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# Controlling multiparticle system on the line. II. Periodic case ☆

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ABSTRACT

As in [A. Sarychev, Controlling multiparticle system on the line. I, J. Differential Equations 246 (12) (2009) 4772–4790] we consider classical system of interacting particles $\mathcal{P}_1, \ldots, \mathcal{P}_n$ on the line with only neighboring particles involved in interaction. On the contrast to [A. Sarychev, Controlling multiparticle system on the line. I, J. Differential Equations 246 (12) (2009) 4772–4790] now *periodic boundary conditions* are imposed onto the system, i.e. $\mathcal{P}_1$ and $\mathcal{P}_n$ are considered neighboring. Periodic Toda lattice would be a typical example. We study possibility to control periodic multiparticle systems by means of forces applied to just few of its particles; mainly we study system controlled by single force. The free dynamics of multiparticle systems in periodic and nonperiodic case differ substantially. We see that also the controlled periodic multiparticle system does not mimic its nonperiodic counterpart.

Main result established is global controllability by means of single controlling force of the multiparticle system with a generic potential of interaction. We study the nongeneric potentials for which controllability and accessibility properties may lack. Results are formulated and proven in Sections 2, 3.



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## 1. Introduction

Consider classical system of $n$ interacting particles $\mathcal{P}_1, \dots, \mathcal{P}_n$ moving on the line with only neighboring particles being involved in the interaction. Let $q_k$ be the coordinate of the $k$th particle and $p_k$—its momentum. We assume the potential of this interaction to be

$$\Phi(q_1 - q_2) + \Phi(q_2 - q_3) + \dots + \Phi(q_{n-1} - q_n) + \Phi(q_n - q_1), \tag{1}$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is real analytic, bounded below function

$$\lim_{y \to +\infty} \Phi(y) = +\infty. \tag{2}$$

The difference with nonperiodic case, studied in [7], is due to the presence of the last addend in (1) which accounts for neighboring of $\mathcal{P}_1$ and $\mathcal{P}_n$. This extra addend leads to a substantial change of dynamics. For example in famous and extensively studied case of Toda lattice, in which interaction potential equal to $\Phi(x) = e^{2x}$, the distances between particles are known [5] to tend to infinity in nonperiodic case, while in periodic case the particles are involved in quasiperiodic motion on compact isoenergetic surfaces. Below we will see that the controlled dynamics in the periodic case also differs from nonperiodic controlled dynamics studied in Part I.

The dynamics of multiparticle system with the potential (1) is described by Hamiltonian system of equations with the Hamiltonian

$$H(q, p) = \frac{1}{2} \sum_{k=1}^{n} p_k^2 + \sum_{j=1}^{n} \Phi(q_j - q_{j+1}). \tag{3}$$

Corresponding equations are

$$\dot{q}_k = \frac{\partial H}{\partial p_k} = p_k, \quad k = 1, \dots, n, \tag{4}$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} = \phi(q_{k-1} - q_k) - \phi(q_k - q_{k+1}), \quad k = 2, \dots, n, \tag{5}$$

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = \phi(q_n - q_1) - \phi(q_1 - q_2). \tag{6}$$

In (5), (6) and further on $\phi = \Phi'$ is the derivative of $\Phi$. Besides for unification of notation we assume in (3), (5) and (6)

$$q_0 = q_n, \qquad q_{n+1} = q_1.$$

The control will be realized by a force, which we choose to act on the particle $\mathcal{P}_1$. In the presence of the control Eq. (6) becomes

$$\dot{p}_1 = \phi(q_n - q_1) - \phi(q_1 - q_2) + u(t), \tag{7}$$

where $u(\cdot)$ stays for the controlling force. Eqs. (4), (5) remain unchanged. We call the model *single-forced periodic multiparticle system*.

We wish to study controllability properties of the controlled multiparticle periodic system (4), (5), (7).

**Definition 1.** System (4), (5), (7) is globally controllable if for each given pair of points $\tilde{x} = (\tilde{q}, \tilde{p})$, $\hat{x} = (\hat{q}, \hat{p})$ of its state space there exists an admissible (measurable essentially bounded) control $u(\cdot)$ which steers the system from $\tilde{x}$ to $\hat{x}$ in time $\theta > 0$.

The controlled multiparticle system (4), (5), (7) is a particular case of control-affine system of the form

$$\frac{dx}{dt} = f(x) + g(x)u, \tag{8}$$

where the *controlled vector field* $g$ and the uncontrolled vector field $f$—the *drift*—are defined as

$$g = \partial/\partial p_1, \qquad f = \sum_{k=1}^{n} p_k \frac{\partial}{\partial q_k} + \sum_{k=1}^{n} \bigl(\phi(q_{k-1} - q_k) - \phi(q_k - q_{k+1})\bigr) \frac{\partial}{\partial p_k}. \tag{9}$$

In Part I we observed that in nonperiodic case global controllability is in general non-achievable by means of one controlling force and is achievable by means of two controlling forces applied to the "extreme" particles $\mathcal{P}_1$ and $\mathcal{P}_n$. It is immediate to conclude (see Section 2.1) that the periodic multiparticle system is also globally controllable by means of two forces.

We are going to prove instead that *in the periodic case for a generic potential $\Phi$ global controllability is achievable by means of single controlling force* (Theorem 10 in Section 3). The proof is split into two parts. First we establish full dimensionality of the *orbit* of a single-forced multiparticle system.

An orbit $\mathcal{O}_{\tilde{x}}$ of control system is the minimal invariant manifold for the control system, whenever one starts from the initial point $\tilde{x}$ and proceeds with controlled motion in direct (positive) and reverse (negative) time. We will prove that for a *generic potential $\Phi$* the orbits of the control system (8)–(9) coincide with the state space $\mathbb{R}^{2n}$. This is done (Section 2.1) by verification of *bracket generating property* of the couple of vector fields $\{f, g\}$. This property may fail for some potentials; in Section 2.2 we provide an example of *low-dimensional orbits for a specific potential $\Phi$*. In Section 2.3 we return for a moment to nonperiodic case and provide an example of *low-dimensional orbit for nonperiodic* system whenever controlling force is applied to a particle $\mathcal{P}_j$ with $j \neq 1$, $j \neq n$.

Once full dimensionality of an orbit is established, one has to deal with another difficulty. Positive invariant set of a control system (*attainable set*) is often a proper subset of the respective orbit. The reason for this is actuation of the drift vector field $f$, which may drive the system in certain direction without a possibility to compensate this drift by action of any control.

In some exceptional cases such compensation is possible. One of these cases is represented by Bonnard–Lobry theorem [3], whose main assumption is *recurrence property of dynamics of the noncontrolled motion*.

In the nonperiodic case, treated in [7], we arranged a simple design of feedback controls which modified the noncontrolled dynamics in such a way that all its trajectories became recurrent. Such design was only possible with two controls available.

In the periodic case we get instead a property of *constrained recurrence* for the dynamics of noncontrolled motion: the dynamics is recurrent on a hyperplane of zero momentum $\Pi$: $p_1 + \cdots + p_n = 0$. The hyperplane is invariant with respect to free dynamics, but is not invariant with respect to controlled dynamics. Therefore one cannot remain in $\Pi$, whenever nonzero control is employed, and we cannot use the recurrence property when one is outside $\Pi$. We will adapt the technique of *Lie extensions* for overcoming this difficulty and establishing global controllability.

## 2. Orbits and accessibility property for single-forced multiparticle system

We study single-forced periodic multiparticle system, or, the same, control-affine system (8)–(9) in the *state space* $\mathbb{R}^{2n}$.

We start with computation in the next subsection of the orbits of this control-affine system. Recall that one obtains orbit $\mathcal{O}_{\tilde{x}}$ by taking vector fields $f^{u^j} = f + u^j g$ with $u^j \in \mathbb{R}$ constant, and acting on $\tilde{x} \in \mathbb{R}^{2n}$ by the compositions

$$P = e^{t_1 f^{u^{j_1}}} \circ \cdots \circ e^{t_N f^{u^{j_N}}}, \quad t_1, \ldots, t_N \in \mathbb{R}, \tag{10}$$

where $e^{tX}$ stays for the flow of the vector field $X$.

According to Nagano theorem [2,4] an orbit $\mathcal{O}_{\tilde{x}}$ is an immersed submanifold of $\mathbb{R}^{2n}$ and the tangent space to this manifold at a point $x \in \mathcal{O}_{\tilde{x}}$ is obtained by evaluation at $x$ of the vector fields from the Lie algebra $\mathrm{Lie}\{f, g\}$ generated by $f$ and $g$.

**Definition 2.** A family $\mathcal{F}$ of vector fields is called bracket generating at point $x \in \mathbb{R}^{2n}$ if the evaluation at $x$ of the vector fields from $\mathrm{Lie}\{F\}$ coincides with $\mathbb{R}^{2n}$.

An *attainable set* $\mathcal{A}_{\tilde{x}}$ of the system (8) from $\tilde{x}$ is the set of points to which the system can be steered from $\tilde{x}$ by means of an admissible (measurable, bounded) control. If we require in addition the transfer time to be equal, or respectively, $\leqslant$ than $T$, then we obtain time-$T$ (respectively time-$\leqslant T$) attainable set $\mathcal{A}^T_{\tilde{x}}$ (respectively $\mathcal{A}^{\leqslant T}_{\tilde{x}}$). Obviously $\mathcal{A}^T_{\tilde{x}} \subset \mathcal{A}^{\leqslant T}_{\tilde{x}} \subset \mathcal{A}_{\tilde{x}}$; also $\mathcal{A}_{\tilde{x}}$ is contained in the orbit $\mathcal{O}_{\tilde{x}}$.

If one employs piecewise-constant controls, i.e. takes only positive times $t_j > 0$ in the compositions (10), then one gets *positive orbit* $\mathcal{O}^+_{\tilde{x}}$ of the system. In general it is proper subset of $\mathcal{O}_{\tilde{x}}$ and is far from being a manifold. Obviously $\mathcal{O}^+_{\tilde{x}} \subset \mathcal{A}_{\tilde{x}}$.

**Remark 3.** It is known from A.J. Krener theorem [2,4], that for each point $\tilde{x}$ positive orbit $\mathcal{O}^+_{\tilde{x}}$ possesses nonvoid relative interior in the orbit $\mathcal{O}_{\tilde{x}}$, and moreover $\mathcal{O}^+_{\tilde{x}}$ is contained in the closure of its relative interior. A consequence of this theorem is the useful fact (see [2]) that *density* of $\mathcal{O}^+_{\tilde{x}}$ in the orbit $\mathcal{O}_{\tilde{x}}$ implies the coincidence of $\mathcal{O}^+_{\tilde{x}}$ and $\mathcal{A}_{\tilde{x}}$ with $\mathcal{O}_{\tilde{x}}$.

### *2.1. Orbits of single-forced periodic multiparticle system*

In [7] we proved for single-forced nonperiodic multiparticle system that all the orbits coincide with $\mathbb{R}^{2n}$. In the periodic case this holds for generic potentials. We prove this fact in the present subsection and provide counterexamples in Sections 2.2, 2.3.

**Theorem 4.** *For generic potentials $\Phi$ (namely, for nondegenerate potentials of Definition 7) the system of vector fields $\{f, g\}$ is bracket generating at each point of the state space $\mathbb{R}^{2n}$; therefore $\forall \tilde{x} \in \mathbb{R}^{2n}$ the orbit $\mathcal{O}_{\tilde{x}}$ of single-forced multiparticle periodic system (8)–(9) through $\tilde{x}$ coincides with $\mathbb{R}^{2n}$.*

The proof is structured in two lemmas, first of which mimics similar result for double-forced *nonperiodic* multiparticle system.

Assume for the moment that periodic multiparticle system is controlled by *two* forces applied to the particles $\mathcal{P}_1$ and $\mathcal{P}_n$, i.e. we gain an additional controlled vector field $g^n = \frac{\partial}{\partial p_n}$. The additional controlling force appears in Eq. (5) indexed by $k = n$, which now will take form

$$\dot{p}_n = \phi(q_{n-1} - q_n) - \phi(q_n - q_1) + v(t). \tag{11}$$

**Lemma 5.** *The family of vector fields $\{f, g, g^n\}$ is bracket generating at each point of $\mathbb{R}^{2n}$.*

**Proof.** The conclusion of the lemma follows from the fact that feedback transformation

$$u \mapsto -\phi(q_n - q_1) + \tilde{u}, \qquad v \mapsto \phi(q_n - q_1) + \tilde{v}$$

transforms Eqs. (7), (11) into respective equations of double-forced *nonperiodic* multiparticle system, whose orbits coincide with $\mathbb{R}^{2n}$ by results of [7]. This transformation does not affect bracket generating property, hence the vector fields $\{f, g, g^n\}$ form a bracket generating system. $\square$

Recall the notation: for a vector field $X$ operator $\mathrm{ad}\, X$ acts on another vector field $Y$ as $\mathrm{ad}\, XY = [X, Y]$. The proof of Theorem 4 would be accomplished by the following lemma.

**Lemma 6.** *For a generic potential $\Phi$, for each point $\tilde{x} \in \mathbb{R}^{2n}$:*

$$\mathrm{Span}\{f(x), g(x), \mathrm{ad}^2 fg(x), [\mathrm{ad}\, fg, \mathrm{ad}^2 fg](x)\} \supset \mathrm{Span}\left\{f(x), \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_n}\right\}, \tag{12}$$

*for all $x$ of an open dense subset of the orbit $\mathcal{O}_{\tilde{x}}$ of the system* (8)–(9).

For those potentials $\Phi$, for which the conclusion of Lemma 6 is valid, one easily gets the statement of Theorem 4 proven. Indeed since the system $\{f, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_n}\}$ is bracket generating at each point, then by Lemma 6 the system of vector fields $\{f, g\}$ is bracket generating at some point of each orbit $\mathcal{O}_{\tilde{x}}$. The dimension $\dim \mathrm{Lie}_x\{f, g\}$ of the evaluation at $x$ of the Lie algebra $\mathrm{Lie}\{f, g\}$ is known to be constant along $\mathcal{O}_{\tilde{x}}$ (see [2,4]). Hence we conclude that $\{f, g\}$ is bracket generating at each point of $\mathbb{R}^{2n}$ and all orbits of the system (8)–(9) coincide with the state space $\mathbb{R}^{2n}$.

**Proof of Lemma 6.** By direct computation $[f, g] = \mathrm{ad}\, fg = -\frac{\partial}{\partial q_1}$. Computing the iterated Lie brackets $\mathrm{ad}^2 fg, [\mathrm{ad}^2 fg, \mathrm{ad}\, fg]$ we get

$$\mathrm{ad}^2 fg = \phi'(q_1 - q_2)\left(\frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1}\right) + \phi'(q_n - q_1)\left(\frac{\partial}{\partial p_n} - \frac{\partial}{\partial p_1}\right), \tag{13}$$

$$[\mathrm{ad}^2 fg, \mathrm{ad}\, fg] = \phi''(q_1 - q_2)\left(\frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1}\right) - \phi''(q_n - q_1)\left(\frac{\partial}{\partial p_n} - \frac{\partial}{\partial p_1}\right). \tag{14}$$

We would arrive to the needed conclusion at each point $x \in \mathcal{O}_{\tilde{x}}$ where the determinant

$$\begin{aligned}
&\det\begin{pmatrix} \phi'(q_1 - q_2) & \phi'(q_n - q_1) \\ \phi''(q_1 - q_2) & -\phi''(q_n - q_1) \end{pmatrix} \\
&\quad = -\phi'(q_1 - q_2)\phi''(q_n - q_1) - \phi'(q_n - q_1)\phi''(q_1 - q_2) \\
&\quad = \left(\phi'(q_1 - q_2)\right)^2 \frac{\partial}{\partial q_1}\frac{\phi'(q_n - q_1)}{\phi'(q_1 - q_2)}
\end{aligned} \tag{15}$$

is nonvanishing.

As far as the vector field $[g, f] = \frac{\partial}{\partial q_1}$ is tangent to any orbit $\mathcal{O}_{\tilde{x}}$ of (8) then we get the conclusion of the lemma whenever the determinant (15) (an analytic function) *does not vanish identically with respect to* $q_1$. It will vanish identically only if the relation

$$\phi'(q - q_2) = c\phi'(q_n - q) \tag{16}$$

would hold identically with respect to $q$ (by which we substituted $q_1$) with $c$ constant. Seeing now $q_2, q_n, c$ as parameters, we treat (16) as a functional equation.

Substituting $q = \frac{q_n + q_2}{2} - t$ into (16) we obtain the relation

$$\forall t\colon\quad \phi'\left(\frac{q_n - q_2}{2} - t\right) = c\phi'\left(\frac{q_n - q_2}{2} + t\right),$$

wherefrom $c = \pm 1$. Then

$$c = \pm 1, \qquad \phi'(t) = f(t - b), \quad f \text{ is even or odd, according to the sign of } c, \tag{17}$$

and $b = \frac{q_n - q_2}{2}$.

(See [6] for an alternative description of solution of (16).)

**Definition 7.** We call potential $\Phi$ nondegenerate, if its second derivative $\Phi''(t)$ is *not* of the form $f(t-b)$ with $f$ being either even or odd function.

**Remark 8.** Evidently generic potential is nondegenerate.

For *generic* potentials $\Phi$ (e.g. for nondegenerate) the inclusion (12) holds on an open dense subset of any orbit and then these orbits coincide with the state space $\mathbb{R}^{2n}$. $\square$

It is interesting to know whether there exist potentials $\Phi$, for which the system (8)–(9) possesses low-dimensional orbits. In the next two subsections we provide such examples.

*2.2. Low-dimensional orbits of single-forced periodic multiparticle system*

Consider a trimer—periodic three-particle system with the Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \Phi(q_1 - q_2) + \Phi(q_2 - q_3) + \Phi(q_3 - q_1)$$

and the *controlled* dynamics

$$\begin{gathered} \dot q_i = p_i, \quad i = 1, 2, 3, \\ \dot p_1 = \phi(q_3 - q_1) - \phi(q_1 - q_2) + u, \\ \dot p_2 = \phi(q_1 - q_2) - \phi(q_2 - q_3), \qquad \dot p_3 = \phi(q_2 - q_3) - \phi(q_3 - q_1), \end{gathered} \tag{18}$$

where $\phi(q) = \Phi'(q)$. We assume the derivative $\phi'(q)$ to be of the form (17) with $f$ *even*.[1] Then $\phi(t) = F(t-b) + c$, where $F(t)$ is a primitive of even function $f(t)$, and therefore can be chosen an *odd function*. In this case $\phi(t) = F(t-b) + \phi(b)$.

From the differential equations for $q_2, p_2, q_3, p_3$ in (18) we derive

$$\begin{aligned} \frac{d}{dt}(q_3 - q_2) &= (p_3 - p_2), \\ \frac{d}{dt}(p_3 - p_2) &= 2\phi(q_2 - q_3) - \phi(q_1 - q_2) - \phi(q_3 - q_1) \\ &= 2F(q_2 - q_3 - b) - F(q_1 - q_2 - b) - F(q_3 - q_1 - b). \end{aligned}$$

Assuming in addition $F(-3b) = 0$, or equivalently $\phi(b) = \phi(-2b)$, we check immediately that the 4-*dimensional plane*

$$\Pi_b\colon \quad p_3 - p_2 = 0, \quad q_3 - q_2 = 2b,$$

is an invariant manifold for the control system (18). Indeed, along $\Pi_b$

$$\begin{gathered} F(q_2 - q_3 - b) = F(-3b) = 0, \\ F(q_1 - q_2 - b) + F(q_3 - q_1 - b) = F(q_1 - q_2 - b) + F(q_2 - q_1 + b) = 0. \end{gathered}$$

Hence $\forall \tilde x \in \Pi_b$ the orbit $\mathcal{O}_{\tilde x}$ of the control system (18) is contained in $\Pi_b$.

[1] One can prove that whenever $c = -1$ in (16) and respectively $f$ is odd in (17) the orbits coincide with $\mathbb{R}^{2n}$.

### 2.3. Low-dimensional orbits of nonperiodic multiparticle system

In Part I we mentioned that nonperiodic single-forced multiparticle system may possess low-dimensional orbits for some choices of $\phi$, whenever the control force is applied to a particle different from $\mathcal{P}_1, \mathcal{P}_n$. Here we provide such example obtained by a variation on the example of the previous subsection.

For a *nonperiodic trimer* with the controlling force acting on the particle $\mathcal{P}_2$ the dynamic equations are

$$\dot{q}_i = p_i, \quad i = 1, 2, 3,$$
$$\dot{p}_1 = -\phi(q_1 - q_2), \qquad \dot{p}_2 = \phi(q_1 - q_2) - \phi(q_2 - q_3) + u, \qquad \dot{p}_3 = \phi(q_2 - q_3). \tag{19}$$

Then

$$\frac{d}{dt}(q_3 - q_1) = p_3 - p_1, \qquad \frac{d}{dt}(p_3 - p_1) = \phi(q_2 - q_3) + \phi(q_1 - q_2). \tag{20}$$

Let us choose the function $\phi(t) = f(t-b)$, $f$—odd function. We claim that the 4-dimensional plane $\Pi'_b$: $q_3 - q_1 = -2b$, $p_3 - p_1 = 0$, is invariant for the control system (19). Indeed restricting the second one of Eqs. (20) to $\Pi'_b$ we conclude

$$\frac{d}{dt}(p_3 - p_1) = \phi(q_2 - q_3) + \phi(q_3 + 2b - q_2) = f(q_2 - q_3 - b) + f(q_3 - q_2 + b) = 0,$$

independently of a choice of control $u(\cdot)$. The first one of Eqs. (20) restricted to $\Pi'_b$ implies: $\frac{d}{dt}(q_3 - q_1) = (p_3 - p_1) = 0$. Thus the 4-dimensional plane $\Pi'_b$ contains the orbits $\mathcal{O}_x$ of the control system (19) for each $x \in \Pi'_b$.

## 3. Controllability of periodic multiparticle system by means of a single force

In [7] we designed special feedback controls which imposed *recurrent behavior* on dynamics of *nonperiodic double-forced* multiparticle system. This allowed us to apply Bonnard–Lobry theorem [3] for proving global controllability. The same procedure can be repeated for *double-forced periodic case*.

**Proposition 9.** *Periodic multiparticle system is globally controllable by means of controlling forces applied to the particles* $\mathcal{P}_1, \mathcal{P}_n$.

We are aiming though at a stronger result.

**Theorem 10.** *Periodic multiparticle system with nondegenerate* (*see Definition* 7) *interaction potential* $\Phi$ *is globally controllable by means of a single force.*

**Remark 11.** There are no a priori constraints imposed on the magnitude of the controlling force in the formulation of Theorem 10.

In the rest of this contribution we *prove Theorem* 10.

### 3.1. Lie extensions

The following definition is slight modification of the notion of Lie saturation introduced by V. Jurdjevic [4].

**Definition 12.** Let $\mathcal{F}$ be a family of analytic vector fields, and $\mathrm{Lie}(\mathcal{F})$ be the Lie algebra generated by $\mathcal{F}$. Lie extension $\hat{\mathcal{F}}$ of $\mathcal{F}$ is a family $\hat{\mathcal{F}} \subseteq \mathrm{Lie}(\mathcal{F})$ such that for each $\hat{x}$

$$\operatorname{clos} \mathcal{A}_{\hat{\mathcal{F}}}(\hat{x}) \subseteq \operatorname{clos} \mathcal{A}_{\mathcal{F}}(\hat{x}). \tag{21}$$

Any vector field from a Lie extension is called *compatible* with $\mathcal{F}$.

We specify some types of Lie extensions.

**Proposition 13.** *A closure* $\operatorname{clos}(\mathcal{F})$ *of* $\mathcal{F}$ *in the Whitney* $C^\infty$*-topology is a Lie extension.*

This assertion follows from classical result on continuous dependence of the solutions of ODE on initial data and the right-hand side.

An important kind of extension which underlies theory of relaxed or sliding mode controls is introduced by the following

**Proposition 14.** *For a control system* $\mathcal{F}$ *its conic hull*

$$\operatorname{cone}(\mathcal{F}) = \left\{ \sum_{j=1}^{N} \alpha_j f^j \;\middle|\; \alpha_j \in C^\infty(\mathbb{R}^n),\ f^j \in \mathcal{F},\ N \in \mathbb{N},\ \alpha_j \geqslant 0,\ j = 1, \ldots, N \right\}$$

*is a Lie extension.*

To introduce another type of Lie extension we define *normalizer* of $\mathcal{F}$.

**Definition 15.** (See [4].) Diffeomorphism $P$ is a normalizer for the family $\mathcal{F}$ of vector fields if $\forall \hat{x}$:

$$P\big(\mathcal{A}_{\mathcal{F}}\big(P^{-1}(\hat{x})\big)\big) \subseteq \operatorname{clos} \mathcal{A}_{\mathcal{F}}(\hat{x}).$$

The following sufficient criterion is useful for finding normalizers.

**Proposition 16.** *(See [4].) Diffeomorphism* $P$ *is a normalizer for the family* $\mathcal{F}$ *if both* $P(\hat{x})$ *and* $P^{-1}(\hat{x})$ *belong to* $\operatorname{clos}(\mathcal{A}_{\mathcal{F}}(\hat{x}))$, $\forall \hat{x}$.

Recall that *adjoint action of diffeomorphism* $P$ *on a vector field* $f$ results in another vector field defined as

$$\operatorname{Ad} P f(x) = P_*^{-1}\big|_{P(x)} f\big(P(x)\big).$$

**Proposition 17.** *The set*

$$\tilde{\mathcal{F}} = \{\operatorname{Ad} P f \mid f \in \mathcal{F},\ P\text{—normalizer of } \mathcal{F}\}$$

*is a Lie extension of* $\mathcal{F}$.

For control-affine of the form (8) the family of vector fields, which determines polidynamics of such system, is $\mathcal{F} = \{f + gu \mid u \in \mathbb{R}\}$.

According to Propositions 13, 14 the vector fields

$$\pm g = \lim_{\theta \to 0} \theta^{-1}\big(f + g(\pm\theta)\big) \tag{22}$$

are contained in the closure of the conic hull of $\mathcal{F}$ and therefore are compatible with $\mathcal{F}$.

By Proposition 16 each diffeomorphism $e^{ug}$ is a normalizer of $\mathcal{F}$ and hence there holds the following

**Lemma 18.** *The vector fields* $\{\mathrm{Ad}\, e^{\pm ug} f \mid u \in \mathbb{R}\}$ *are compatible with the control system* (8).

*3.2. Lie extension for single-forced periodic multiparticle system*

We will employ Lie extensions for proving Theorem 10.

Direct computation of $e^{u\,\mathrm{ad}\, g} f$ for the vector fields (9) results in vector fields

$$b_u = e^{u\,\mathrm{ad}\, g} f = f + u[g, f];$$

it suffices to note that $\mathrm{ad}^2 gf = [g, [g, f]] = 0$.

Consider vector field $-f$ and join it to the vector fields $b_1, b_{-1}$. The three vector fields are contained in 2-distribution $\mathcal{D}$ spanned by $f$ and $[g, f]$.

Above we introduced the plane of zero momentum $\Pi$: $P = p_1 + \cdots + p_n = 0$, which is invariant for the vector field $f$.

**Lemma 19.** *The hyperplane* $\Pi$ *is invariant for* 2*-distribution* $\mathcal{D}$*, which is bracket generating on* $\Pi$.

**Proof.** By direct computation (see formulae (13), (14)) one checks that distribution $\mathcal{D}$ is tangent to $\Pi$. For a nondegenerate $\Phi$:

$$\mathrm{Span}\{\mathrm{ad}^2 fg, [\mathrm{ad}^2 fg, \mathrm{ad}\, fg]\} = \mathrm{Span}\{Y^2, Y^n\},$$

where $Y^2 = \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1}$, $Y^n = \frac{\partial}{\partial p_n} - \frac{\partial}{\partial p_1}$. Again by direct computation

$$[Y^2, f] = Z^2 = \frac{\partial}{\partial q_2} - \frac{\partial}{\partial q_1}, \qquad [Y^n, f] = Z^n = \frac{\partial}{\partial q_n} - \frac{\partial}{\partial q_1},$$
$$[Z^2, f] = \frac{\partial}{\partial p_3} - \frac{\partial}{\partial p_1}, \qquad [Z^n, f] = \frac{\partial}{\partial p_{n-1}} - \frac{\partial}{\partial p_1} \quad (\mathrm{mod}\ \mathrm{Span}\{Y^2, Y^n\}).$$

We can arrive to the second conclusion of the lemma by induction. □

The conic hull of the triple of vector fields $\{-f, b_1, b_{-1}\}$ coincides with $\mathcal{D}$. Hence by Rashevsky–Chow theorem [2,4] for each $\tilde{x} \in \Pi$ positive orbit $\mathcal{O}^+_{\tilde{x}}$ of this triple is dense in the orbit of $\mathcal{D}$, equal to $\Pi$. By Remark 3 it must coincide with $\Pi$.

Note that

$$e^{tb_{\pm 1}} = e^{t\,\mathrm{Ad}(e^{\pm g}) f} = \mathrm{Ad}(e^{\pm g}) e^{tf} = e^{\pm g} \circ e^{tf} \circ e^{\mp g}.$$

According to the aforesaid each point of $\Pi$ is attainable from another point of $\Pi$ by means of composition of diffeomorphisms from the family

$$\{e^g \circ e^{tf} \circ e^{-g},\ e^{-g} \circ e^{tf} \circ e^g,\ e^{-tf},\ t \geqslant 0\}; \tag{23}$$

$\Pi$ is invariant under the action of diffeomorphisms (23).

We wish to achieve global controllability on $\Pi$ without having recourse to $e^{-tf}$.

**Proposition 20.** *Each point of* $\Pi$ *is attainable from another point of* $\Pi$ *by means of compositions of diffeomorphisms from the family*

$$\{e^g \circ e^{tf} \circ e^{-g},\ e^{-g} \circ e^{tf} \circ e^g,\ e^{tf},\ t \geqslant 0\}. \tag{24}$$

The proof of Proposition 20, postponed to Section 3.4, follows the line of the proof of Bonnard–Lobry theorem (see [2,3]) and is based on the recurrence property of the free motion of the multiparticle system in the plane $\Pi$. Meanwhile taking it conclusion for granted we accomplish the proof of global controllability.

### 3.3. *Proof of global controllability*

By direct computation one checks that for controlled motion the total momentum $P$ varies according to the equation $\dot{P} = u(t)$. Taking two points $(\tilde{q}, \tilde{p}), (\bar{q}, \bar{p})$ in the state space, we can steer, say in time 1, the system (8) from $(\tilde{q}, \tilde{p})$ to some point $(\tilde{q}^0, \tilde{p}^0)$ of $\Pi$ by application of a constant control $\tilde{u}$. Considering the reverse time dynamics $\dot{P} = -u(t)$ one ensures the possibility to steer the system (8) in time $-1$ from the point $(\bar{q}, \bar{p})$ to a point $(\bar{q}^0, \bar{p}^0)$ of the plane $\Pi$ by means of another constant control $\bar{u}$. In direct time the system (8) would shift in time 1 from $(\bar{q}^0, \bar{p}^0)$ to $(\bar{q}, \bar{p})$ under the action of $\bar{u}$.

According to Proposition 20 one can steer the point $(\tilde{q}^0, \tilde{p}^0)$ to the point $(\bar{q}^0, \bar{p}^0)$ by a composition of diffeomorphisms of the form $e^g, e^{-g}, e^{tf}, \ t \geqslant 0$. Then this composition of diffeomorphisms preceded by time-1 action of the control $\tilde{u}$ and succeeded by time-1 action of the control $\bar{u}$ steers the system from $(\tilde{q}, \tilde{p})$ to $(\bar{q}, \bar{p})$ in the state space.

According to the limit relation (22) we can approximate arbitrarily well the diffeomorphisms $e^{\pm g}$ in the composition, we have just described, by diffeomorphisms $e^{\theta^{-1}(f \pm g\theta)}$ with sufficiently large $\theta > 0$; these latter diffeomorphisms are elements of admissible flows $e^{t(f \pm g\theta)}$. Hence one can steer the point $(\tilde{q}, \tilde{p})$ by an admissible control to a point $(\bar{q}', \bar{p}')$ which is arbitrarily close to $(\bar{q}, \bar{p})$. As far as $(\bar{q}, \bar{p})$ is arbitrarily chosen, one concludes that the attainable set of the system (8)–(9) from each point $(\tilde{q}, \tilde{p}) \in \mathbb{R}^{2n}$ is dense in $\mathbb{R}^{2n}$. Given bracket generating property of the pair (9) for a generic potential $\Phi$, we conclude according to Remark 3 that this attainable set coincides with $\mathbb{R}^{2n}$.

### 3.4. *Proof of Proposition 20*

First note that all points of $\Pi$ are *nonwandering* for the vector field $f$, defined by (9). Recall that a point $x \in \mathbb{R}^{2n}$ is nonwandering for $f$ (see [1, §6.2]) if for each neighborhood $U \supset x$ and each $T > 0$ there exists $t > T$ such that $e^{tf}(U) \cap U \neq \emptyset$. We will prove in a moment (Lemma 21).

Basing on this property we conclude that for each point $x \in \Pi$ and any $t > 0$ the points $e^{-tf}(x)$ (contained in $\Pi$) are arbitrarily well approximable by points $e^{\tau f}(x)$ with $\tau > 0$.

Acting by a composition of diffeomorphisms $P_N \circ \cdots \circ P_1$ belonging to the family (23) on a point $\tilde{x} \in \Pi$ we pick the factors $P_i = e^{-t_i f}$ $(t > 0)$. Each diffeomorphism $P_i$ is applied to a point $y_i = (P_{i-1} \circ \cdots \circ P_1)(\tilde{x})$ which belongs to $\Pi$. By nonwandering property in $\Pi$ we can approximate the action of $P_i$ on this point by an action on it of some diffeomorphism $\hat{P}_i = e^{\theta_i f}$, $\theta > 0$.

Thus we proved that positive orbit of the family (24) is dense in $\Pi$ (which is positive orbit of the family (23)) and hence coincides with $\Pi$ given the fact that $\{\mathrm{Ad}(e^g)f, f\}$, restricted to $\Pi$, form a bracket generating pair of vector fields on $\Pi$.

**Lemma 21.** *Each point of the hyperplane $\Pi$ is nonwandering for the vector field $f$.*

We will derive this property from Poincaré theorem [1, §3.4]. Indeed the hyperplane $\Pi$ of zero momentum is invariant for the Hamiltonian vector field $f$; according to [1, §3.4] one can introduce a volume form on $\Pi$, which is preserved by the flow of $f$.

Let us introduce the planes $\Pi_Q = \{\sum_{i=1}^n p_i = 0, \ \sum_{i=1}^n q_i = Q\}$ and consider the Lebesgue sets $\{H^p \leqslant c\}$ of the Hamiltonian (3). We will prove in a moment (Lemma 22) that intersections of the Lebesgue sets with each $\Pi_Q$ are compact.

Taking this for granted we see that for each $a, c > 0$ the sets

$$\bigcup_{|Q| \leqslant a} \Pi_Q \cap \{H^p \leqslant c\}$$

are compact and invariant with respect to the volume-preserving (and Hamiltonian-preserving) flow of the vector field $f$. We are under conditions of Poincaré theorem according to which $\forall Q, c$ points of $\Pi_Q \cap \{H^p \leqslant c\}$ are nonwandering. It rests to note that each point of $\Pi$ is included in some set $\Pi_Q \cap \{H^p \leqslant c\}$.

**Lemma 22.** *Intersections of the Lebesgue sets of the Hamiltonian $H^p$ with the planes $\Pi_Q = \{\sum_{i=1}^n p_i = 0, \sum_{i=1}^n q_i = Q\}$ are compact.*

**Proof.** Closedness of the Lebesgue sets $\{H^p \leqslant c\}$ is obvious; we prove their boundedness.

Since $\sum_{j=1}^{n-1} \Phi(q_j - q_{j+1}) + \Phi(q_n - q_1)$ is bounded below, say by $-B \leqslant 0$, then the inequality $H^p \leqslant c$ implies the constraints:

$$\|p\|^2 \leqslant c + B, \qquad \sum_{j=1}^{n-1} \Phi(q_j - q_{j+1}) + \Phi(q_n - q_1) \leqslant c. \tag{25}$$

By lower boundedness of the function $\Phi$ and by the growth conditions (2) we derive from the second one of the relations (25)

$$q_1 - q_2 \leqslant b \wedge \cdots \wedge q_{n-1} - q_n \leqslant b \wedge q_n - q_1 \leqslant b, \tag{26}$$

for some constant $b$.

Summing the first $k$ inequalities at the right-hand side of the implication (26) we conclude

$$q_1 \leqslant q_k + (k-1)b, \quad k = 1, \dots, n, \tag{27}$$

while summing $n + 1 - k$ inequalities, starting from the last one, we obtain

$$q_k - (n+1-k)b \leqslant q_1, \quad k = 1, \dots, n. \tag{28}$$

If we restrict our consideration onto the plane $\Pi_Q$ and sum separately the inequalities (27) and (28) we get

$$nq_1 \leqslant Q + b(n-1)n/2, \qquad nq_1 \geqslant Q - b(n+1)n/2.$$

Due to invariance with respect to the permutations of particles we conclude

$$n^{-1}Q - b(n+1)/2 \leqslant q_j \leqslant n^{-1}Q + b(n-1)/2,$$

for each coordinate $q_j$ of a point $(q, p) \in \Pi \cap \{H^p \leqslant c\}$. $\square$

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