(Q) \| \leqslant \nu \| S - Q \| \|$, *and, as a bonus, if* S *is in* $\mathscr{S}_r$, *then* $\mathscr{G}(S)_t$, $\mathscr{G}(S)_{Tt}$ *and* $\mathscr{G}(S)_{\theta t}$ *exist are continuous and* $\| \mathscr{G}(S)_t \|$, $\| \mathscr{G}(S)_{Tt} \|$, $\| \mathscr{G}(S)_{\theta t}/T \| \leqslant r$.

First, estimates on $\psi(t, \tau, \theta, T, \mu, \epsilon; S)$ and its various derivatives are needed, where S is in $\mathscr{S}_r$. From

$$\psi_\theta(t, \tau, \theta) = \exp \left(T \int_\tau^t (f_\theta(u, \psi, S(u, \psi)) + f_y(u, \psi, S(u, \psi))\, S_\theta(u, \psi))\, du\right),$$

we obtain

$$| \psi_\theta(t, \tau, \theta) \leqslant e^{T|t-\tau|(M(r+\delta)r+M)} \leqslant e^{\nu T|t-\tau|}. \tag{4.4}$$

Similarly,

$$T\psi_T(t, \tau, \theta) = T \int_\tau^t (\psi(t, \tau, \theta)/\psi(v, \tau, \theta))[1 + f(v, \psi, S(v, \psi)) + f_y(v, \psi, S(v, \psi)) \cdot TS_T(v, \psi) + Tf_T(v, \psi, S(v, \psi))]\, dv$$

gives

$$| T\psi_T(t, \tau, \theta)| \leqslant 3(e^{\nu T|t-\tau|} - 1)/(r + \delta) \leqslant 6MT \mid t - \tau \mid e^{\nu T|t-\tau|}. \tag{4.5}$$

Continuing, we obtain

$$| \psi_\tau(t, \tau, \theta)/T \mid \leqslant 3Me^{\nu T|t-\tau|}, \tag{4.6}$$

$$| T\psi_{T\theta}(t, \tau, \theta)| \leqslant 15MT \mid t - \tau \mid e^{2\nu T|t-\tau|}, \tag{4.7}$$

$$| \psi_{\theta\theta}(t, \tau, \theta)| \leqslant (5/2)\, e^{2\nu T|t-\tau|}, \tag{4.8}$$

$$| T(T\psi_T)_T(t, \tau, \theta)| \leqslant 42MT \mid t - \tau \mid e^{\nu T|t-\tau|} + 90MT \mid t - \tau \mid e^{\nu T|T-\tau|}(e^{\nu T|t-\tau|} - 1)/(r + \delta), \tag{4.9}$$

$$| \psi_{T\tau}(t, \tau, \theta)| \leqslant 6Me^{\nu T|t-\tau|} + 45M^2T \mid t - \tau \mid e^{2\nu T|t-\tau|}, \tag{4.10}$$

$$| \psi_{\theta\tau}(t, \tau, \theta)/T \mid \leqslant 8Me^{2\nu T|t-\tau|}. \tag{4.11}$$

The existence of the derivatives above follows from the continuity of $S_{\theta\theta}$, $S_{T\theta}$, S_{TT} [4, Chapter 1]. The estimates concerning τ derivatives follow easily from

$$\psi_\tau(t, \tau, \theta) = -T\psi_\theta(t, \tau, \theta)(1 + f(\tau, \theta, S(\tau, \theta))).$$

The following estimate is needed to deal with $\mathscr{G}(S)$ where S is in $\mathscr{S}_r$.

$$\begin{aligned} \| \Phi(\theta)\, P\Phi^{-1}(\psi(v, t, \theta))\| &\leqslant Me^{-\beta T(t-v)}, \qquad t \geqslant v, \\ \| \Phi(\theta)(I - P)\, \Phi^{-1}(\psi(v, t, \theta))\| &\leqslant Me^{-\beta T(v-t)}, \qquad v \geqslant t. \end{aligned} \tag{4.12}$$

These follow from (iv) and

$$\begin{aligned} 0 \leqslant T(t-\tau)(1 - M(r+\delta)) &\leqslant \psi(t, \tau, \theta) - \theta, \qquad t \geqslant \tau, \\ \psi(t, \tau, \theta) - \theta \leqslant T(t-\tau)(1 - M(r+\delta)) &\leqslant 0, \qquad t \leqslant \tau, \end{aligned}$$

where the last inequalities are consequences of

$$\begin{aligned} | \psi(t, \tau, \theta) - \theta - T(t-\tau)| &= \left| T \int_\tau^t f(u, \psi(u, \tau, \theta), S(u, \psi(u, \tau, \theta)))\, du \right| \\ &\leqslant TM(r + \delta) \mid t - \tau \mid. \end{aligned}$$

Equation (3.3) immediately gives

$$\begin{aligned} \| \mathscr{G}(S)\| &\leqslant T \left\{ \int_{-\infty}^t Me^{-\beta T(t-v)} + \int_t^\infty Me^{-\beta T(v-t)} \right\} [M(r^2 + \delta)]\, dv \\ &\leqslant 2M^2(r^2 + \delta)/\beta \leqslant r. \end{aligned} \tag{4.13}$$

Now

$$\begin{aligned} \mathscr{G}(S)_\theta = {} & A(\theta)\, \mathscr{G}(S) \\ & - T \left\{ \int_{-\infty}^t \Phi(\theta)\, P\Phi^{-1}(\psi)\, A(\psi)\, \psi_\theta - \int_t^\infty \Phi(\theta)(I - P)\, \Phi^{-1}(\psi)\, A(\psi)\, \psi_\theta \right\} \\ & \times [h(v, \psi, S(v, \psi))]\, dv \\ & + T \left\{ \int_{-\infty}^t \Phi(\theta)\, P\Phi^{-1}(\psi)\, \psi_\theta - \int_t^\infty \Phi(\theta)(I - P)\, \Phi^{-1}(\psi)\, \psi_\theta \right\} \\ & \times [h_\theta(v, \psi, S(v, \psi)) + h_y(v, \psi, S(v, \psi))\, S_\theta(v, \psi)]\, dv, \end{aligned}$$

so

$$\begin{aligned} & \| \mathscr{G}(S)_\theta - A(\theta)\, \mathscr{G}(S)\| \\ & \quad \leqslant T \left\{ \int_{-\infty}^t M^2 e^{-(\beta-\gamma)T(t-v)} + \int_t^\infty M^2 e^{-(\beta-\gamma)T(v-t)} \right\} [M(r^2 + \delta)]\, dv \\ & \qquad + T \left\{ \int_{-\infty}^t Me^{-(\beta-\gamma)T(t-v)} + \int_t^\infty Me^{-(\beta-\gamma)T(v-t)} \right\} \\ & \qquad \times [M(r^2 + \delta) + M(r + \delta)r]\, dv \\ & \quad \leqslant (2M^3 + 4M^2)(r^2 + \delta)/(\beta - \gamma) \end{aligned}$$

and

$$\begin{aligned}\| \mathscr{G}(S)_\theta \| &\leqslant \| \mathscr{G}(S)_\theta - A(\theta)\, \mathscr{G}(S)\| + \| A(\theta)\, \mathscr{G}(S)\| \\ &\leqslant (2M^3 + 4M^2)(r^2+\delta)/(\beta-\gamma) + 2M^3(r^2+\delta)/\beta \\ &\leqslant 8M^3(r^2+\delta)/(\beta-\gamma) \\ &\leqslant r. \end{aligned} \tag{4.14}$$

Next

$$\begin{aligned} T\mathscr{G}(S)_T = \mathscr{G}(S) - T \Big\{ &\int_{-\infty}^{t} \Phi(\theta)\, P\Phi^{-1}(\psi)\, A(\psi)\, T\psi_T \\ &- \int_t^{\infty} \Phi(\theta)(I-P)\, \Phi^{-1}(\psi)\, A(\psi)\, T\psi_T \Big\} [h(v,\psi,S(v,\psi))]\, dv \\ &+ T \Big\{ \int_{-\infty}^{t} \Phi(\theta)\, P\Phi^{-1}(\psi) - \int_t^{\infty} \Phi(\theta)(I-P)\, \Phi^{-1}(\psi) \Big\} \\ &\times [h_\theta(v,\psi,S(v,\psi))\, T\psi_T + h_y(v,\psi,S(v,\psi)) \\ &\times (S_\theta(v,\psi)\, T\psi_T + TS_T(v,\psi)) + Th_T(v,\psi,S(v,\psi))]\, dv. \end{aligned} \tag{4.15}$$

Thus

$$\begin{aligned} &\| T\mathscr{G}(S)_T - \mathscr{G}(S)\| \\ &\quad\leqslant T \Big\{ \int_{-\infty}^{t} M^2 e^{-\beta T(t-v)}(3)(e^{\gamma T(t-v)} - 1)/(r+\delta) \\ &\qquad + \int_t^{\infty} M^2 e^{-\beta T(v-t)}(3)(e^{\gamma T(v-t)} - 1)/(r+\delta) \Big\} [M(r^2+\delta)]\, dv \\ &\qquad + T \Big\{ \int_{-\infty}^{t} M e^{-\beta T(t-v)}(3)(e^{\gamma T(t-v)} - 1)/(r+\delta) \\ &\qquad + \int_t^{\infty} M e^{-\beta T(v-t)}(3)(e^{\gamma T(v-t)} - 1)/(r+\delta) \Big\} [M(r^2+\delta) + M(r+\delta)r]\, dv \\ &\qquad + T \Big\{ \int_{-\infty}^{t} M e^{-\beta T(t-v)} + \int_t^{\infty} M e^{-\beta T(v-t)} \Big\} [M(r^2+\delta) + M(r+\delta)r]\, dv \\ &\quad\leqslant (12M^4 + 24M^3)(r^2+\delta)/(\beta(\beta-\gamma)) + 4M^2(r^2+\delta)/\beta, \end{aligned}$$

and

$$\begin{aligned} &\| T\mathscr{G}(S)_T \| \\ &\quad\leqslant \| T\mathscr{G}(S)_T - \mathscr{G}(S)\| + \| \mathscr{G}(S)\| \\ &\quad\leqslant (12M^4 + 24M^3)(r^2+\delta)/(\beta(\beta-\gamma)) + 4M^2(r^2+\delta)/\beta + 2M^2(r^2+\delta)/\beta \\ &\quad\leqslant 42\, M^4(r^2+\delta)/(\beta(\beta-\gamma)) \qquad (4.16) \\ &\quad\leqslant r. \end{aligned}$$

Continuing similarly gives

$$\| T\mathscr{G}(S)_{T\theta} \| \leqslant 284M^5(r^2+\delta)/(\beta-2\gamma)^2 \leqslant r, \tag{4.17}$$

$$\| \mathscr{G}(S)_{\theta\theta} \| \leqslant 53M^4(r^2+\delta)/(\beta-2\gamma) \leqslant r, \tag{4.18}$$

$$\| \mathscr{G}(S)_{\theta t}/T \| \leqslant 134M^5(r^2+\delta)/(\beta-2\gamma) \leqslant r, \tag{4.19}$$

$$\| T(T\mathscr{G}(S)_T)_T \| \leqslant 4256M^6(r^2+\delta)/(\beta-2\gamma)^3 \leqslant r, \tag{4.20}$$

$$\| \mathscr{G}(S)_{Tt} \| \leqslant 765M^6(r^2+\delta)/(\beta-2\gamma)^2 \leqslant r. \tag{4.21}$$

It should be pointed out that the t derivatives of $\mathscr{G}(S)$ exist and are continuous because of (3.3), where $\psi = \psi(v, t, \theta)$, and the needed τ-derivatives of $\psi(t, \tau, \theta, T, \mu, \epsilon; S)$ exist and are continuous.

The contraction is now needed and only the key estimates will be given. Let $\| S \|$, $\| Q \|$, $\| S_\theta \|$, $\| Q_\theta \| \leqslant r$ and let $\psi = \psi(t, \tau, \theta, T, \mu, \epsilon; S)$ and $\phi = \psi(t, \tau, \theta, T, \mu, \epsilon; Q)$. Then

$$\psi(t, \tau, \theta) - \phi(t, \tau, \theta) = T \int_\tau^t (f(v, \psi, S(v, \psi)) - f(v, \phi, Q(v, \phi)))\, dv$$

implies

$$| \psi(t, \tau, \theta) - \phi(t, \tau, \theta)| \leqslant MT \| S - Q \| \, | t - \tau | + \gamma T \left| \int_\tau^t | \psi(v, \tau, \theta) - \phi(v, \tau, \theta)| \, dv \right|$$

and by Gronwall's inequality we obtain

$$| \psi(t, \tau, \theta) - \phi(t, \tau, \theta)| \leqslant \| S - Q \| (e^{\gamma T |t-\tau|} - 1)/(r + \delta).$$

Next, for $0 \leqslant u \leqslant 1$, we obtain

$$\begin{aligned} &| \phi(v, t, \theta) + u(\psi(v, t, \theta) - \phi(v, t, \theta) - \theta - T(v - t))| \\ &= \left| (1-u)T \int_t^v f(\sigma, \phi, Q(\sigma, \phi))\, d\sigma + uT \int_t^v f(\sigma, \psi, S(\sigma, \psi))\, d\sigma \right| \\ &\leqslant (1-u)\, TM(r+\delta) \, | v - t | + uTM(r+\delta) \, | v - t | \\ &= MT(r+\delta) \, | v - t |, \end{aligned}$$

and this implies

$$\theta - \phi(v, t, \theta) - u(\psi(v, t, \theta) - \phi(v, t, \theta)) \geqslant T(t - v)(1 - M(r+\delta)) \geqslant 0, \qquad t \geqslant v,$$

$$0 \geqslant -T(v - t)(1 - M(r+\delta)) \geqslant \theta - \phi(v, t, \theta) - u(\psi(v, t, \theta) - \phi(v, t, \theta)), \qquad v \geqslant t.$$

For $v \leqslant t$

$$\begin{aligned}
&\| \Phi(\theta)\, P\Phi^{-1}(\psi(v, t, \theta)) - \Phi^{-1}(\phi(v, t, \theta))\| \\
&\quad = \left\| \Phi(\theta) P \int_0^1 \Phi^{-1\prime}(\psi + u(\psi - \phi))(\psi - \phi)\, du \right\| \\
&\quad = \left\| -\int_0^1 \Phi(\theta)\, P\Phi^{-1}(\phi + u(\psi - \phi)\, A(\phi + u(\psi - \phi))\, du(\psi - \phi) \right\| \\
&\quad \leqslant M^2 \| S - Q \| \left[(e^{\gamma T(t-v)} - 1)/(r + \delta)\right] e^{-\beta T(t-v)}. \qquad (4.22)
\end{aligned}$$

Likewise, for $v \geqslant t$,

$$\| \Phi(\theta)(I - P)(\Phi^{-1}(\psi(v, t, \theta)) - \Phi^{-1}(\phi(v, t, \theta))\| \leqslant M^2 \| S - Q \| \cdot \left[(e^{\gamma T(v-t)} - 1)/(r + \delta)\right] e^{-\beta T(v-t)}$$

The contraction estimate is now easily made and found to be

$$\begin{aligned}
\| \mathscr{G}(S) - \mathscr{G}(Q)\| &\leqslant \{(4M^4 + 8M^3)(r^2 + \delta)/(\beta(\beta - \gamma)) + 2M^2(r + \delta)/\beta\} \| S - Q \| \\
&\leqslant \{14M^4(r + \delta)/(\beta(\beta - \gamma))\} \| S - Q \| = \nu \| S - Q \|,
\end{aligned} \qquad (4.23)$$

where $0 < \nu < 1$ by (4.3).

If $S_0 \equiv 0$ and $S_{n+1} = \mathscr{G}(S_n)$, then $S = \lim S_n$ exists uniformly on K_δ and S is continuous. Consider $\{S_{n\theta}(t, \theta, T, \mu, \epsilon)\}_{n=1}^{\infty}$ for fixed μ, ϵ. By estimates (4.17), (4.18), and (4.19) this sequence is uniformly bounded and equicontinuous on each compact subset of $R^1 \times R^1 \times (0, \infty)$ and an application of the Ascoli lemma to $\{S_{n\theta}\}$ shows that S_θ exists, is continuous in (t, θ, T), and $\| S_\theta \| < r$. Further, it follows that

$$\| S_\theta(t, \theta_1, T, \mu, \nu) - S_\theta(t, \theta_2, T, \mu, \epsilon)\| \leqslant r \mid \theta_1 - \theta_2 \mid$$

from.(4.18). Similarly, it is shown that S_T exists as a continuous function of (t, θ, T) for each fixed μ, ϵ, $\| TS_T \| \leqslant r$ and

$$\| T_1 S_T(t, \theta, T_1, \mu, \epsilon) - T_2 S_T(t, \theta, T_2, \mu, \epsilon\| \leqslant r \mid \ln(T_2/T_1)|.$$

Since $\| S \|$, $\| S_\theta \| \leqslant r$ we see that

$$\| S - \mathscr{G}(S)\| \leqslant \| S - S_{n+1} \| + \| \mathscr{G}(S_n) - \mathscr{G}(S)\| \leqslant \| S - S_{n+1} \| + \nu \| S_n - S \|$$

for all n, and thus $S = \mathscr{G}(S)$. The assertion about S_t now follows from the fact that $\psi(t, \tau, \theta, T, \mu, \epsilon; S)$ is C^1 in (t, τ, θ, T) because S is C^1 in (θ, T) and $\psi = \psi(v, t, \theta, T, \mu, \epsilon; S)$ in (3.3) and $S = \mathscr{G}(S)$ (and incidentally, $\| S_t \| \leqslant r$).

From $S = \mathscr{G}(S)$, we obtain $\| S/\epsilon \| [\leqslant (2M^2/\beta)(\| S \| \| S/\epsilon \| + 1) \leqslant$

$(2M^2r/\beta)\| S/\epsilon \| + 2M^2/\beta$ which gives $\| S/\epsilon \| \leqslant 2M^2/(\beta - 2M^2r)$, where $\beta - 2M^2r > 0$ as a consequence of (4.3). Again using $S = \mathscr{G}(S)$ and (4.15) we obtain in a similar manner to that of (4.16)

$$\| TS_T/\epsilon \| \leqslant \| S/\epsilon \| + ((12M^4 + 24M^3)/(\beta(\beta - \gamma)))(r\| S/\epsilon \| + 1) \\ + (4M^2/\beta)(r \| S/\epsilon \| + 1).$$

By choosing N large enough we obtain (4.1) and (4.2).

The computation of S_ϵ is straight forward [(4.1) is conveniently used] and will be omitted.

5. The Stability Manifolds

The stability analysis is very much like that of Palmer [12, pp. 372–392] and will be sketched. First, if $(\psi(t), y(t))$ is a solution of (2.1) with $\| y(t)\| \leqslant r$ for all $t \geqslant \tau$, where r and ϵ satisfy $1 - M(r + | \epsilon |) > 0$, then $\Phi(\psi(t))$ satisfies the exponential dichotomy (iv) for $t, s \geqslant \tau$ with α replaced by

$$\alpha T(1 - M(r + | \epsilon |)).$$

Consequently, $y(t)$ satisfies the integral equation

$$y(t) = \Phi(\psi(t))\, P\Phi^{-1}(\theta)\rho + T \int_\tau^t \Phi(\psi(t))\, P\Phi^{-1}(\psi(s))\, h(s, \psi(s), y(s))\, ds$$

$$- T \int_t^\infty \Phi(\psi(t))(I - P)\, \Phi^{-1}(\psi(s))\, h(s, \psi(s), y(s))\, ds, \tag{5.1}$$

where $\theta = \psi(\tau)$ and $\rho = y(\tau)$. Consider now, Eq. (5.1) together with the integral equation

$$\psi(t) = \theta + T(t - \tau) + T \int_\tau^t f(v, \psi(v), y(v))\, dv. \tag{5.2}$$

Let r and δ be chosen so that $0 < r \leqslant \bar{r}$, $0 < \delta \leqslant \bar{\epsilon}$, and

$$0 < 96M^4(r^2 + \delta)/q \leqslant r, \tag{5.3}$$

where $q = \alpha(1 - 2M^2(r + \delta))$. Then successive approximations in (5.1)–(5.2) lead to a solution $(\psi(t, \tau, \theta, \rho), y(t, \tau, \theta, \rho))$ if $\| \rho \| \leqslant r$ and $| \epsilon | \leqslant \delta$. [With $\psi_0 = \theta + T(t - \tau)$, $y_0 = \Phi(\theta)\, P\Phi^{-1}(\theta)\rho$, the author has obtained estimates of the form $| \psi_{n+1}(t) - \psi_n(t)|$, $\| y_{n+1}(t) - y_n(t)\| \leqslant L2^{-n}e^{qT(t-\tau)/2}$, where L is a constant independent of T. The limit (ψ, y) satisfies $| \psi(t) - \psi_n(t)|$,

$\| y(t) - y_n(t)\| \leqslant L2^{-n}e^{qT(t-\tau)/2}$. The exponential dichotomies that occur have exponential decay $e^{-qT(s-t)}$ and consequently (ψ, y) can be shown to satisfy (5.1)–(5.2).] Please note that r and δ have been chosen independent of T. $y(\tau)$ satisfies

$$\Phi(\theta)\, P\Phi^{-1}(\theta)\, y(\tau, \tau, \theta, \rho) = \Phi(\theta)\, P\Phi^{-1}(\theta)\rho,$$

which shows that for each (τ, θ), the positive stability manifold has dimension k, where k is the rank of P and also of $\Phi(\theta)\, P\Phi^{-1}(\theta)$. Of course, complimentary results hold for the negative stability manifold.

Next we will show that solutions bounded on half-lines tend to the surface exponentially. Let r and δ be chosen as in (4.3). Let (ϕ, z) be a solution (2.1) such that $|\epsilon| \leqslant \delta$ and $\| z(t)\| \leqslant r$ for all $t \geqslant \tau$. Let $y(t) = S(t, \phi(t))$. To estimate $\| z(t) - y(t)\|$ we need the partial-differential equation which $S(t, \theta)$ satisfies, namely,

$$S_t + S_\theta T(1 + f(t, \theta, S)) = TA(\theta)\, S + Tg(t, \theta, S).$$

Thus

$$\begin{aligned} z' - y' &= TA(\phi)\, z + Tg(t, \phi, z) - S_t(t, \phi) - S_\theta(t, \phi)\, \phi' \\ &= TA(\phi)\, z + Tg(t, \phi, z) + S_\theta(t, \phi)\, T(1 + f(t, \phi, y)) \\ &\quad - TA(\phi)\, y - Tg(t, \phi, y) - S_\theta(t, \phi)\, T(1 + f(t, \phi, z)) \\ &= \phi' A(\phi)(z - y) + Tl(t), \end{aligned}$$

where

$$\begin{aligned} l(t) &= g(t, \phi, z) - g(t, \phi, y) - f(t, \phi, z)\, A(\phi)(z - y) \\ &\quad - S_\theta(t, \phi, z)(f(t, \phi, z) - f(t, \phi, y)). \end{aligned}$$

It follows that

$$\| l(t)\| \leqslant 3M^2(r + \delta)\| z(t) - y(t)\|.$$

Let $\tau \leqslant s \leqslant t$. Then

$$\begin{aligned} z(t) - y(t) &= \Phi(\phi(t))\, P\Phi^{-1}(\phi(s))(z(s) - y(s)) \\ &\quad + T \int_s^t \Phi(\phi(t))\, P\Phi^{-1}(\phi(v))\, l(v)\, dv \\ &\quad - T \int_t^\infty \Phi(\phi(t))(I - P)\, \Phi^{-1}(\phi(v))\, l(v)\, dv, \end{aligned}$$

which gives

$$\| z(t) - y(t)\| \leqslant M \| z(s) - y(s)\| e^{-\beta T(t-s)}$$
$$+ 3M^3(r + \delta)T \int_s^t \| z(v) - y(v)\| e^{-\beta T(t-v)} \, dv$$
$$+ 3M^3(r + \delta)T \int_t^\infty \| z(v) - y(v)\| e^{-\beta T(v-t)} \, dv. \quad (5.4)$$

Here $6M^3(r + \delta)/\beta \leqslant \nu/2 < 1/2$, where $\nu = 14M^4(r + \delta)/(\beta(\beta - \gamma)) < 1$ [cf. below (4.3)]. The inequality (5.4) has occurred several times in the literature, see Chang [3, pp. 442–423], Coppel [5, p. 80], and Palmer [12, pp. 391–392], and with $6M^3(r + \delta)/\beta < 1/2$, their method gives

$$\| z(t) - y(t)\| \leqslant 2M \| z(s) - y(s)\| e^{-\beta T(t-s)/2}.$$

Lastly, the methods of Palmer can be adopted easily to show (ϕ, z)tends exponentially to a solution $(\psi(t), S(t, \psi(t)))$ of (2.1) on the surface S.

6. Periodic Solutions

Let (2.1) arise from (1.1). If $S(t, \theta, T, \epsilon)$ is as in Theorem 2, then $H(t, \theta, T, \epsilon) = u(\theta) + Z(\theta) S(t, \theta, T, \epsilon)$. The result of Section 5 pertinent to the present discussion is that if $x(t) = u(\theta(t)) + Z(\theta(t)) y(t)$ is a pT-periodic solution to (1.1), where p is a positive integer, such that $\| y(t)\| \leqslant r$, then $x(t)$ must be on the surface $x = H$, that is, $y(t) = S(t/T, \theta(t))$ and $\theta(t)$ is a pT-periodic solution (on the torus) to

$$\theta' = k(t/T, \theta, T, \epsilon). \quad (6.1)$$

Consequently, the discussion of pT-periodic solutions to (1.1) in the neighborhood $N_r = \{x : x = u(\theta) + Z(\theta)y, \| y \| \leqslant r, 0 \leqslant \theta \leqslant 1\}$ of $x = u(\theta) : 0 \leqslant \theta \leqslant 1$ is reduced to consideration of (1.6). Note that N_r is independent of T.

Now

$$k(t, \theta, T, \epsilon) = 1 + \epsilon m(t, \theta, T, \epsilon),$$

where m is the continuous function defined by

$$m(t, \theta, T, \epsilon) = F(\theta, S(t, \theta, T, \epsilon)) S(t, \theta, T, \epsilon)/\epsilon + U(t, \theta, S(t, \theta, T, \epsilon), \epsilon)$$

for $\epsilon \neq 0$ and $m(t, \theta, T, 0) = F(\theta, 0) S_\epsilon(t, \theta, T, 0) + U(t, \theta, 0,0)$. By (4.1) and (4.2) we have

$$| m(t, \theta, T, \epsilon)|, \qquad | Tm_T(t, \theta, T, \epsilon)| \leqslant M_3 \quad (6.1)$$

for all (t, θ, T, ϵ) in $R^1 \times R^1 \times (0, \infty) \times [-\delta, \delta]$, where M_3 is a constant [of course, independent of (t, θ, T, ϵ)]. From the results of Poincare, Bohl, and Denjoy [4, Chapter 17] we have that if $\theta(t)$ is a pT-periodic solution of (1.6), then there is a unique integer q such that

$$\theta(pT) = \theta(0) + q. \tag{6.2}$$

Let A_{pq} be the set of points in the T–ϵ plane corresponding to solutions of (1.6) satisfying (6.2) when $|\epsilon| < \bar{\delta} = \min(\delta, 1/M_3)$. Then $\{A_{pq} : p, q$ relatively prime positive integers$\}$ is a disjoint collection and $\bigcup_{p,q} A_{pq}$ accounts for all periodic solutions to (1.6) and, of course, to (1.1) in N_r when $|\epsilon| < \bar{\delta}$.

The methods of [1] together with the inequality (6.1) can be used to prove that for each relatively prime pair (p, q) there exists two continuous functions of ϵ, T_m^{pq}, T_M^{pq} defined for $|\epsilon| < \delta$ such that

$$\frac{q}{p} \cdot \frac{1}{1 + |\epsilon| M_3} \leqslant T_m^{pq}(\epsilon) \leqslant T_M^{pq}(\epsilon) \leqslant \frac{q}{p} \cdot \frac{1}{1 - |\epsilon| M_3}$$

and

$$A_{pq} = \{(T, \epsilon) : |\epsilon| < \bar{\delta},\ T_m^{pq}(\epsilon) \leqslant T \leqslant T_M^{pq}(\epsilon)\}.$$

Further

$$D^+T_m^{pq}(0) = D^-T_M^{pq}(0), \qquad D^+T_M^{pq}(0) = D^-T_m^{pq}(0),$$

where D^+ and D^- denote right and left derivatives. Examples show that this description is best possible.

In [2] lesser results are obtained with less work on the problem of characterizing periodic solutions to (1.1).

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