ELSEVIER

Discrete Mathematics 161 (1996) 297–300

DISCRETE MATHEMATICS

Note

A cyclically 6-edge-connected snark of order 118

Martin Kochol*

Institute for Informatics, Slovak Academy of Sciences, P.O. Box 56, Dúbravská cesta 9, 840 00 Bratislava 4, Slovak Republic

Received 22 December 1993; revised 23 September 1994

Abstract

We present a cyclically 6-edge-connected snark of order 118, thereby illustrating a new method of constructing snarks.

1. Introduction

It is well known that the edges of any cubic graph can be coloured with 3 or 4 colours in such a way that incident edges receive distinct colours (see e.g. [6]). Cubic graphs whose edges cannot be coloured with 3 colours are called *snarks.* Two well-known snarks are shown in Fig. 1. The Petersen graph P and the *flower snark* I_5 of Isaacs [2]. They will be used in our construction.

The history, motivation and various constructions of snarks are surveyed by Watkins and Wilson [7]. So far, only one infinite family of cyclically 6-edge-connected snarks is known. It contains the flower snarks of Isaacs [2] having $20+8k$ $k(\geqslant 1)$ vertices.

In this note we illustrate a new method of constructing snarks by producing a cyclically 6-edge-connected snark of order 118. The idea can be easily generalized to obtain similar snarks of any even order $\geqslant 118$ (see [3]). A detailed discussion about this method and its ties with some known constructions can be found in [3–5] and other papers which are in the stage of preparation.

*This work was supported by the SAV Grant No. 2/1138/94.

SSDI 0012-365X(95)00237-5

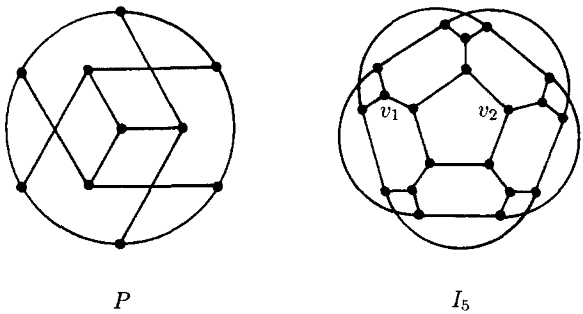


Fig. 1.

2. Construction

Following the notation of Fiol [1] we consider graphs with semiedges. They are called multipoles.

More formally, a *multipole* $M = (V, E, S)$ consists of a set of vertices $V = V(M)$, a set of edges $E = E(M)$ and a set of *semiedges* $S = S(M)$. Each semiedge is incident either with one vertex or with another semiedge making up the so-called *isolated edge*. In Fig. 2 is shown a multipole with seven semiedges and two isolated edges. All multipoles considered here are cubic, i.e. any vertex is incident with just three edges or semiedges.

Let $0, a, b, c$ denote the elements (0,0), (0,1), (1,0), (1,1) of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ respectively. Then by a *3-edge-colouring* of a multipole M we mean a mapping $\varphi : E(M) \cup S(M) \to \{a, b, c\}$ such that:

- $\varphi(e_1) \neq \varphi(e_2)$ for any two (semi)edges e_1, e_2 with a vertex in common;
- $\varphi(s_1) = \varphi(s_2)$ for any two incident semiedges s_1, s_2 making up an isolated edge.

The following Parity lemma is well known (see e.g. [1, 2, 7]).

Lemma 1. *Let M be a multipole with m semiedges and let it be 3-edge-coloured by the elements a, b, c. If m_i denotes the number of semiedges with colour i ($i = a, b, c$), then* $m_a \equiv m_b \equiv m_c \equiv m \pmod 2$.

From Lemma 1 it follows that any 3-edge-colouring φ of a multipole M satisfies

$$\sum_{e \in S(M)} \varphi(e) = 0, \tag{1}$$

where the addition is carried out in $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let A be the multiple depicted in Fig. 3. It has 6 semiedges which are partitioned into two 3-element sets S_1 and S_2. The multipole A was created by deleting two vertices v_1 and v_2 from the Isaacs' flower snark I_5 (see Fig. 1) and retaining the

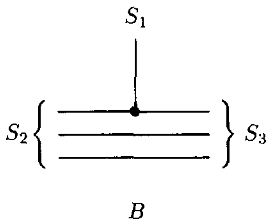


Fig. 2.

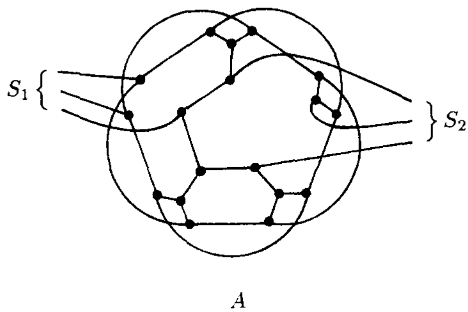


Fig. 3.

resulting semiedges. These are grouped into two sets S_1 and S_2 which arise by the deletion of v_1 and v_2 respectively. Let φ be a 3-edge-colouring of A. Using the addition in $\mathbb{Z}_2 \times \mathbb{Z}_2$ we can define

$$\bar{\varphi}(S_i) = \sum_{e \in S_i} \varphi(e) \quad (i = 1, 2). \tag{2}$$

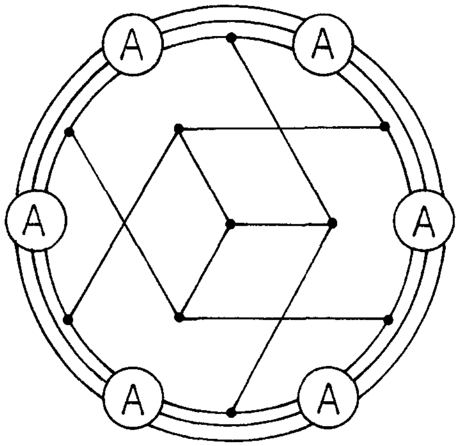
Lemma 2. *For any 3-edge-colouring φ of A we have $\bar{\varphi}(S_1) = \bar{\varphi}(S_2) \neq 0$.*

Proof. By (1) and (2), $\bar{\varphi}(S_1) = \bar{\varphi}(S_2)$. If $\bar{\varphi}(S_1) = \bar{\varphi}(S_2) = 0$, then φ gives rise to a 3-edge-colouring of the snark I_5. Since no such 3-edge-colouring exists, the result follows. □

Let B be the multipole depicted in Fig. 2 whose 7 semiedges are partitioned into three sets S_1, S_2 and S_3. Using formula (2) again, for any 3-edge-colouring φ of B we define $\bar{\varphi}(S_i)$, $i = 1, 2, 3$. Our last lemma is an immediate consequence of (1).

Lemma 3. *Let φ be a 3-edge-colouring of B such that $\bar{\varphi}(S_i) \neq 0$ for any $i = 1, 2, 3$. Then $\bar{\varphi}(S_1)$, $\bar{\varphi}(S_2)$ and $\bar{\varphi}(S_3)$ are pairwise distinct.*

Let P be the Petersen graph, shown in Fig. 1, and C be a cycle of P with length 6. We construct a new cubic graph G_{118} by the following process: replace every vertex from C by a copy of B and every edge from C by a copy of A, leaving the rest of P unchanged. Join the corresponding semiedges of the copies of A and B as indicated in Fig. 4 to obtain G_{118}. The latter graph is cubic, has 118 vertices and we can easily check that it is cyclically 6-edge-connected. We claim that G_{118} is a snark. Indeed, it follows from Lemmas 2 and 3 that any 3-edge-colouring of G_{118} would provide a 3-edge-colouring of P, which is impossible since P is a snark.



G_{118}

S_1 

S_2 - a symbolic representation of A

Fig. 4.

Acknowledgements

The author would like to thank M. Škoviera for reading the manuscript and his comments.

References

[1] M.A. Fiol, A Boolean algebra approach to the construction of snarks, in: Y. Alavi, G. Chartrand, O.R. Oellermann and A.J. Schwenk, eds, Graph Theory, Combinatorics, and Applications (Wiley, New York, 1991) 493–524.

[2] R. Isaacs, Infinite families of non-trivial trivalent graphs which are not Tait colorable, Amer. Math. Monthly 82 (1975) 221–239.

[3] M. Kochol, Construction of cyclically 6-edge-connected snarks, Technical Report TR-II-SAS-07/93-5, Slovak Academy of Sciences, Bratislava, Slovakia, 1993.

[4] M. Kochol, Snarks with large girths, Technical Report TR-II-SAS-10/93-11, Slovak Academy of Sciences, Bratislava, Slovakia, 1993.

[5] M. Kochol, Snarks without small cycles, J. Combin. Theory Ser. B 67 (1996) 34–47.

[6] V.G. Vizing, On an estimate of the chromatic class of a p-graph, Diskret. Analiz 3 (1964) 25–30.

[7] J.J. Watkins and R.J. Wilson, A survey of snarks, in: Y. Alavi, G. Chartrand, O.R. Oellermann and A.J. Schwenk, eds, Graph Theory, Combinatorics, and Applications (Wiley, New York, 1991) 1129–1144.