Some properties of the $c$-nilpotent multiplier of Lie algebras

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**A B S T R A C T**

In this article, we give a sufficient condition for the $c$-nilpotent multiplier of a Lie algebra to be finite dimensional. Also, we show that the $c$-nilpotent multipliers of perfect Lie algebras are isomorphic.

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**1. Introduction and preliminary**

Throughout this article, all Lie algebras are considered over some fixed field $\Lambda$ and $[ , ]$ denotes the Lie bracket. Let $L$ be a Lie algebra presented as the quotient of a free Lie algebra $F$ by an ideal $R$. Then the $c$-nilpotent multiplier of $L$, $c \geq 1$, is defined to be the abelian Lie algebra $M^{(c)}(L) = (R \cap \gamma_{c+1}(F))/\gamma_{c+1}(R, F)$, where $\gamma_{c+1}(F)$ denotes the $(c+1)$-th term of the lower central series of $F$ and $\gamma_1(R, F) = R$, $\gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$ (see [7]). The Lie algebra $M^{(1)}(L) = M(L)$ is more known as the Schur multiplier of $L$ (see [2,3,6] or [8] for more information on the Schur multiplier of Lie algebras). One may check that $M^{(c)}(L)$ is independent of the choice of the free presentation of $L$. Furthermore, if we set $\gamma_{c+1}^*(L) = \gamma_{c+1}(F)/\gamma_{c+1}(R, F)$, then it is readily deduced from the short exact sequence

$$0 \longrightarrow M^{(c)}(L) \longrightarrow \gamma_{c+1}^*(L) \longrightarrow \gamma_{c+1}(L) \longrightarrow 0$$

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and the invariance of $\mathcal{M}^{(c)}(L)$ that $\gamma_{c+1}^*(M)$ is an invariant of $L$. It is obvious that the image of the canonical homomorphism $\gamma_{c+1}^*(M) \rightarrow \gamma_{c}^*(M)$ is ideal in $\gamma_{c}^*(M)$, and $\gamma_{c+1}^*(M) = 1$ if and only if $M$ is nilpotent of class $c$ and $\mathcal{M}^{(c)}(L) = 0$.

A Lie algebra $L$ is said to be c-capable if there exists a Lie algebra $K$ with $L \cong K/Z_c(K)$, where $Z_c(K)$ is the c-th centre of $K$. Evidently, $L$ is 1-capable if and only if it is an inner derivation Lie algebra, and $L$ is c-capable ($c \geq 2$) if and only if it is an inner derivation Lie algebra of a $(c-1)$-capable Lie algebra. Now, we define $Z^*_c(L)$ to be the smallest ideal $T$ of $L$ such that $L/T$ is c-capable. It is obvious that $Z^*_c(L)$ is a characteristic ideal of $L$ contained in $Z_c(L)$, and $Z^*_c(L/Z^*_c(L)) = 0$.

It has been shown in [7] that the dimension of c-nilpotent multiplier of an arbitrary Lie algebra are not necessary isomorphic. For example, if $L$ is a finite dimensional abelian Lie algebra then $\mathcal{M}^{(c)}(L) \not\cong \mathcal{M}^{(d)}(L)$ whenever $c \neq d$. In the next result, we prove that c-nilpotent multipliers of perfect Lie algebras are indeed isomorphic to the Schur multipliers.

**Theorem A.** Let $L$ be a Lie algebra.

(i) If $L/Z^*_c(L)$ is finite dimensional, then both Lie algebras $\gamma_{c+1}^*(M)$ and $\mathcal{M}^{(c)}(L)$ are finite dimensional.

(ii) If $L/Z^*_c(L)$ is nilpotent (resp. solvable), then $\gamma_{c+1}^*(M)$ is nilpotent (resp. solvable).

In general, the c-nilpotent multipliers of an arbitrary Lie algebra are not necessary isomorphic. For example, if $L$ is a finite dimensional abelian Lie algebra then $\mathcal{M}^{(c)}(L) \not\cong \mathcal{M}^{(d)}(L)$ whenever $c \neq d$. In the next result, we prove that c-nilpotent multipliers of perfect Lie algebras are indeed isomorphic to the Schur multipliers.

**Theorem B.** Let $L$ be a prefect Lie algebra. Then the canonical homomorphisms

$$\gamma_{c+1}^*(M) \xrightarrow{\cong} \gamma_{c}^*(M) \quad \text{and} \quad \mathcal{M}^{(c)}(L) \xrightarrow{\cong} \mathcal{M}(L)$$

are isomorphisms for $c \geq 1$.

To prove the above results we need to recall and develop some details on crossed modules and exterior products. A crossed module is a homomorphism of Lie algebras $\lambda : M \rightarrow L$ with a Lie algebra action $(l, m) \mapsto lm$ of $L$ on $M$ satisfying (i) $\lambda(lm) = [l, \lambda(m)]$, (ii) $\lambda(m)m' = [m, m']$, for all $m, m' \in M$, $l \in L$. If $M$ is an ideal of $L$, then the inclusion map $M \hookrightarrow L$ is a crossed module. Given a free presentation $L \cong F/R$, one readily verifies

**Lemma 1.1.** The canonical homomorphism $\mu_c : \gamma_c^*(L) \rightarrow L$ is a crossed module in which an element $l$ acts on an element $\bar{f} = f + \gamma_c(R, F)$ in $\gamma_c^*(L)$ by $l \bar{f} = [l, \bar{f}]$, where $l$ is any lift of $l$ in $F/\gamma_c(R, F)$.

Let $\lambda : M \rightarrow K$ and $\mu : L \rightarrow K$ be two crossed modules. There are actions of $M$ on $L$ and of $L$ on $M$ given by $m l = \lambda(m)l$ and $l m = \mu(l)m$. We take $M$ (and $L$) to act on itself by Lie multiplication. The non-abelian exterior product $M \wedge L$ is defined in [4] as the Lie algebra generated by the symbols $m \wedge l$ $(m \in M, l \in L)$ subject to the relations

$$c(m \wedge l) = cm \wedge l = m \wedge cl, \quad mm' \wedge l = m \wedge m' l - m' \wedge m l,$$

$$(m + m') \wedge l = m \wedge l + m' \wedge l, \quad m \wedge l' = l m \wedge l - l m \wedge l',$$

$$m \wedge (l + l') = m \wedge l + m \wedge l', \quad [(m \wedge l), (m' \wedge l')] = -l m \wedge m' l'$$

for all $c \in A$, $m, m' \in M$ and $l, l' \in L$.

Any Lie algebra $L$ acts on itself by Lie multiplication and so we can always form the exterior product $L \wedge L$. In [5], it is shown that the commutator map $\kappa_L : L \wedge L \rightarrow L$ defined on generators by
\( l_1 \wedge l_2 \longmapsto [l_1, l_2] \), together with the action of \( L \) on \( L \wedge L \) given by \( \lambda (l_1 \wedge l_2) = [\lambda l_1, l_2] + l_1 \wedge [\lambda l_2, l_1] \), is a crossed module. One thus gets the triple exterior product \((L \wedge L) \wedge L\), and applying this process gives \( \bigwedge^{c+1} L = \cdots \cdot \cdot \cdot (L \wedge L) \wedge \cdots \cdot \cdot \cdot L \), \( c \geq 1 \), involving \((c + 1)\) copies of \( L \). Note that the image of \( \kappa \) is equal to the derived subalgebra of \( L, L^2 \), and its kernel is a central subalgebra of \( L \wedge L \), which is isomorphic to \( \mathcal{M}(L) \).

The following results are useful in our investigation.

**Lemma 1.2.** Let \( L \) be a Lie algebra and \( c \geq 1 \). Then:

(i) There is an epimorphism

\[
\kappa : \gamma_c^s(L) \wedge L \longrightarrow \gamma_{c+1}^s(L), \quad x \wedge y \longmapsto [\tilde{x}, \tilde{y}]
\]

where \( \tilde{x} \) and \( \tilde{y} \) are lifts in \( F/\gamma_{c+1}(R, F) \) of \( x \in \gamma_c^s(L) \) and \( y \in L \).

(ii) The Lie algebra \( \gamma_{c+1}^s(L) \) is a homomorphic image of \( \bigwedge^{c+1} L \).

**Proof.** The part (i) is clear and the part (ii) is a straightforward consequence of [7, Proposition 1.4(i)]. \( \square \)

**Proposition 1.3.** (See [4]) For any Lie algebra \( L \), the exterior product \( L \wedge L \) is isomorphic to \( \gamma_2^s(L) \).

**2. Proof of theorems**

To prove Theorem A, we first present some different forms for the ideal \( Z_c^s(L) \) of a Lie algebra \( L \).

**Proposition 2.1.** The ideal \( Z_c^s(L) \) of a Lie algebra \( L \) is the intersection of all subalgebras of the form \( \theta(Z_c(K)) \), where \( \theta : K \longrightarrow L \) is an epimorphism with \( \ker \theta \subseteq Z_c(K) \).

**Proof.** Set \( A = \bigcap \{ \theta(Z_c(L)) \mid \theta : K \longrightarrow L \} \) is an epimorphism with \( \ker \theta \subseteq Z_c(K) \). By the definition of \( Z_c^s(L) \), there exists a Lie algebra \( K \) together with an epimorphism \( \theta : K \longrightarrow L/Z_c^s(L) \) such that \( \ker \theta = Z_c(K) \). Suppose \( H = \{ (l, k) \in L \oplus K \mid \theta(k) = l + Z_c^s(L) \} \) and \( \phi : H \longrightarrow L \) denotes the projective map. It is readily verified that \( \phi \) is an epimorphism with \( \ker \phi \subseteq Z_c(H) \) and \( \phi(Z_c(H)) \subseteq Z_c^s(L) \). It therefore follows that \( A \subseteq Z_c^s(L) \). To prove the reverse containment, we first show that if \( \{ N_i \mid i \in I \} \) is a family of ideals of \( Z_c^s(L) \) such that each \( L/N_i \) is \( c \)-capable, then so is \( L/\bigcap_{i \in I} N_i \). For each \( i \in I \), let \( 0 \longrightarrow Z_c(K_i) \longrightarrow K_i \longrightarrow L/N_i \longrightarrow 0 \) indicate the assumption that \( L/N_i \) is \( c \)-capable. Put \( N = \bigcap_{i \in I} N_i \) and \( K = \{ (k_i) \in \prod_{i \in I} K_i \mid \exists \eta \in L \) such that \( \theta_i(k_i) = l + N_i \forall i \in I \} \), where \( \prod_{i \in I} K_i \) denotes the Cartesian product of the Lie algebras \( K_i \). One may see that \( Z_c(K) = \prod_{i \in I} Z_c(K_i) \). For each \( i \in I \), we can choose elements \( k_{i, i} \in K_i \) such that \( \delta_i(k_{i, i}) = l + N \). Consequently, \( k_i = (k_{i, i}) \in K \) and the map \( L/N \longrightarrow K/Z_c(K) \) given by \( l + n \longmapsto k_i + Z_c(K) \) is an isomorphism. The conclusion is that \( L/N \) is \( c \)-capable.

Now, let \( \eta : B \longrightarrow L \) be an epimorphism with \( \ker \eta \subseteq Z_c(B) \). Using the isomorphism \( L/\eta(Z_c(B)) \cong B/Z_c(B) \) and the assertion above, we conclude that \( L/A \) is \( c \)-capable and thus \( Z_c^s(L) = A \), as required. \( \square \)

Using the above proposition, we obtain another representation of \( Z_c^s(L) \) by free presentations as follows:

**Corollary 2.2.** Let \( 0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0 \) be a free presentation of a Lie algebra \( L \). Then \( Z_c^s(L) = \pi(Z_c(F/\gamma_{c+1}(R, F))) \), where \( \pi \) is the natural epimorphism induced by \( \pi \).
**Proof.** Let \( \phi : K \to L \) be an epimorphism with \( \ker \phi \subseteq Z_c(K) \). As \( F \) is free, there exists a homomorphism \( \beta' : F \to K \) such that \( \phi \beta' = \pi \). It is easily checked that \( \beta'(R) \subseteq \ker \phi \) and \( \beta'(\gamma_{c+1}(R, F)) = 0 \). Hence \( \beta' \) induces a homomorphism \( \beta : F/\gamma_{c+1}(R, F) \to K \) such that the following diagram is commutative:

\[
\begin{array}{c}
0 \to R/\gamma_{c+1}(R, F) \to F/\gamma_{c+1}(R, F) \xrightarrow{\pi} L \to 0 \\
\beta_1 \downarrow \quad \beta \downarrow \quad 1_1 \downarrow \\
0 \to \ker \phi \to K \to L \to 0,
\end{array}
\]

where \( \beta_1 \) is the restriction of \( \beta \) to \( R/\gamma_{c+1}(R, F) \). Obviously, \( K = \ker \phi + \Im \beta \) and hence \( \beta((Z_c(F/\gamma_{c+1}(R, F)))) \subseteq Z_c(K) \). One deduces that \( \pi((Z_c(F/\gamma_{c+1}(R, F)))) \subseteq \phi(Z_c(K)) \). Now, the result follows from Proposition 2.1 \( \Box \)

In the following, we show that \( Z^*_c(L) \) is the largest ideal of \( L \) such that the Lie algebras \( \gamma^*_c(L) \) and \( \gamma^*_c(L/Z^*_c(L)) \) are isomorphic.

**Proposition 2.3.** Let \( L \) be a Lie algebra with an ideal \( N \). Then \( N \subseteq Z^*_c(L) \) if and only if the quotient homomorphism \( L \to L/N \) induces an isomorphism \( \gamma^*_c(L) \cong \gamma^*_c(L/N) \).

**Proof.** Let \( \gamma^*_c(L) \) and \( \gamma^*_c(L/N) \) be defined in terms of free presentations \( L \cong F/R \) and \( L/N \cong F/S \) in which \( S \) is the preimage of \( N \) in \( F \). The kernel of the natural map \( \gamma^*_c(L) \to \gamma^*_c(L/N) \) is then \( \gamma^*_c(S/F)/\gamma^*_c(R,F) \). Therefore, it suffices to verify that \( \gamma^*_c(S,F) = \gamma^*_c(R,F) \) if and only if \( N \subseteq Z^*_c(L) \). Set \( \bar{F} = F/\gamma^*_c(R,F) \), \( \bar{R} = R/\gamma^*_c(R,F) \) and \( \bar{S} = S/\gamma^*_c(R,F) \). Then \( \gamma^*_c(S,F) = \gamma^*_c(R,F) \) is equivalent to \( \bar{S} \subseteq \bar{Z}_c(\bar{F}) \). Invoking Corollary 2.2, \( \bar{\pi}(\bar{Z}_c(\bar{F})) = Z^*_c(L) \). Consequently, it may be inferred that \( \bar{\pi}(\bar{S}) \subseteq Z^*_c(L) \) if and only if \( \bar{S} \subseteq \bar{Z}_c(\bar{F}) \). Taking into account that \( \bar{\pi}(\bar{S}) = N \), the result follows. \( \Box \)

As an immediate consequence of Proposition 2.3 and [7, Corollary 2.2], we have

**Corollary 2.4.** Let \( N \) be an ideal of a finite dimensional Lie algebra \( L \) which lies in \( Z_c(L) \). Then \( N \subseteq Z^*_c(L) \) if and only if \( \dim(M^O(L/N)) = \dim(M^O(L)) + \dim(N \cap \gamma^*_c(L)) \).

Now, we are ready to prove Theorem A.

**Proof of Theorem A.** (i) By Lemma 1.2(ii) and Proposition 2.3, we obtain an epimorphism \( \wedge^{c+2}(L/Z^*_c(L)) \to \gamma^*_c(L/Z^*_c(L)) \cong \gamma^*_c(L) \). It is shown in [5] that the exterior product \( M \wedge N \) of two crossed modules is finite dimensional if both \( M \) and \( N \) are finite dimensional. Consequently, if \( L/Z^*_c(L) \) is finite dimensional then so is \( \wedge^{c+2}(L/Z^*_c(L)) \). The result follows.

(ii) It follows from [9, Theorem 2.2] and an argument similar to that used in the proof of part (i). \( \Box \)

In readiness for the proof of Theorem B, we recall from [8] the concept of the universal central extension of a Lie algebra.

Let \( e_i : 0 \to M_i \to K_i \xrightarrow{\theta_i} L \to 0 \), \( i = 1, 2 \), be central extensions of a Lie algebra \( L \). Then we say that the extension \( e_1 \) covers (uniquely covers) the extension \( e_2 \) if there exists a homomorphism (or a unique homomorphism) \( \phi_1 : K_1 \to K_2 \) such that \( \theta_2 \phi_1 = \theta_1 \). Now, the central extension \( e_1 \) is called universal if it covers uniquely any central extension of \( L \).

We have the following results regarding the universal central extension of perfect Lie algebras.
Lemma 2.5. (See [8].) Using the above notations, the following statements hold:

(i) If the central extensions \( e_1 \) and \( e_2 \) are universal, then there is an isomorphism \( K_1 \rightarrow K_2 \) that carries \( M_1 \) onto \( M_2 \).
(ii) If \( e_1 \) is universal, then \( K_1 \) and \( L \) are both perfect.
(iii) If \( K_1 \) is perfect, then \( e_1 \) covers \( e_2 \) if and only if \( e_1 \) uniquely covers \( e_2 \).

Proposition 2.6. (See [8].) Let \( L \) be a perfect Lie algebra. Then

\[
0 \rightarrow \mathcal{M}(L) \rightarrow \gamma_2^*(L) \xrightarrow{\mu_2} L \rightarrow 0
\]

is the universal central extension of \( L \).

From the above conclusions and Proposition 1.3, we deduce that

Corollary 2.7. For any perfect Lie algebra \( L \), \( 0 \rightarrow \mathcal{M}(L) \rightarrow L \wedge L \xrightarrow{K_2} L \rightarrow 0 \) is the universal central extension of \( L \).

Now we are able to prove Theorem B.

Proof of Theorem B. By virtue of Lemma 2.5(i), it is enough to show that the exact sequence

\[
0 \rightarrow \mathcal{M}^{(i)}(L) \rightarrow \gamma_{c+1}^*(L) \xrightarrow{\mu_{c+1}} L \rightarrow 0
\]

is the universal central extension of \( L \). Put \( \delta = \mu_{c+1}^{\kappa_{c+1}} \) and \( B = \ker \delta \). We claim that the exact sequence \( 0 \rightarrow B \rightarrow \gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L) \xrightarrow{\delta} L \rightarrow 0 \) is the universal central extension of \( L \). For any \( b \in B \), \( \kappa_{c+1}(b) \in Z(\gamma_{c+1}^*(L)) \) and then the image of the inner derivation map \( ad_{\gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L)}(b) \) is central, whence the map is a homomorphism. As the central extension \( e \) is universal, Lemma 2.5(ii) indicates that the Lie algebra \( \gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L) \) is perfect. Hence the map \( ad_{\gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L)}(b) \) must be zero, implying \( b \in Z(\gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L)) \). We therefore conclude that \( B \) is contained in the centre of \( \gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L) \). Now, assume that \( 0 \rightarrow C \rightarrow P \xrightarrow{\alpha} L \rightarrow 0 \) is an arbitrary central extension of \( L \). Evidently, \( T = \{(x, p) \in \gamma_{c+1}^*(L) \oplus P \mid \mu_{c+1}(x) = \sigma(p)\} \) is a subalgebra of \( \gamma_{c+1}^*(L) \oplus P \) and \( 0 \rightarrow 0 \oplus C \rightarrow T \xrightarrow{\lambda} \gamma_{c+1}^*(L) \rightarrow 0 \) a central extension of \( \gamma_{c+1}^*(L) \), in which \( \lambda \) denotes the natural projection. Thanks to the universality of the extension \( e \), we can find a homomorphism \( \alpha : \gamma_{c+1}^*(L) \rightarrow T \) with \( \kappa_{c+1} = \lambda \alpha \). If \( \gamma : T \rightarrow P \) is the natural projection, then \( \sigma(\gamma \alpha) = \delta \) and this proves our claim.

Therefore, by the induction hypothesis and Lemma 2.5(i), there exists an isomorphism \( \varphi : \gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L) \rightarrow \gamma_{c+1}^*(L) \) such that \( \mu_{c+1} \varphi = \delta \). It is readily verified that the following diagram is commutative:

\[
\begin{array}{ccc}
\gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L) & \xrightarrow{1_{\gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L)}} & \gamma_{c+1}^*(L) \wedge L \\
\varphi & & \kappa \\
& & \\
\gamma_{c+1}^*(L) & & \gamma_{c+2}^*(L)
\end{array}
\]
where $\beta$ is the canonical homomorphism and $\kappa$ is homomorphism due to Lemma 1.2. As $L$ is perfect, the crossed module $\mu$ is surjective. Consequently both $1_{\gamma_{c+1}^*}(L) \wedge \mu_{c+1}$ and $\kappa$ are isomorphisms, implying that $\gamma_{c+2}^*(L)$ is isomorphic to $\gamma_{c+1}^*(L)$. This completes the induction and the proof of the theorem. $\square$

In [1], it is proved that the multiplier of a cover of a finite dimensional perfect Lie algebra is zero (also see [8]). Now, Theorem B shows this result for $c$-nilpotent multipliers of covers of finite dimensional perfect Lie algebras.

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**References**