

# A VAN KAMPEN SPECTRAL SEQUENCE FOR HIGHER HOMOTOPY GROUPS

CHRISTOPHER R. STOVER†

(Received 4 January 1988)

## 1. INTRODUCTION

THE AIM of this paper is to construct

(i) a spectral sequence that converges strongly to the homotopy groups of the one point union  $X \vee Y$  of a pair of spaces  $X$  and  $Y$ , and whose  $E^2$  term depends only on the homotopy groups of  $X$  and  $Y$  and on the actions of the primary homotopy operations [25, XI] on these groups, and

(ii) a generalization of (i) to a “van Kampen spectral sequence” that converges strongly to the homotopy groups of the homotopy direct limit [3, XII] of a diagram of pointed connected spaces.

At the end of the paper we also discuss a related spectral sequence that converges strongly to the homotopy groups of the smash product  $X \wedge Y$  of a pair of spaces  $X$  and  $Y$ .

To give a more precise description of the  $E^2$  terms of the wedge spectral sequence and its generalization, we need to consider a category whose objects are modelled on the graded set of the homotopy groups of a space with action by the primary homotopy operations. Call such objects  $\Pi$ -algebras (4.2). Then we can state

1.1 THEOREM. *There is a first quadrant spectral sequence  $E_{p,q}^r$ , functorial in pointed connected CW complexes  $X$  and  $Y$ , converging strongly to the homotopy groups  $\pi_*(X \vee Y)$  of the one point union of  $X$  and  $Y$ , and such that the columns of the  $E^2$  term have the following interpretation:*

(i)  $E_{0,*}^2 = \pi_* X \amalg \pi_* Y$  is the coproduct of  $\pi_* X$  and  $\pi_* Y$  in the category of  $\Pi$ -algebras, and

(ii)  $E_{p,*}^2 = \pi_* X \amalg_p \pi_* Y$  is the value on the pair of  $\Pi$ -algebras  $(\pi_* X, \pi_* Y)$  of the  $p$ -th derived functor in the sense of Quillen [19, 20] of the coproduct functor.

1.2 THEOREM. *Let  $\mathbf{I}$  be a small category. There is a first quadrant spectral sequence, functorial in  $\mathbf{I}$ -diagrams  $\underline{X}$  of pointed connected CW complexes  $X_i$ ,  $i \in \mathbf{I}$ , that converges strongly to the homotopy groups  $\pi_* \text{holim } \underline{X}$  of the homotopy direct limit of  $\underline{X}$ . The columns of the  $E^2$  term have the following interpretation:*

(i)  $E_{0,*}^2 = \lim (\pi_* \underline{X})$  is the direct limit of the  $\mathbf{I}$ -diagram  $\pi_* \underline{X}$  of  $\Pi$ -algebras  $\pi_* X_i$ , and

(ii)  $E_{p,*}^2 = \overline{\lim}_p (\pi_* \underline{X})$  is the value on the diagram of  $\Pi$ -algebras  $\pi_* \underline{X}$  of the  $p$ -th derived functor of the direct limit functor.

---

†Supported by graduate fellowships from the National Science Foundation and the Alfred P. Sloan Foundation.

1.3 *Recovery of the van Kampen Theorem.* The  $E^2$  term of spectral sequence 1.1 for  $\pi_*(X \vee Y)$  contains a single non-abelian group  $E_{0,1}^2$  in the lower left corner, and it is straightforward to demonstrate that fundamental group of the wedge

$$\pi_1(X \vee Y) = E_{0,1}^\infty = E_{0,1}^2 = \pi_1 X * \pi_1 Y$$

is the coproduct of the fundamental groups of the spaces  $X$  and  $Y$  as the van Kampen theorem requires.

From the spectral sequence 1.2 for  $\pi_* \text{holim} \underline{X}$  we similarly recover that the fundamental group of a homotopy direct limit of pointed connected spaces

$$\pi_1 \text{holim} \underline{X} = E_{0,1}^\infty = E_{0,1}^2 = \varinjlim \pi_1 X$$

is the direct limit of the fundamental groups of the spaces in the diagram. This includes the results of [6] and in particular the often used statement of the van Kampen theorem for a pushout.

1.4 *Extension of a Result of Hilton.* Functoriality of homotopy groups implies that  $\pi_n X \oplus \pi_n Y$  is a direct summand in  $\pi_n(X \vee Y)$  for  $n \geq 2$ . For sufficiently connected  $X$  and  $Y$  the remainder term was computed by Hilton [9] in the first dimension where it is non-trivial and up to extension in the second. Hilton's result may be recovered, and indeed extended one dimension further, as the "corner" of the wedge spectral sequence 1.1.

1.5 *Remark.* In 1.1–1.2 we lose some of the determinism of the classical van Kampen theorem in order to obtain an extension that considers higher homotopy groups specifically. A different approach may be found in [4], [5] where other functors generalizing the fundamental group are defined and shown to preserve certain direct limits.

1.6 *On The Proof of 1.1.* We describe our strategy for constructing the spectral sequence 1.1 for the homotopy groups of a wedge of spaces. The key step is to produce a functorial "simplicial resolution by wedges of spheres" (2.4) of a pointed connected CW complexes  $X$ . This is a simplicial diagram  $V_* X$  of pointed connected CW complexes and topological maps such that the associated simplicial diagram  $\pi_* V_* X$  of  $\Pi$ -algebras is a "free simplicial resolution" (5.4) of the  $\Pi$ -algebra  $X$ . Among other things, this means that each space  $V_p X$  in the diagram has the homotopy type of a wedge of spheres, and that the homotopy type of  $X$  can be recovered as the realization (3.2) of  $V_* X$ .

Suppose  $X$  and  $Y$  are spaces with simplicial resolutions  $V_* X$  and  $V_* Y$ . Let  $V_* X \vee V_* Y$  denote the simplicial space that is obtained by applying the wedge functor in each dimension  $p$  to the spaces  $V_p X$  and  $V_p Y$ . Associated to this simplicial space is a first quadrant homotopy spectral sequence (3.4) analogous to the spectral sequence of a first quadrant bicomplex in homological algebra [17, XI 6]. It is straightforward to show the convergence of this spectral sequence to the homotopy groups of  $X \vee Y$  and to identify the  $E^2$  term as derived functors.

1.7 *On The Proof of 1.2.* To produce the van Kampen spectral sequence for the homotopy groups of a general homotopy direct limit it turns out that we cannot, in spite of functoriality, get away with resolving each space in the diagram separately. Once we have decided how to "resolve the whole diagram at once," however, we can take the homotopy direct limit in every dimension and the spectral sequence of the resulting simplicial space has the desired properties.

1.8 *A Variation for Smash Products.* By proceeding as for the wedge spectral sequence but taking the smash rather than the wedge of the spaces  $V_p X$  and  $V_p Y$  in each dimension of the simplicial resolutions for  $X$  and  $Y$  we obtain

1.9 THEOREM. *There is a first quadrant spectral sequence, functorial in pointed connected CW complexes  $X$  and  $Y$ , converging strongly to the homotopy groups  $\pi_*(X \wedge Y)$  of the smash product of  $X$  and  $Y$ , and such that the columns of the  $E^2$  term have the following interpretation:*

- (i)  $E_{0,*}^2 = \pi_* X \otimes \pi_* Y$  is the tensor product (7.3) of the  $\Pi$ -algebras  $\pi_* X$  and  $\pi_* Y$ , and
- (ii)  $E_{p,*}^2 = \pi_* X \otimes_{\pi_*} \pi_* Y$  is the value on the pair of  $\Pi$ -algebras  $(\pi_* X, \pi_* Y)$  of the  $p$ -th derived functor of the tensor product functor.

We remark that if spaces are compared to chain complexes and simplicial spaces to bicomplexes, then the derivation of theorem 1.9 resembles the derivation of the Kunnetth spectral sequence in homological algebra [17, XII].

1.10 *Organization of the Paper.* The construction of the “simplicial resolution of a space by wedges of spheres” is given in section 2. In section 3 we set up the wedge spectral sequence 1.1 and show convergence. To interpret the  $E^2$  term we need to discuss  $\Pi$ -algebras and recall the machinery of derived functors. This is done in sections 4 and 5. The general van Kampen spectral sequence 1.2 is derived in section 6. We consider the variation for smash products in section 7 and conclude in an appendix with a discussion of the properties of the tensor product functor for  $\Pi$ -algebras.

## 2. SIMPLICIAL RESOLUTION OF A SPACE BY WEDGES OF SPHERES

In this section we construct a natural simplicial diagram  $V_* X$  (2.4) of spaces over  $X$  such that the associated simplicial  $\Pi$ -algebra  $\pi_* V_* X$  is a “free simplicial resolution” (5.4) of the  $\Pi$ -algebra  $\pi_* X$ .

2.1 *Preliminaries.* Let  $\mathcal{CW}_*$  denote the category of *connected CW complexes with basepoint*. In what follows a *space* is an object of this category unless stated otherwise.

By a *simplicial space*  $Y_*$  we mean as usual [14, 2.1] a collection  $\{Y_n\}_{n \geq 0}$  of spaces together with face and degeneracy maps

$$d_j: Y_n \rightarrow Y_{n-1}, \quad s_j: Y_n \rightarrow Y_{n+1}$$

defined for  $0 \leq j \leq n$  and satisfying the simplicial identities [14, 1.1].

A simplicial space  $Y_*$  will be called *cellular* if each degeneracy map  $s_j: Y_n \rightarrow Y_{n+1}$  of  $Y_*$  includes  $Y_n$  as a subcomplex of  $Y_{n+1}$ .

An *augmentation* of a simplicial space  $Y_*$  by a space  $Y_{-1}$  is a (face) map

$$d_0: Y_0 \rightarrow Y_{-1}$$

such that  $d_0 d_0 = d_0 d_1: Y_1 \rightarrow Y_{-1}$ .

With this terminology, the “simplicial resolution” we intend to produce for a space  $X$  is a certain cellular simplicial space  $V_* X$  with augmentation by  $X$ . This simplicial space is generated by a cotriple  $(\mathcal{V}, \varepsilon, \beta)$  on the category of spaces. As the first step in defining this cotriple we describe

2.2 *The Functor  $\mathcal{V}$ .* Let  $S^n \in \mathcal{CW}_*$  be the standard  $n$ -sphere, and let  $D^{n+1}$  be the reduced cone on  $S^n$ . Given a space  $X \in \mathcal{CW}_*$ , let  $\mathcal{V}X$  be the pointed connected CW complex obtained as follows:

- (i) take a wedge of spheres  $S_f^n$  indexed by all positive integers  $n$  and all maps  $f: S^n \rightarrow X$ , and  
(ii) whenever an indexing map  $f$  is null homotopic, attach to the sphere  $S_f^n$  a disk  $D_h^{n+1}$  for every null homotopy  $h: D^{n+1} \rightarrow X$  of  $f$ .  $\mathcal{V}X$  is thus a pushout

$$\begin{array}{ccc} \bigvee_{n \geq 1} \bigvee_{h \in \text{Hom}(D^{n+1}, X)} S_h^n | S^n & \rightarrow & \bigvee_{n \geq 1} \bigvee_{h \in \text{Hom}(D^{n+1}, X)} D_h^{n+1} \\ \downarrow & & \downarrow \\ \bigvee_{n \geq 1} \bigvee_{f \in \text{Hom}(S^n, X)} S_f^n & \longrightarrow & \mathcal{V}X \end{array}$$

where the maps are the obvious ones induced by the inclusions  $S^n \subset D^{n+1}$ .

2.3 The Cotriple  $(\mathcal{V}, \varepsilon, \beta)$ . The construction  $\mathcal{V}$  is functorial and comes with two natural maps  $\varepsilon: \mathcal{V}X \rightarrow X$  and  $\beta: \mathcal{V}X \rightarrow \mathcal{V}^2X$ . These are defined as follows on the subcomplexes  $S_f^n$  and  $D_h^{n+1}$  out of which  $\mathcal{V}X$  is assembled:

- (i)  $\varepsilon: \mathcal{V}X \rightarrow X$  sends  $S_f^n$  into  $X$  by the indexing map  $f$  and  $D_h^{n+1}$  into  $X$  by the indexing map  $h$ , and  
(ii)  $\beta: \mathcal{V}X \rightarrow \mathcal{V}^2X$  takes  $S_f^n$  homeomorphically to the copy of  $S^n$  in  $\mathcal{V}^2X$  that is indexed by the inclusion  $S_f^n \subset \mathcal{V}X$  and takes  $D_h^{n+1}$  homeomorphically to the copy of  $D^{n+1}$  in  $\mathcal{V}^2X$  that is indexed by the inclusion  $D_h^{n+1} \subset \mathcal{V}X$ .

It is easily verified that  $(\mathcal{V}, \varepsilon, \beta)$  is a cotriple [10], [8], i.e. that

$$(\varepsilon \mathcal{V})\beta = \text{id} = (\mathcal{V} \varepsilon)\beta \quad \text{and} \quad (\beta \mathcal{V})\beta = (\mathcal{V} \beta)\beta.$$

2.4 *The Simplicial Resolution.* The cotriple  $(\mathcal{V}, \varepsilon, \beta)$  produces by the standard method [10, §2] a cellular simplicial space  $V_*X = (V_pX, d_j, s_j)$  augmented by  $V_{-1}X = X$  such that

$$V_pX = \mathcal{V}^{p+1}X$$

for all  $p \geq 0$ , and the face and degeneracy maps are given by

$$\begin{aligned} d_j &= \mathcal{V}^j \varepsilon \mathcal{V}^{p-j}: V_pX \rightarrow X \quad \text{and} \\ s_j &= \mathcal{V}^j \beta \mathcal{V}^{p-j}: V_pX \rightarrow V_{p+1}X \end{aligned}$$

for all  $0 \leq j \leq p$ .

We end this section with two key propositions (2.5 and 2.6) about  $V_*X$ . The first says, among other things, that each space  $V_pX$  has the homotopy type of a wedge of spheres, and the second that each simplicial group  $\pi_q V_*X$  has the homotopy type of the discrete group  $\pi_q X$ . Together these two propositions will imply that  $\pi_* V_*X$  is a free simplicial resolution of the  $\Pi$ -algebra  $\pi_* X$  in the sense of 5.4 below.

2.5 PROPOSITION. *Let  $X \in \mathcal{C}\mathcal{W}_*$  be a space. There exist contractible subcomplexes  $C_p \subset V_pX$ ,  $p \geq 0$ , such that*

- (i) *For all  $p \geq 0$  the quotient  $V_pX/C_p$  is a bouquet of spheres of positive dimensions.*  
(ii)  *$s_j c_p \in C_{p+1}$  for all degeneracy maps  $s_j: V_pX \rightarrow V_{p+1}X$  and all points  $c_p \in C_p$ .*  
(iii) *For all  $0 \leq j \leq p$  the map  $\bar{s}_j: V_p/C_p \rightarrow V_{p+1}/C_{p+1}$  that is induced by  $s_j$  is an inclusion of bouquets.*

*Proof.* Recall (2.2) that the space  $V_0X = \mathcal{V}X$  is assembled out of certain spheres  $S_f^n$  and disks  $D_h^{n+1}$ . Let  $C_0$  be a subcomplex of  $V_0X$  that is obtained by choosing, for each sphere  $S_f^n$  whose index map  $f$  is null homotopic, exactly one of the disks  $D_h^{n+1}$  that are attached to  $S_f^n$ . The quotient  $V_0X/C_0$  is then a bouquet of spheres of two kinds: those of the form

$D_h^{n+1}/S_h^n/S^n$  where  $D_h^{n+1}$  is not in  $C_0$ , and those of the form  $S_f^n$  where  $f$  is not null homotopic. Since the simplicial space  $V_*X$  is cellular, a straightforward induction shows that we can similarly choose  $C_1 \subset V_1X = \mathcal{V}V_0X$ ,  $C_2 \subset V_2X = \mathcal{V}V_1X$ , etc. so that (ii) and (iii) are satisfied.  $\square$

If we apply the  $q$ -th homotopy group functor  $\pi_q$  to each space  $V_pX$  in the simplicial space  $V_*X$  we obtain for  $q \geq 1$  a simplicial group  $\pi_q V_*X$ , which itself [14, §17] has homotopy groups  $\pi_p \pi_q V_*X$  for all  $p \geq 0$ . These are abelian by [14, 17.3] unless  $p=0$  and  $q=1$ .

## 2.6 PROPOSITION.

(i)  $\pi_p \pi_q V_*X = 0$  for all  $p \geq 1$  and  $q \geq 1$ .

(ii) The map

$$\pi_0 \pi_q V_*X \rightarrow \pi_q X$$

that is induced by the augmentation  $d_0: V_0X \rightarrow X$  is an isomorphism of groups for all  $q \geq 1$ .

*Proof.* For fixed  $q$  the homotopy groups  $\pi_p \pi_q V_*X$  can be computed [14, 17.4] as the homology groups of the (not necessarily abelian) chain complex  $(N_p \pi_q V_*X, \partial)$  where

$$N_p \pi_q V_*X = \pi_q V_p X \cap \ker d_1 \cap \dots \cap \ker d_p$$

and  $\partial$  is the restriction of  $d_0$ . The proposition thus follows from Lemma 2.7 below.  $\square$

2.7 LEMMA. Let  $\phi \in \pi_q V_p X$ ,  $p \geq 0$  and  $q \geq 1$ , be such that

$$d_0 \phi = d_1 \phi = \dots = d_p \phi = 0 \in \pi_q V_{p-1} X.$$

Then there exists  $\gamma \in \pi_q V_{p+1} X$  such that

$$d_0 \gamma = \phi \quad \text{and} \quad d_1 \gamma = \dots = d_{p+1} \gamma = 0 \in \pi_q V_p X.$$

*Proof.* Choose a representative

$$f: S^q \rightarrow V_p X$$

for  $\phi$ . Let  $\gamma \in \pi_q V_{p+1} X$  be the class of the inclusion  $S_j^q \subset \mathcal{V}V_p X = V_{p+1} X$ . To see that  $d_0 \gamma = \phi$ , note that the zero face,

$$d_0 = \varepsilon: \mathcal{V}V_p X \rightarrow V_p X,$$

takes  $S_j^q$  into  $V_p X$  by  $f$ . To see that  $d_j \gamma = 0$  for  $j \geq 1$ , observe first that the identification

$$d_j = \mathcal{V}d_{j-1}: \mathcal{V}V_p X \rightarrow \mathcal{V}V_{p-1} X$$

implies that  $d_j \gamma$  is equal to the class of the inclusion  $S_{d_{j-1}f}^q \subset \mathcal{V}V_{p-1} X = V_p X$ . Now by hypothesis on  $\phi$ , the composition

$$d_{j-1}f: S^q \rightarrow V_{p-1} X$$

extends to a null homotopy

$$h: D^{q+1} \rightarrow V_{p-1} X.$$

Hence the inclusion  $D_h^{q+1} \subset \mathcal{V}V_{p-1} X$  is a null homotopy for the inclusion  $S_{d_{j-1}f}^q \subset \mathcal{V}V_{p-1} X = V_p X$ , and  $d_j \gamma = 0$  as desired.  $\square$

### 3. THE SPECTRAL SEQUENCE FOR A WEDGE

In this section we prove

**3.1 THEOREM.** *There is a first quadrant spectral sequence, functorial in pointed connected CW complexes  $X$  and  $Y$ , converging to  $\pi_*(X \vee Y)$ , and with  $E_{p,q}^2 = \pi_p \pi_q(V_*X \vee V_*Y)$ , where  $V_*X \vee V_*Y$  denotes the simplicial space that is obtained by applying the wedge functor in each dimension  $p$  to the spaces  $V_pX$  and  $V_pY$  in the simplicial resolutions (2.4) of  $X$  and  $Y$ , and  $\pi_p \pi_q(V_*X \vee V_*Y)$  is the  $p$ -th homotopy group of the simplicial group  $\pi_q(V_*X \vee V_*Y)$ .*

This result will become Theorem 1.1 after reinterpretation of the  $E^2$  term as derived functors has been carried out in Sections 4 and 5.

Theorem 3.1 in an application of some standard results about simplicial spaces which we now review.

**3.2 Realization of Simplicial Spaces.** Let  $\Delta_n$  be the topological  $n$ -simplex. Given a simplicial space  $Y_* \in sCW_*$ , the realization  $\Delta Y_*$  is the pointed connected topological space obtained [23] from

$$\bigvee_{n \geq 0} Y_n \times \Delta_n / * \times \Delta_n$$

by making certain identifications that are defined using the face and degeneracy maps of  $Y_*$ . The space  $\Delta Y_*$  will be a CW complex if  $Y_*$  is cellular (2.1).

**3.3 Realization and Homotopy Equivalences.** Let  $f: Y_* \rightarrow Y'_*$  be a map of simplicial spaces such that each

$$f_n: Y_n \rightarrow Y'_n$$

is a homotopy equivalence. If  $Y_*$  and  $Y'_*$  are both cellular, then the induced map

$$\Delta f: \Delta Y_* \rightarrow \Delta Y'_*$$

of realizations is a homotopy equivalence.

For a proof of 3.3 see [15, A.4], [21] or [24, A.1].

Using 3.3 one can easily translate a homotopy spectral sequence that is defined on bisimplicial sets [2, B.4] to simplicial spaces and obtain the following (see the appendix of [5] for the argument):

**3.4 THEOREM.** *There is a first quadrant spectral sequence, functorial in cellular simplicial spaces  $Y_*$ , converging strongly to the homotopy groups of the realization  $\pi_* \Delta Y_*$ , and such that  $E_{p,q}^2 = \pi_p \pi_q Y_*$ .*

In view of 2.6 an immediate corollary to 3.4 is

**3.5 PROPOSITION.** *Let  $X \in \mathcal{CW}_*$  and let  $V_*X$  be the cellular simplicial space of 2.4. Then the augmentation map induces a homotopy equivalence*

$$\Delta V_*X \rightarrow X \quad \square$$

**3.6 Proof of Theorem 3.1.** Apply the spectral sequence of 3.4 to the simplicial space  $V_*X \vee V_*Y$ . By 3.5 there is a homotopy equivalence

$$\Delta(V_*X \vee V_*Y) = \Delta V_*X \vee \Delta V_*Y \rightarrow X \vee Y$$

which implies convergence to  $\pi_*(X \vee Y)$  as claimed. □

4.  $\Pi$ -ALGEBRAS

In this section we define the category of  $\Pi$ -algebras (4.2), whose objects are models for the graded set of the homotopy groups of a space together with the actions on these groups by the primary homotopy operations. We also describe the sense in which a *free  $\Pi$ -algebra* (4.6) is obtained from the homotopy groups of a bouquet of spheres. We begin with a formal approach to

4.1 *Homotopy Groups with Operations.* Choose a space  $W$  in each homeomorphism class of finite bouquets of pointed spheres of positive dimension. Let  $\Pi$  be the small category whose objects are the spaces  $W$  and whose morphisms are the pointed homotopy classes of maps between them. The point space  $*$  representing the empty wedge is included as an object of  $\Pi$ .

Given a space  $X \in \mathcal{CW}_*$ , we want to capture the information contained in the pointed sets  $\pi_n X$ , their group structures, and the actions on them by the primary homotopy operations [25, XI]. This can be done by considering the contravariant functor from  $\Pi$  to pointed sets

$$[-, X]: \Pi \rightarrow \text{Sets}_*$$

that is obtained by taking pointed homotopy classes of maps from the objects of  $\Pi$  into  $X$ . This functor has the property that, for every object  $W = \bigvee_{i=1}^k S^{n_i} \in \Pi$  ( $k \geq 0$ ), the natural map

$$[\bigvee_{i=1}^k S^{n_i}, X] \rightarrow \prod_{i=1}^k [S^{n_i}, X] \in \text{Sets}_*$$

induced by the inclusions  $S^{n_i} \hookrightarrow \bigvee_{i=1}^k S^{n_i}$  is an isomorphism. It follows that the functor  $[-, X]$  on  $\Pi$  can be thought of as the collection of pointed sets  $\pi_n X$ ,  $n \geq 1$ , endowed with finitary operations corresponding to morphisms in  $\Pi$ . With this interpretation we will write  $\pi_* X$  for the functor  $[-, X]$  from  $\Pi$  to  $\text{Sets}_*$ .

4.2  *$\Pi$ -Algebras.* We define a  $\Pi$ -algebra  $A$  to be a contravariant functor

$$A: \Pi \rightarrow \text{Sets}_*$$

such that, for every object  $W = \bigvee_{i=1}^k S^{n_i} \in \Pi$  ( $k \geq 0$ ), the natural map

$$A(\bigvee_{i=1}^k S^{n_i}) \rightarrow \prod_{i=1}^k A(S^{n_i}) \in \text{Sets}_*$$

induced by the inclusions  $S^{n_i} \hookrightarrow \bigvee_{i=1}^k S^{n_i}$  is an isomorphism.

A general  $\Pi$ -algebra  $A$  may also be thought of as the collection of sets  $A(S^n)$  for  $n \geq 1$  with operations.

Clearly  $\Pi$ -algebras form a category  $\Pi\text{-Alg}$ , and we have a functor

$$\pi_*: \text{HoCW}_* \rightarrow \Pi\text{-Alg}$$

from the category of pointed connected CW complexes and pointed homotopy classes of maps to  $\Pi$ -algebras.

4.3 *Limits.* As usual with categories of universal algebras,  $\Pi\text{-Alg}$  inherits *inverse limits* from the category of pointed sets, and has *direct limits* [16, IX.1].

4.4 *Remark.* The formalism of definition 4.2 makes  $\Pi\text{-Alg}$  a (graded) *algebraic theory* in the sense of Lawvere [13], [18], [22].

4.5 *Free  $\Pi$ -algebras.* Let  $GrSets_*$  be the category of collections  $K = \{K_n\}$  of pointed sets  $K_n$  indexed by positive integers  $n \geq 1$ . Given a graded pointed set  $K$ , define the *free  $\Pi$ -algebra  $\mathcal{F}K$  generated by  $K$*  to be

$$\mathcal{F}K = \pi_* \bigvee_{n \geq 1} \bigvee_{u \in K_n^-} S_u^n,$$

where  $K_n^-$  denotes the set  $K_n - \{*\}$ . We identify a generator  $u_0 \in K_q^-$  with the element of

$$(\mathcal{F}K)(S^q) = \pi_q \bigvee_{n \geq 1} \bigvee_{u \in K_n^-} S_u^n$$

that is represented by the inclusion  $S_{u_0}^q \subset \bigvee_{n \geq 1} \bigvee_{u \in K_n^-} S_u^n$ .

A map out of a free  $\Pi$ -algebra is determined by its value on generators. More precisely, we have

4.6 PROPOSITION. *For any  $\Pi$ -algebra  $A$ , let  $\mathcal{U}A = \{A(S^n)\}$  be the graded pointed set obtained from  $A$  by “forgetting operations.” Then for any graded pointed set  $K$ , the natural restriction*

$$\text{Hom}_{\Pi\text{-Alg}}(\mathcal{F}K, A) \rightarrow \text{Hom}_{GrSets_*}(K, \mathcal{U}A)$$

is a bijection. In other words,  $\mathcal{F}$  is a left adjoint to  $\mathcal{U}$ .

*Proof.* If  $K$  is a finite graded pointed set, then  $\bigvee_{n \geq 1} \bigvee_{u \in K_n^-} S_u^n$  is an object of  $\Pi$  and the usual argument by universal example gives the result. For the general case it suffices to show that the  $\Pi$ -algebra  $\mathcal{F}K$  is the direct limit of those subalgebras that are free on finite graded pointed subsets of  $K$ , but this follows by compactness of spheres.  $\square$

4.7 Remark. Proposition 4.6 gives a triple on the category of graded pointed sets. The category  $\Pi\text{-Alg}$  may be thought of as the category of algebras over this triple [16. IV.2].

## 5. THE $E^2$ TERM FOR THE WEDGE SPECTRAL SEQUENCE

In this section we complete the proof of Theorem 1.1 by proving

5.1 THEOREM. *Let  $X$  and  $Y$  be pointed connected CW complexes and let  $E_{p,q}^r$  be the spectral sequence of 3.1 converging to  $\pi_*(X \vee Y)$  and with*

$$E_{p,q}^2 = \pi_p \pi_q(V_* X \vee V_* Y).$$

*Then the columns of the  $E^2$  term have the interpretation*

- (i)  $E_{0,*}^2 = \pi_* X \amalg \pi_* Y$  is the coproduct of  $\pi_* X$  and  $\pi_* Y$  in the category of  $\Pi$ -algebras, and
- (ii)  $E_{p,*}^2 = \pi_* X \amalg_p \pi_* Y$  is the value on the pair of  $\Pi$ -algebras  $(\pi_* X, \pi_* Y)$  of the  $p$ -th derived functor (5.6) of the coproduct functor.

The derived functors  $\amalg_p$  will be defined in 5.6 below after a preliminary discussion of simplicial  $\Pi$ -algebras, their homotopy groups, and free simplicial resolutions. With this machinery in place Theorem 5.1 will follow at once.

5.2 *Simplicial  $\Pi$ -algebras.* Let  $A_*$  be a simplicial object in the category of  $\Pi$ -algebras. By definition 4.2,  $A_*$  can be thought of as a contravariant functor

$$A_*: \Pi \rightarrow sSets_*$$

from  $\mathbf{\Pi}$  to pointed simplicial sets such that, for every object  $W = \bigvee_{i=1}^k S^{n_i} \in \mathbf{\Pi}$  ( $k \geq 0$ ), the natural map

$$A_*(\bigvee_{i=1}^k S^{n_i}) \rightarrow \prod_{i=1}^k A_*(S^{n_i}) \in sSets_*$$

is an isomorphism.

**5.3 PROPOSITION.** *Given a simplicial  $\mathbf{\Pi}$ -algebra  $A_*$  and integer  $p \geq 0$ , let  $\pi_p A_*$  denote the composition*

$$\pi_p A_*: \mathbf{\Pi} \xrightarrow{A_*} sSets_* \xrightarrow{\pi_p} Sets_*$$

where  $\pi_p$  is the  $p$ -th homotopy group functor [14, 3.6 & 16.7]. Then  $\pi_p A_*$  is a  $\mathbf{\Pi}$ -algebra.

*Proof.* By 5.2 and definition 4.2 it is enough to observe that the functor  $\pi_p$  preserves finite products. □

**5.4 Free Simplicial Resolutions.** A free simplicial resolution of a  $\mathbf{\Pi}$ -algebra  $A$  is a simplicial  $\mathbf{\Pi}$ -algebra  $F_*$  augmented by  $A$  with the following properties:

- (i) there exist graded pointed subsets  $K_p = \{K_{p,n}\} \subset F_p$  for  $p \geq 0$  such that each  $F_p$  is the free  $\mathbf{\Pi}$ -algebra generated by  $K_p$ ,
- (ii)  $s_j u_p \in K_{p+1}$  whenever  $0 \leq j \leq p$  and  $u_p \in K_p$ ,
- (iii)  $\pi_p F_* = 0$  for  $p \geq 1$ , and
- (iv) the augmentation of  $F_*$  by  $A$  induces an isomorphism of  $\mathbf{\Pi}$ -algebras

$$\pi_0 F_* \xrightarrow{\cong} A.$$

**5.5 Existence of Free Simplicial Resolutions.** The cotriple associated to the adjoint pair of 4.6 gives a functorial free simplicial resolution by standard methods [10]. Others can be constructed by a process analogous to that of attaching cells to kill homotopy groups [20, II 4 Prop. 3].

**5.6 Derived Functors of the Coproduct Functor.** Suppose  $A$  and  $B$  are  $\mathbf{\Pi}$ -algebras with free simplicial resolutions  $F_* A$  and  $F_* B$ . Let  $F_* A \amalg F_* B$  denote the simplicial  $\mathbf{\Pi}$ -algebra formed by taking the coproduct in every dimension  $p$  of the  $\mathbf{\Pi}$ -algebras  $F_p A$  and  $F_p B$ . The  $p$ -th derived functor  $\amalg_p$  of the coproduct functor is then defined on the pair  $(A, B)$  by

$$A \amalg_p B = \pi_p(F_* A \amalg F_* B).$$

**5.7 PROPOSITION.** *For all  $p \geq 0$  the definition of 5.6 provides a well defined functor  $\amalg_p$  from the category of pairs of  $\mathbf{\Pi}$ -algebras to the category of  $\mathbf{\Pi}$ -algebras.*

*Proof.* This is almost an immediate application of the standard Quillen-André theory of derived functors [19, § 1], [20, II 4], [1]. The only difference from the usual situation is that the target category is not abelian, so that in defining  $\amalg_p$  in 5.6 we used homotopy groups of simplicial objects rather than homology groups of differential graded ones. This causes no difficulties in view of Lemma 5.8 below. □

**5.8 LEMMA.** *Let  $f, g: A_* \rightarrow B_*$  be maps of simplicial  $\mathbf{\Pi}$ -algebras that are homotopic in the sense of [11], [12] (described also in [7, 1.9–1.10] [20, II 1.6–1.7]). Then for all  $p \geq 0$*

$$\pi_p f = \pi_p g: \pi_p A_* \rightarrow \pi_p B_*.$$

*Proof.* Use definition 5.3 and the corresponding result for simplicial sets [7, 2.3], [14, §14–16].  $\square$

5.9 PROPOSITION. *The zeroth derived functor  $A \amalg_0 B$  is naturally isomorphic to the coproduct of  $\Pi$ -algebras  $A \amalg B$ .*

*Proof.* For any simplicial  $\Pi$ -algebra  $C_*$ , the zeroth homotopy group  $\Pi$ -algebra  $\pi_0 C_*$  may be identified with  $\varinjlim C_p$ , the direct limit of the simplicial diagram of  $\Pi$ -algebras  $C_p$ . Hence we have

$$\begin{aligned} \pi_0(F_* A \amalg F_* B) &= \varinjlim (F_p A \amalg F_p B) \\ &= (\varinjlim F_p A) \amalg (\varinjlim F_p B) \\ &= A \amalg B. \end{aligned} \quad \square$$

5.10 *Proof of Theorem 5.1.* By 4.6 the free  $\Pi$ -algebra  $\pi_*(V_p X \vee V_p Y)$  is the coproduct of the free  $\Pi$ -algebras  $\pi_* V_p X$  and  $\pi_* V_p Y$  for all  $p \geq 0$ . Thus we have

$$E_{p,*}^2 = \pi_p \pi_*(V_* X \vee V_* Y) = \pi_p(\pi_* V_* X \amalg \pi_* V_* Y).$$

The theorem is now immediate from 5.6 and 5.9 since the simplicial  $\Pi$ -algebra  $\pi_* V_* X$  is a free simplicial resolution (5.4) of the  $\Pi$ -algebra  $\pi_* X$  by 2.5 and 2.6.  $\square$

## 6. A SPECTRAL SEQUENCE FOR HOMOTOPY DIRECT LIMITS

Our goal in this section is to show how the proof in sections 2 through 5 of Theorem 1.1 on the homotopy groups of a wedge can be generalized to yield Theorem 1.2, i.e.

THEOREM 1.2. *Let  $\mathbf{I}$  be a small category. There is a first quadrant spectral sequence, functorial in  $\mathbf{I}$ -diagrams  $\underline{X}$  of pointed connected CW complexes  $X_i$ ,  $i \in \mathbf{I}$ , that converges strongly to the homotopy groups  $\pi_* \text{holim } \underline{X}$  of the homotopy direct limit [3, XII] of  $\underline{X}$ . The columns of the  $E^2$  term have the following interpretation:*

- (i)  $E_{0,*}^2 = \varinjlim (\pi_* \underline{X})$  is the direct limit of the  $\mathbf{I}$ -diagram  $\pi_* \underline{X}$  of  $\Pi$ -algebras  $\pi_* X_i$ , and
- (ii)  $E_{p,*}^2 = \varinjlim_p (\pi_* \underline{X})$  is the value on the diagram of  $\Pi$ -algebras  $\pi_* \underline{X}$  of the  $p$ -th derived functor of the direct limit functor.

To motivate our proof of 1.2, note that we are claiming that the  $E^2$  term consists of derived functors that are defined on the category of  $\mathbf{I}$ -diagrams of  $\Pi$ -algebras. Hence we will be considering free simplicial resolutions in this category, and in particular must decide what it means for an  $\mathbf{I}$ -diagram of  $\Pi$ -algebras  $\underline{F}$  to be “free.” The answer we give (6.3) is that  $\underline{F}$  should be a “free diagram (6.1) of free  $\Pi$ -algebras.” With this notion of free object, the construction of the derived functors  $\varinjlim_p$  in 6.5–6.9 parallels the construction of the derived functors  $\amalg_p$  in section 5.

Given a diagram  $\underline{X}$  of spaces, we have an associated diagram  $\pi_* \underline{X}$  of  $\Pi$ -algebras. The main step in proving 1.2 is to show that some free simplicial resolution of  $\pi_* \underline{X}$  can be realized by a simplicial object in the category of diagrams of spaces. By modifying the construction  $V_*$  of section 2, we produce (6.10–6.11) a “simplicial resolution  $V_* \underline{X}$  of  $\underline{X}$  by free diagrams of wedges of spheres” that has the required properties. Applying the functor  $\text{holim}$  in each dimension to  $V_* \underline{X}$  gives a simplicial space augmented by  $\text{holim } \underline{X}$ . The homotopy spectral sequence of this simplicial space is readily seen (6.14) to satisfy the claims of 1.2.

6.1 *Free Diagrams in Categories.* Given a small category  $\mathbf{I}$ , let  $\mathbf{I}_0$  be the “object category” formed by stripping  $\mathbf{I}$  of all non-identity morphisms. Let  $\mathcal{C}$  be a category with coproducts.

For any  $\mathbf{I}_0$ -diagram  $\{X_i\}_{i \in \mathbf{I}}$ , define an  $\mathbf{I}$ -diagram  $\mathcal{D}\{X_i\}$  as follows:

(i) for each object  $i_0 \in \mathbf{I}$ , put

$$\mathcal{D}\{X_i\}_{i_0} = \coprod_{i \in \mathbf{I}, \alpha: i \rightarrow i_0} X_i, \text{ and}$$

(ii) for each morphism  $\beta: i_0 \rightarrow i_1$  of  $\mathbf{I}$ , let

$$\mathcal{D}\{X_i\}\beta: \mathcal{D}\{X_i\}_{i_0} \rightarrow \mathcal{D}\{X_i\}_{i_1}$$

be the obvious map induced by sending an index  $i \rightarrow i_0$  to the composition  $i \xrightarrow{\alpha} i_0 \xrightarrow{\beta} i_1$ .

In light of the following adjunction result, we call  $\mathcal{D}\{X_i\}$  the *free  $\mathbf{I}$ -diagram on  $\{X_i\}$* :

6.2 PROPOSITION. *Given an  $\mathbf{I}$ -diagram  $\underline{Y}$ , let  $\mathcal{O}\underline{Y} = \{Y_i\}_{i \in \mathbf{I}}$  be the  $\mathbf{I}_0$ -diagram obtained by “forgetting morphisms.” Then the natural map*

$$\text{Hom}_{\mathcal{C}^{\mathbf{I}}}(\mathcal{D}\{\underline{X}_i\}, Y) \rightarrow \text{Hom}_{\mathcal{C}^{\mathbf{I}_0}}(\{X_i\}, \mathcal{O}\underline{Y})$$

that sends a map of diagrams

$$f: \mathcal{D}\{X_i\} \rightarrow \underline{Y}$$

to the collection of maps of spaces

$$\{X_{i_0}, \text{id}: i_0 \rightarrow i_0\} \rightarrow \mathcal{D}\{X_i\}_{i_0} \xrightarrow{f_{i_0}} \underline{Y}_{i_0} \}_{i_0 \in \mathbf{I}}$$

is a bijection of sets. □

6.3 *Free Diagrams of  $\Pi$ -algebras.* Suppose that for each  $i \in \mathbf{I}$  we are given a collection  $K_i = \{K_{i,n}\}$  of pointed sets  $K_{i,n}$  indexed by positive integers  $n \geq 1$ . We define the *free  $\mathbf{I}$ -diagram of  $\Pi$ -algebras generated by the  $K_i$*  to be the free diagram  $\mathcal{D}\{\mathcal{F}K_i\}$  where each  $\mathcal{F}K_i$  is the free  $\Pi$ -algebra on generators  $K_i$  as in 4.5. We identify a generator  $u \in K_{i_0, n_0} \subset \mathcal{F}(K_{i_0})(S^{n_0})$  with its image under the inclusion

$$\mathcal{F}K_{i_0, \text{id}: i_0 \rightarrow i_0} \hookrightarrow \mathcal{D}\{\mathcal{F}K_i\}_{i_0}.$$

Combining 4.6 and 6.2 we see that

6.4 PROPOSITION. *A map out of  $\mathcal{D}\{\mathcal{F}K_i\}$  is freely determined by its value on generators, i.e. there is a natural bijection*

$$\text{Hom}(\mathcal{D}\{\mathcal{F}K_i\}, \underline{A}) \xrightarrow{\cong} \text{Hom}(\{K_i\}, \mathcal{U}\mathcal{O}\underline{A}).$$

where  $\underline{A}$  is any  $\mathbf{I}$ -diagram of  $\Pi$ -algebras and  $\mathcal{U}$  is the forgetful functor of 4.6.

6.5 *Free Simplicial Resolutions* (cf. 5.4). A *free simplicial resolution* of an  $\mathbf{I}$ -diagram of  $\Pi$ -algebras  $\underline{A}$  is a simplicial  $\mathbf{I}$ -diagram of  $\Pi$ -algebras  $\underline{F}_\bullet$  augmented by  $\underline{A}$  with the following properties:

(i) there exist graded pointed subsets  $K_{p,i} = \{K_{p,i,n}\} \subset \underline{F}_p i$  for  $p \geq 0$  and  $i \in \mathbf{I}$  such that each  $\underline{F}_p$  is the free  $\mathbf{I}$ -diagram of  $\Pi$ -algebras generated by the collection  $\{K_{p,i}\}_{i \in \mathbf{I}}$  as in 6.3,

- (ii)  $s_j u_{p,i} \in K_{p+1,i}$  whenever  $0 \leq j \leq p$  and  $u_{p,i} \in K_{p,i}$ ,
- (iii)  $\pi_p F_\bullet = \underline{0}$  for  $p \geq 1$  where  $\underline{0}$  is the I-diagram of trivial  $\Pi$ -algebras, and
- (iv) the augmentation induces an isomorphism of I-diagrams of  $\Pi$ -algebras

$$\pi_0 \underline{F}_\bullet \xrightarrow{\cong} \underline{A}.$$

6.6 *Existence of Free Simplicial Resolutions.* The cotriple associated to the adjoint pair of 6.4 gives a functorial free simplicial resolution as in 5.5.

6.7 *Derived Functors of Direct Limit.* Suppose  $\underline{A}$  is an I-diagram of  $\Pi$ -algebras with free simplicial resolution  $\underline{F}_\bullet$ . Let  $\varinjlim \underline{F}_\bullet$  denote the simplicial  $\Pi$ -algebra formed by taking the direct limit in every dimension  $p$  of the I-diagram of  $\Pi$ -algebras  $\underline{F}_p$ . The  $p$ -th derived functor  $\varinjlim_p$  of the direct limit functor is then defined on the diagram  $\underline{A}$  by

$$\varinjlim_p \underline{A} = \pi_p \varinjlim \underline{F}_\bullet.$$

Corresponding to 5.7 and 5.9 we have

6.8 PROPOSITION. For all  $p \geq 0$  the definition of 6.7 provides a well defined functor  $\varinjlim_p$  from the category of I-diagrams of  $\Pi$ -algebras to the category of  $\Pi$ -algebras.  $\square$

6.9 PROPOSITION. The zeroth derived functor  $\varinjlim_0 \underline{A}$  is naturally isomorphic to the direct limit  $\varinjlim \underline{A}$ .  $\square$

6.10 *The Basic Cotriple for Diagrams of Spaces.* Given an I-diagram of spaces  $\underline{X} \in \mathcal{CW}_*^1$ , let  $\underline{\mathcal{V}}\underline{X}$  denote the free I-diagram on  $\{\mathcal{V}X_i\}_{i \in \mathbf{I}}$  where  $\mathcal{V}$  is the functor from spaces to spaces of 2.2. In other words, we are defining  $\underline{\mathcal{V}} = \mathcal{D}\mathcal{V}\mathcal{O}$  where  $\mathcal{D}: \mathcal{CW}_*^1 \rightarrow \mathcal{CW}_*^1$  and  $\mathcal{O}: \mathcal{CW}_*^1 \rightarrow \mathcal{CW}_*^0$  are the adjoint pair of 6.2. By the standard argument [10,4.2] the cotriple  $(\underline{\mathcal{V}}, \underline{\varepsilon}, \underline{\beta})$  of 2.3 induces a cotriple

$$(\underline{\mathcal{V}}, \underline{\varepsilon}, \underline{\beta}) = (\mathcal{D}\mathcal{V}\mathcal{O}, \eta \circ \mathcal{D}\varepsilon\mathcal{O}, \mathcal{D}\mathcal{V}\chi\mathcal{V}\mathcal{O} \circ \mathcal{D}\beta\mathcal{O})$$

where  $\eta: \mathcal{D}\mathcal{O} \rightarrow \text{id}$  and  $\chi: \text{id} \rightarrow \mathcal{O}\mathcal{D}$  are the natural transformations defined by the adjointness of  $\mathcal{D}$  and  $\mathcal{O}$ .

6.11 *Simplicial Resolution of Diagrams of Spaces.* As in 2.4 the basic cotriple  $(\underline{\mathcal{V}}, \underline{\varepsilon}, \underline{\beta})$  produces an I-diagram  $\underline{V}_\bullet \underline{X}$  of cellular simplicial spaces together with an augmentation map  $d_0: \underline{V}_0 \underline{X} \rightarrow \underline{X}$ .

It is straightforward to verify the following two propositions about  $\underline{V}_\bullet \underline{X}$  that correspond to 2.5 and 2.6:

6.12 PROPOSITION. Let  $\underline{X} \in \mathcal{CW}_*$  be an I-diagram of spaces. There exist I-diagrams  $\underline{C}_p$ ,  $p \geq 0$ , of contractible subcomplexes  $\underline{C}_p i \subset \underline{V}_p \underline{X}_i$ ,  $i \in \mathbf{I}$ , such that

(i) For all  $p \geq 0$  the quotient  $\underline{V}_p \underline{X} / \underline{C}_p$  is the free diagram on a collection of bouquets of spheres of positive dimensions.

(ii)  $s_j c_{p,i} \in \underline{C}_{p+1,i}$  for all degeneracy maps  $s_j: \underline{V}_p \underline{X} \rightarrow \underline{V}_{p+1} \underline{X}$  and all points  $c_{p,i} \in \underline{C}_{p,i}$ .

(iii) For all  $0 \leq r \leq p$  the map  $\bar{s}_j: \underline{V}_p \underline{X} / \underline{C}_p \rightarrow \underline{V}_{p+1} \underline{X} / \underline{C}_{p+1}$  that is induced by  $s_j$  is the free diagram on a collection of inclusions of bouquets.  $\square$

6.13 PROPOSITION. For all  $i \in \mathbf{I}$  the simplicial space  $(\underline{V} \cdot \underline{X})_i$  satisfies

- (i)  $\pi_p \pi_q (\underline{V} \cdot \underline{X})_i = 0$  for  $p \geq 0, q \geq 1$ .
- (ii) The map

$$\pi_0 \pi_q (\underline{V} \cdot \underline{X})_i \rightarrow \pi_q \underline{X}_i$$

induced by the augmentation  $d_0: \underline{V}_0 \underline{X} \rightarrow \underline{X}$  is an isomorphism of groups for all  $q \geq 1$ .  $\square$

6.14 Proof of Theorem 1.2. Suppose  $\underline{X}$  is an  $\mathbf{I}$ -diagram of spaces and  $\underline{V} \cdot \underline{X}$  is the simplicial resolution of 6.11. Let  $\underline{\text{holim}} \underline{V} \cdot \underline{X}$  denote the simplicial space that is obtained by applying the homotopy direct limit functor [3, XII] in each dimension  $p$  to the  $\mathbf{I}$ -diagram of spaces  $\underline{V}_p \underline{X}$ . The homotopy spectral sequence (3.4) of the simplicial space  $\underline{\text{holim}} \underline{V} \cdot \underline{X}$  converges strongly to  $\pi_* \Delta \underline{\text{holim}} \underline{V} \cdot \underline{X}$ . The columns of the  $E^2$  term are

$$E_{p,*}^2 = \pi_p \pi_* \underline{\text{holim}} \underline{V} \cdot \underline{X}.$$

We show

- (i) There is a natural homotopy equivalence

$$\Delta \underline{\text{holim}} \underline{V} \cdot \underline{X} \rightarrow \underline{\text{holim}} \underline{X}$$

- (ii) The columns of the  $E^2$  term can be identified as the derived functors

$$E_{p,*}^2 = \varinjlim_p \pi_* \underline{X}.$$

*Proof* (i). By 6.13 and 3.4, the map of spaces

$$\Delta \underline{V} \cdot \underline{X}_i \rightarrow \underline{X}_i$$

that is induced by the augmentation of  $\underline{V} \cdot \underline{X}$  is a homotopy equivalence for all  $i \in \mathbf{I}$ . It follows from 6.15 below that we obtain a homotopy equivalence

$$\Delta \underline{\text{holim}} \underline{V} \cdot \underline{X} = \underline{\text{holim}} \Delta \underline{V} \cdot \underline{X} \rightarrow \underline{\text{holim}} \underline{X}$$

as desired.

*Proof* (ii). By 6.16 below, the natural map of spaces [3, XII 2.5]

$$\underline{\text{holim}} \underline{V}_p \underline{X} \rightarrow \varinjlim_p \underline{V}_p \underline{X}$$

is a homotopy equivalence for all  $p \geq 0$ . Using 6.12 and 4.6, moreover, it is not hard to see that the free  $\Pi$ -algebra  $\pi_* \varinjlim_p \underline{V}_p \underline{X}$  is the direct limit of the free diagram of free  $\Pi$ -algebras  $\pi_* \underline{V}_p \underline{X}$ . Thus

$$E_{p,*}^2 = \pi_p \pi_* \underline{\text{holim}} \underline{V} \cdot \underline{X} = \pi_p \pi_* \varinjlim_p \underline{V} \cdot \underline{X} = \pi_p \varinjlim_p \pi_* \underline{V} \cdot \underline{X}.$$

The result is now immediate from 6.7 since the simplicial diagram of  $\Pi$ -algebras  $\pi_* \underline{V} \cdot \underline{X}$  is a free simplicial resolution (6.5) of the diagram of  $\Pi$ -algebras  $\pi_* \underline{X}$  by 6.12 and 6.13.  $\square$

6.15 PROPOSITION [3, XII 4.2]. If  $f: \underline{X} \rightarrow \underline{Y}$  is a map of  $\mathbf{I}$ -diagrams of pointed connected CW complexes such that each  $f_i: \underline{X}_i \rightarrow \underline{Y}_i$  is a homotopy equivalence, then the induced map

$$\underline{\text{holim}} f: \underline{\text{holim}} \underline{X} \rightarrow \underline{\text{holim}} \underline{Y}$$

is a homotopy equivalence.

*Proof.* Use simplicial replacement [3, XII 5.2] and 3.3.  $\square$

6.16 LEMMA. Let  $\mathcal{D}\{X_i\}$  be the free  $\mathbf{I}$ -diagram (6.1) on spaces  $X_i \in \mathcal{C}\mathcal{W}_*$ ,  $i \in \mathbf{I}$ . Then there are identifications

$$\begin{array}{ccc} \underline{\text{holim}} \mathcal{D}\{X_i\} & \xrightarrow{\rho} & \underline{\text{lim}} \mathcal{D}\{X_i\} \\ \parallel & & \parallel \\ \bigvee_{i \in \mathbf{I}} \left( \frac{\mathbf{I} \setminus i \times X_i}{\mathbf{I} \setminus i \times * } \right) & \xrightarrow{q} & \bigvee_{i \in \mathbf{I}} X_i \end{array}$$

where  $\rho$  is the natural map of [3, XII 2.5],  $\mathbf{I} \setminus i$  is the contractible pointed space of [3, XI 2.7], and  $q$  is induced by collapsing each  $\mathbf{I} \setminus i$  to its basepoint. In particular,  $\rho$  is a homotopy equivalence.

*Proof.* Straightforward from the definition of  $\underline{\text{holim}}$  [3, XII 2.1]. □

## 7. A SPECTRAL SEQUENCE FOR SMASH PRODUCTS

In this section we imitate the construction of the homotopy spectral sequence of a wedge to obtain

**THEOREM 1.9** *There is a first quadrant spectral sequence, functorial in pointed connected CW complexes  $X$  and  $Y$ , converging strongly to the homotopy groups  $\pi_*(X \wedge Y)$  of the smash product of  $X$  and  $Y$ , and such that the columns of the  $E^2$  term have the following interpretation:*

- (i)  $E_{0,*}^2 = \pi_* X \otimes \pi_* Y$  is the tensor product (7.3) of the  $\Pi$ -algebras  $\pi_* X$  and  $\pi_* Y$ , and in general
- (ii)  $E_{p,*}^2 = \pi_* X \otimes_p \pi_* Y$  is the value on the pair of  $\Pi$ -algebras  $(\pi_* X, \pi_* Y)$  of the  $p$ -th derived functor of the tensor product functor.

The “tensor product” functor for  $\Pi$ -algebras that is referred to in Theorem 1.9 has properties analogous to the tensor product of modules over a commutative graded ring, as we show in the Appendix below. For the proof of theorem 1.9 in this section, however, we only need to give a definition of the tensor product (7.3). We begin with the special case of

**7.1 Tensor Products of Free  $\Pi$ -algebras.** Let  $\mathcal{F}\{K_n\}$  and  $\mathcal{F}\{L_n\}$  be the free  $\Pi$ -algebras generated by graded pointed sets  $\{K_n\}$  and  $\{L_n\}$ . With respect to these generators we define the tensor product  $\mathcal{F}\{K_n\} \otimes \mathcal{F}\{L_n\}$  to be the free  $\Pi$ -algebra on the graded set  $\{J_n = \bigvee_{p+q=n} (K_p \wedge L_q)\}$ .

Equivalently, if

$$W = \bigvee_{n \geq 1} \bigvee_{u \in K_n^-} S_u^n \quad \text{and} \quad X = \bigvee_{n \geq 1} \bigvee_{v \in L_n^-} S_v^n$$

are the bouquets of spheres with  $\mathcal{F}\{K_n\} = \pi_* W$  and  $\mathcal{F}\{L_n\} = \pi_* X$ ; then

$$\mathcal{F}\{K_n\} \otimes \mathcal{F}\{L_n\} = \pi_*(W \wedge X).$$

**7.2 Functoriality.** Suppose we are given another pair of free  $\Pi$ -algebras

$$\mathcal{F}\{K'_n\} = \pi_* W' \quad \text{and} \quad \mathcal{F}\{L'_n\} = \pi_* X'$$

with

$$W' = \bigvee_{n \geq 1} \bigvee_{u' \in K_n'^-} S_{u'}^n \quad \text{and} \quad X' = \bigvee_{n \geq 1} \bigvee_{v' \in L_n'^-} S_{v'}^n.$$

Let  $f: \mathcal{F}\{K_n\} \rightarrow \mathcal{F}\{K'_n\}$  and  $g: \mathcal{F}\{L_n\} \rightarrow \mathcal{F}\{L'_n\}$  be maps of  $\Pi$ -algebras that do not necessarily take generators to generators. There exist nevertheless unique homotopy classes of maps

$$\mathcal{R}f: W \rightarrow W' \quad \text{and} \quad \mathcal{R}g: X \rightarrow X'$$

such that  $f = \pi_* \mathcal{R}f: \pi_* W \rightarrow \pi_* W'$  and  $g = \pi_* \mathcal{R}g: \pi_* X \rightarrow \pi_* X'$ . Hence the smash product of maps of spaces

$$\mathcal{R}f \wedge \mathcal{R}g: W \wedge X \rightarrow W' \wedge X'$$

induces a  $\Pi$ -algebra map

$$f \otimes g: \mathcal{F}\{K_n\} \otimes \mathcal{F}\{L_n\} \rightarrow \mathcal{F}\{K'_n\} \otimes \mathcal{F}\{L'_n\}.$$

Naturality of this construction implies that the definition of  $\otimes$  in 7.1 gives a functor on the category of pairs of free  $\Pi$ -algebras that is independent of choice of generators.

**7.3 Tensor Products of General  $\Pi$ -algebras.** If  $A$  and  $B$  are  $\Pi$ -algebras we define

$$A \otimes B = \pi_0(F.A \otimes F.B)$$

where  $F.A$  and  $F.B$  are free simplicial resolutions (5.4) of  $A$  and  $B$ , and where  $F.A \otimes F.B$  denotes the simplicial  $\Pi$ -algebra that is  $F_p A \otimes F_p B$  in dimension  $p$ . This gives a well defined functor  $\otimes$  on pairs of  $\Pi$ -algebras by the same machinery as was used in 5.7.

Note that, if  $A$  and  $B$  are free, this definition agrees with the tensor product  $A \otimes B$  defined in 7.1 above because in this case the constant simplicial  $\Pi$ -algebras  $(A)_*$  and  $(B)_*$  are free simplicial resolutions of  $A$  and  $B$ .

**7.4 Derived Functors of Tensor Product.** We similarly define the derived functors  $\otimes_p$  of tensor product for  $p \geq 0$  by the formula

$$A \otimes_p B = \pi_p(F.A \otimes F.B).$$

Again this yields well-defined functors from the category of pairs of  $\Pi$ -algebras to the category of  $\Pi$ -algebras as in 5.7.

**7.5 Proof of Theorem 1.9.** Let  $X$  and  $Y$  be pointed connected CW complexes. By analogy with 3.1 consider the homotopy spectral sequence of the simplicial space  $V.X \wedge V.Y$  that is formed by taking the smash product in each dimension of the simplicial resolutions 2.4 of  $X$  and  $Y$ . As in 3.6 we have a homotopy equivalence

$$\Delta(V.X \wedge V.Y) = \Delta V.X \wedge \Delta V.Y \rightarrow X \wedge Y$$

so that the spectral sequence converges strongly to the homotopy groups of  $X \wedge Y$ .

The  $E^2$  term, moreover, has columns of the form

$$E_{p,*}^2 = \pi_p \pi_*(V.X \wedge V.Y) = \pi_p(\pi_* V.X \otimes \pi_* V.Y)$$

which may be identified with  $\pi_* X \otimes_p \pi_* Y$  by 7.4.  $\square$

#### APPENDIX ON TENSOR PRODUCTS OF $\Pi$ -ALGEBRAS

We show here that the functor  $\otimes$  on pairs of  $\Pi$ -algebras is left adjoint to a functor  $\underline{\text{Hom}}$  (A.5), and has other properties one might expect of a tensor product (A.7). As a preliminary we must consider

A.1 *Smash Multiplication in  $\Pi$* . We define a functor

$$\mu: \Pi \times \Pi \rightarrow \Pi$$

as the composition of two functors  $Sm: \Pi \times \Pi \rightarrow \hat{\Pi}$  and  $E: \hat{\Pi} \rightarrow \Pi$ . To define the first, let  $\hat{\Pi}$  be the category whose objects are the same as the objects of  $\Pi \times \Pi$  but whose morphisms  $(W_1, W_2) \rightarrow (W'_1, W'_2)$  are homotopy classes of maps  $W_1 \wedge W_2 \rightarrow W'_1 \wedge W'_2$  between the smash products. The functor  $Sm: \Pi \times \Pi \rightarrow \hat{\Pi}$  is then the obvious one sending a morphism  $(f_1, f_2)$  to  $f_1 \wedge f_2$ . To define the second functor  $E$ , first choose for each object  $(W_1, W_2)$  of  $\hat{\Pi}$  a homotopy equivalence

$$e_{w_1, w_2}: W_1 \wedge W_2 \xrightarrow{\sim} W$$

with the unique (4.1) corresponding object  $W$  of  $\Pi$ . The functor  $E$  is then required to send a morphism  $f: W_1 \wedge W_2 \rightarrow W'_1 \wedge W'_2$  of  $\hat{\Pi}$  to the unique morphism  $Ef: W \rightarrow W'$  of  $\Pi$  such that the diagram

$$\begin{array}{ccc} W_1 \wedge W_2 & \xrightarrow{e_{w_1, w_2}} & W \\ \downarrow f & & \downarrow E_f \\ W'_1 \wedge W'_2 & \xrightarrow{e_{w'_1, w'_2}} & W' \end{array}$$

commutes.

A.2 *Remark*. If the homotopy equivalences  $e_{w_1, w_2}$  are chosen carefully, then the functor  $\mu$  will be associative, i.e.

$$\mu \circ (1 \times \mu) = \mu \circ (\mu \times 1).$$

On the other hand, if  $T: \Pi \times \Pi \rightarrow \Pi \times \Pi$  is the functor that exchanges components, then  $\mu T$  and  $\mu$  agree on objects and are naturally equivalent, but the natural equivalence will involve changes of orientation and permutations of wedge summands.

A.3 *The Functor  $\underline{\text{Hom}}$* . For every object  $W \in \Pi$  the multiplication  $\mu$  defines a functor  $\mu(-, W): \Pi \rightarrow \Pi$ . If  $A: \Pi \rightarrow \text{Sets}_*$  is a  $\Pi$ -algebra, we write  $A^W$  for the composition

$$A^W: \Pi \xrightarrow{\mu(-, W)} \Pi \xrightarrow{A} \text{Sets}_*.$$

It is readily verified that  $A^W$  is a  $\Pi$ -algebra. If  $A = \pi_* X$  for a space  $X$ , then there is a natural isomorphism  $A^W \cong \pi_* \text{map}_*(W, X)$ .

If we allow  $W$  to vary, then we obtain a contravariant functor

$$A^*: \Pi \rightarrow \Pi\text{-Alg}.$$

If  $B$  is another  $\Pi$ -algebra, then taking  $\Pi$ -algebra maps out of  $B$  into the diagram  $A^*$  gives a contravariant functor

$$\underline{\text{Hom}}(B, A) = \text{Hom}(B, A^*): \Pi \rightarrow \text{Sets}_*.$$

A.4 **PROPOSITION**.  $\underline{\text{Hom}}(B, A)$  is a  $\Pi$ -algebra.

*Proof*. It is sufficient to note that, if  $W = \vee_{i=1}^k S^{n_i}$ , then the natural map

$$A^W \rightarrow \prod_{i=1}^k A^{S^{n_i}}$$

is an isomorphism of  $\Pi$ -algebras. □

A.5 PROPOSITION. *There is a natural bijection of sets*

$$\text{Hom}(A \otimes B, C) \xrightarrow{\cong} \text{Hom}(A, \underline{\text{Hom}}(B, C)).$$

*Proof.* The proposition is immediate if  $A$  and  $B$  are free. For the general case let  $F.A$  and  $F.B$  be free simplicial resolutions of  $A$  and  $B$ . Let  $F.A \tilde{\otimes} F.B$  denote the bisimplicial  $\Pi$ -algebra that is  $F_p A \otimes F_q B$  in dimension  $p, q$ . The diagonal simplicial  $\Pi$ -algebra  $\Delta(F.A \tilde{\otimes} F.B)$  is what we have called  $F.A \otimes F.B$ . By a general fact about bisimplicial sets, there is an isomorphism of the  $\Pi$ -algebra  $\pi_0 \Delta(F.A \tilde{\otimes} F.B)$  with the  $\Pi$ -algebra  $\pi_0 \pi_0(F.A \tilde{\otimes} F.B)$  that is obtained by applying  $\pi_0$  first in the  $q$  direction and then in the  $p$  direction. Moreover, it is easy to see that if  $D.$  is any simplicial  $\Pi$ -algebra, then  $\pi_0 D.$  can be identified with the direct limit  $\varinjlim(D_p)$  of the simplicial diagram. Hence we have

$$\begin{aligned} \text{Hom}(A \otimes B, C) &= \text{Hom}(\pi_0 \pi_0(F.A \tilde{\otimes} F.B), C) \\ &= \text{Hom}(\varinjlim_p \varinjlim_q (F_p A \otimes F_q B), C) \\ &= \varprojlim_p \varprojlim_q \text{Hom}(F_p A \otimes F_q B, C) \\ &\cong \text{Hom}(\varinjlim_p F_p A, \text{Hom}(\varinjlim_q F_q B, C^*)) \\ &= \text{Hom}(A, \underline{\text{Hom}}(B, C)) \end{aligned} \quad \square$$

A.6 COROLLARY. *There is a natural isomorphism of  $\Pi$ -algebras*

$$\underline{\text{Hom}}(A \otimes B, C) \cong \underline{\text{Hom}}(A, \underline{\text{Hom}}(B, C)).$$

*Proof.* If  $W$  and  $W'$  are objects of  $\Pi$  then by remark A.2 we may identify the  $\Pi$ -algebras

$$C^{\mu(W', W)} = C(\mu(-, \mu(W', W))) = C(\mu(\mu(-, W'), W)) = (C^W)^{W'}.$$

This gives an identification of the  $\Pi$ -algebras

$$\underline{\text{Hom}}(B, C)^W = \text{Hom}(B, C^{\mu(*, W)}) = \text{Hom}(B, (C^W)^*) = \underline{\text{Hom}}(B, C^W).$$

From this we obtain a natural bijection of sets

$$\text{Hom}(A, \underline{\text{Hom}}(B, C)^W) = \text{Hom}(A, \underline{\text{Hom}}(B, C^W)) \cong \text{Hom}(A \otimes B, C^W)$$

which implies the result. □

A.7 PROPOSITION. *There are natural isomorphisms*

- (i)  $A \otimes B \cong B \otimes A$
- (ii)  $(A \amalg A') \otimes B \cong (A \otimes B) \amalg (A' \otimes B)$
- (iii)  $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ .

*Proof.* (i) is immediate from the definition. (ii) follows because left adjoints preserve direct limits. (iii) is a consequence of the natural bijections of sets

$$\begin{aligned} \text{Hom}((A \otimes B) \otimes C, D) &\cong \text{Hom}(A, \underline{\text{Hom}}(B, \underline{\text{Hom}}(C, D))) \\ &\cong \text{Hom}(A, \underline{\text{Hom}}(B \otimes C, D)) \\ &\cong \text{Hom}(A \otimes (B \otimes C), D) \end{aligned} \quad \square$$

*Acknowledgements*—I am indebted to Daniel M. Kan for suggesting the approach taken in this paper, as well as for general guidance and encouragement. I am also grateful to David Blanc and Jeff Smith for many stimulating conversations.

#### REFERENCES

1. M. ANDRÉ: *Méthode simpliciale en algèbre homologique et algèbre commutative*, Lecture Notes in Mathematics **32**, Springer, Berlin (1967).
2. A. K. BOUSFIELD and E. M. FRIEDLANDER: *Homotopy theory of  $\Gamma$ -spaces, spectra, and bisimplicial sets*, Geometric Applications of Homotopy Theory II, Lecture Notes in Mathematics **658**, Springer, Berlin (1978), 80–130.
3. A. K. BOUSFIELD and D. M. KAN: *Homotopy limits, completions, and localizations*, Lecture Notes in Mathematics **304**, Springer, New York (1974).
4. R. BROWN and P. J. HIGGINS: Colimit theorems for relative homotopy groups, *J. Pure Appl. Alg.* **22** (1981), 11–41.
5. R. BROWN and J. L. LODAY: van Kampen theorems for diagrams of spaces, with an appendix by M. Zisman. *Topology* **26** (1987), 311–335.
6. R. H. CROWELL: On the van Kampen theorem, *Pacific J. Math.* **9** (1959), 43–50.
7. E. B. CURTIS: Simplicial homotopy theory, *Adv. Math.* **6** (1971), 107–209.
8. S. EILENBERG and J. C. MOORE: Adjoint functors and triples, *Illinois J. Math.* **9** (1965), 381–398.
9. P. J. HILTON: On the homotopy groups of unions of spaces, *Comm. Math. Helv.* **29** (1955), 59–92.
10. P. J. HUBER: Homotopy theory in general categories, *Math. Annalen* **144** (1961), 361–385.
11. D. M. KAN: On the homotopy relation for c.s.s. maps, *Boln. Soc. mat. mexi.* **2** (1957), 75–81.
12. D. M. KAN: On c.s.s. categories, *Boln. Soc. mat. mexi.* **2** (1957), 82–94.
13. F. W. LAWVERE: Functorial semantics of algebraic theories, *Proc. Nat. Acad. Sci. U.S.A.* **50** (1963), 869–872.
14. J. P. MAY: *Simplicial objects in algebraic topology*, van Nostrand, Princeton, (1967).
15. J. P. MAY:  $E_\infty$  spaces, group completions, and permutative categories, in G. Segal, ed., *New Developments in Topology*, London Math. Soc. Lecture Notes 11, Cambridge University Press, 1974, pp. 61–93.
16. S. MACLANE: *Categories for the working mathematician*, Graduate texts in mathematics **5**, Springer, New York (1971).
17. S. MACLANE: *Homology*, Springer, Berlin (1973).
18. B. PAREIGIS: *Categories and functors*, Academic Press, New York (1970).
19. D. G. QUILLEN: *On the (co-) homology of commutative rings*, Proc. Symp. Pure Math. XVII, Amer. Math. Soc., Providence, RI (1970), pp. 65–87.
20. D. G. QUILLEN: *Homotopical algebra*, Lecture Notes in Mathematics no. 43, Springer, Berlin (1967).
21. C. L. REEDY: *Homotopy theory of model categories*, preprint.
22. H. SCHUBERT: *Categories*, Springer, New York (1972).
23. G. SEGAL: Classifying spaces and spectral sequences, *Inst. H. Et. Sci. Math.* **34** (1968), 105–112.
24. G. SEGAL: Categories and cohomology theories, *Topology* **13** (1974), 293–312.
25. G. W. WHITEHEAD: *Elements of homotopy theory*, Springer, New York (1978).

University of Chicago  
 5734 University avenue  
 Chicago  
 IL 60637  
 U.S.A.