A Note on Algebra Automorphisms of Triangular Matrices over Commutative Rings

Thomas P. Kezlan
Department of Mathematics
University of Missouri — Kansas City
Kansas City, Missouri 64110-2499

Submitted by George Phillip Barker

ABSTRACT

It is shown that if R is a commutative ring with unity, then every R-algebra, automorphism of the algebra of upper triangular \( n \times n \) matrices over R is inner.

Let \( R \) be a commutative ring with unity and \( \mathcal{T}_n(R) \) the R-algebra of upper triangular \( n \times n \) matrices over \( R \). It was stated in [1] that every R-algebra automorphism of \( \mathcal{T}_n(R) \) is inner provided \( R \) is an integral domain. In I. M. Isaacs's review [2] of this result he mentioned that the proof was in error (although the theorem is true) and also suggested that the hypothesis that \( R \) is an integral domain probably could be weakened or perhaps eliminated altogether. It is the purpose of this note to simultaneously correct the error and generalize the result of [1], thereby confirming the accuracy of Isaacs's conjecture.

**Theorem.** If \( R \) is any commutative ring with unity, then every R-algebra automorphism of \( \mathcal{T}_n(R) \) is inner.

**Proof.** We induct on \( n \), the result being trivial with \( n = 1 \), since the only R-algebra automorphism of \( R \) itself is the identity mapping. Thus, assuming the theorem for matrices of size less than \( n \), let \( \Theta \) be an R-algebra automorphism of \( \mathcal{T} = \mathcal{T}_n(R) \).

Let \( E_{ij} \) denote the standard unit matrices for \( 1 \leq i, j \leq n \) and \( I_k \) the \( k \times k \) identity matrix. We first show that \( \Theta(E_{11}) = T^{-1}E_{11} T \) for some invertible \( T \in \mathcal{T} \).
We use the following notation:

\[ \Theta(E_{kk}) = \begin{bmatrix} e_{ij}^{(k)} \end{bmatrix} \text{ for } k = 1, 2, \ldots, n, \]

\[ \Theta(E_{k,k+1}) = \begin{bmatrix} a_{ij}^{(k)} \end{bmatrix} \text{ for } k = 1, 2, \ldots, n-1. \]

Using

\[ \Theta(E_{k,k+1}) = \Theta(E_{kk}) \Theta(E_{k,k+1}), \quad \Theta(E_{k,k+1}) \Theta(E_{kk}) = 0, \]

and looking at diagonal entries, we obtain

\[ a_{ii}^{(k)} = \sum_{r=1}^{n} e_{ir}^{(k)} a_{ri}^{(k)}, \quad \sum_{r=1}^{n} a_{ir}^{(k)} e_{ri}^{(k)} = 0 \]

for \(1 \leq k \leq n-1\) and \(1 \leq i \leq n\). Recalling that the matrices are upper triangular, we have

\[ a_{ii}^{(k)} = e_{ii}^{(k)} a_{ii}^{(k)} = a_{ii}^{(k)} e_{ii}^{(k)} = 0. \]

Thus the matrices \(\Theta(E_{k,k+1})\) are strictly upper triangular for \(k = 1, 2, \ldots, n-1\).

Now consider the product

\[ \Theta(E_{1n}) = \Theta(E_{12}) \Theta(E_{23}) \cdots \Theta(E_{n-1,n}) \]

\[ = \begin{bmatrix} a_{ij}^{(1)} & a_{ij}^{(2)} & \cdots & a_{ij}^{(n-1)} \end{bmatrix}, \]

the \((i,j)\) entry of which is

\[ \sum_{r_{n-2} = 1}^{n} \sum_{r_{n-3} = 1}^{n} \cdots \sum_{r_{2} = 1}^{n} \sum_{r_{1} = 1}^{n} a_{i1}^{(1)} a_{r_{1}r_{2}}^{(2)} \cdots a_{r_{n-3}r_{n-2}}^{(n-2)} a_{r_{n-2}j}^{(n-1)}. \]

But since the matrices are strictly upper triangular, the nonzero terms in the above sum must have first subscript less than the second; thus there can be only one such term, namely that in which

\[ i = 1, \quad r_{1} = 2, \quad r_{2} = 3, \ldots, \quad r_{n-2} = n-1, \quad j = n. \]
We now know that the \((1, n)\) entry of \(\Theta(E_{1n})\) is \(a_{12}^{(1)}a_{23}^{(2)} \cdots a_{n-1,n}^{(n-1)}\) while all other entries are 0, that is,

\[
\Theta(E_{1n}) = a_{12}^{(1)}a_{23}^{(2)} \cdots a_{n-1,n}^{(n-1)} E_{1n}.
\]

The above is all true for \(\Theta^{-1}\) as well, so

\[
\Theta^{-1}(E_{1n}) = b_{12}^{(1)}b_{23}^{(2)} \cdots b_{n-1,n}^{(n-1)} E_{1n},
\]

where the \(b\)'s are elements of \(R\). Since

\[
E_{1n} = \Theta(\Theta^{-1}(E_{1n})) = a_{12}^{(1)}a_{23}^{(2)} \cdots a_{n-1,n}^{(n-1)} E_{1n},
\]

it follows that \(a_{12}^{(1)}, a_{23}^{(2)}, \ldots, a_{n-1,n}^{(n-1)}\) are units of \(R\).

Using \(\Theta(E_{12}) = \Theta(E_{11}) \Theta(E_{12})\), we get \(a_{12}^{(1)} = e_{11}^{(1)}a_{12}^{(1)}\), and since \(a_{12}^{(1)}\) is a unit, we obtain \(e_{11}^{(1)} = 1\). Similarly, for \(k = 1, 2, \ldots, n-1\) we use \(\Theta(E_{kk+k+1}) = \Theta(E_{kk} + e_{kk+k+1})\) to obtain \(a_{k,k+1}^{(k)} = a_{k,k+1}^{(k)} + e_{k+1,k+1}^{(k)}\), and since \(a_{k,k+1}^{(k)}\) is a unit, we have \(e_{k+1,k+1}^{(k)} = 1\). We now have \(e_{11}^{(1)} = e_{22}^{(2)} = \cdots = e_{nn}^{(n)} = 1\).

Using \(\Theta(E_{kk}) \Theta(E_{qq}) = 0\) for \(q \neq k\), we have \(e_{qq}^{(k)}e_{qq}^{(q)} = 0\) with \(e_{qq}^{(q)} = 1\), whence \(e_{qq}^{(k)} = 0\). In short, the \(k\)th diagonal entry of \(\Theta(E_{kk})\) is 1, while the other diagonal entries are 0, that is, \(\Theta(E_{kk})\) has the form \(\Theta(E_{kk}) = E_{kk} + S_k\), where \(S_k\) is strictly upper triangular.

Taking \(k = 1\), we use the fact that \(\Theta(E_{11})\) is idempotent and \(S_1 E_{11} = 0\) to obtain \(S_1 = E_{11} S_1 + S_1^2\), which upon left multiplication by \(S_1\) yields \(S_1^2 = S_1^3 = \cdots = S_1^n = 0\). Thus \(S_1 = E_{11} S_1\), and therefore we may write

\[
\Theta(E_{11}) = \begin{bmatrix}
1 & c_{12} & c_{13} & \cdots & c_{1n} \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}.
\]

Taking

\[
T = \begin{bmatrix}
1 & c_{12} & c_{13} & \cdots & c_{1n} \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & I_{n-1} & \ddots & \vdots \\
0 & \cdots & 0 & 1 & c_{12} \\
0 & \cdots & 0 & 0 & 1
\end{bmatrix},
\]

gives \(\Theta(E_{11}) = T^{-1} E_{11} T\), and since our aim is to show \(\Theta\) inner, we may assume \(\Theta(E_{11}) = E_{11}\).
We now apply the inductive hypothesis to

$$\mathcal{S} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathcal{F}_{n-1}(R).$$

Note that for $A \in \mathcal{S}$ we have $A \in \mathcal{S}$ iff $E_{11}A = 0 = AE_{11}$. But $E_{11}A = 0 = AE_{11}$ iff $E_{11}\Theta(A) = 0 = \Theta(A)E_{11}$, and hence $A \in \mathcal{S}$ iff $\Theta(A) \in \mathcal{S}$. Inductively, $\Theta$ restricted to $\mathcal{S}$ is inner; say it is induced by $S_0 \in \mathcal{S}$. Let $S = E_{11} + S_0$. Then $S$ is invertible in $\mathcal{S}$, and $\Theta(U) = S^{-1}US$ for all $U \in \mathcal{S}$. Again, since our aim is to show $\Theta$ inner, it suffices to assume $\Theta(U) = U$ for all $U \in \mathcal{S}$.

For $j > 1$ we have $\Theta(E_{1j}) = \Theta(E_{11}E_{1j}E_{jj}) = E_{11}\Theta(E_{1j})E_{jj}$ and hence $\Theta(E_{1j}) = a_{1j}E_{1j}$ for some $a_{1j} \in R$. We shall show that $a_{1j} = a_{12}$ for $j > 1$. Since the above is also true of $\Theta^{-1}$, we have $\Theta^{-1}(E_{1j}) = b_{1j}E_{1j}$ for some $b_{1j} \in R$. Since $\Theta(b_{1j}E_{11}) = \Theta(b_{1j}E_{11}E_{11}) = \Theta(b_{1j}E_{11})E_{11}$, we have $\Theta(b_{1j}E_{11}) = c_{1j}E_{11}$ for some $c_{1j} \in R$. Finally, $E_{1j} = \Theta(b_{1j}E_{1j}) = \Theta(b_{1j}E_{11}E_{1j}) = \Theta(b_{1j}E_{11})\Theta(E_{1j}) = c_{1j}E_{11}a_{1j}E_{1j} = c_{1j}a_{1j}E_{1j}$, whence $c_{1j}a_{1j} = 1$ and $a_{1j}$ is a unit of $R$. Also, $a_{1j}E_{1j} = \Theta(E_{1j}) = \Theta(E_{12}E_{2j}) = a_{12}E_{12}E_{2j}$ [note that $\Theta(E_{2j}) = E_{2j}$, since $E_{2j} \in \mathcal{S}$] = $a_{12}E_{1j}$. Thus $a_{1j} = a_{12}$ for $j > 1$.

Letting

$$D = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{1j} \\ 0 & \cdots & 0 & a_{12} \end{bmatrix},$$

we obtain $D^{-1}E_{ij}D = \Theta(E_{ij})$ for all $i, j$. Thus $\Theta$ is inner, and the proof is complete.

REFERENCES


Received 10 July 1989; final manuscript accepted 21 July 1989