Finite and transversely isotropic elastic cylinders under compression with end constraint induced by friction

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A B S T R A C T

This paper derives an exact solution for the non-uniform stress and displacement fields within a finite, transversely isotropic, and linear elastic cylinder under compression with a kind of radial constraint induced by friction between the end surfaces of the cylinder and the loading platens. The main feature of the present work is the introduction of a general solution form for Lekhnitskii's stress function such that the governing equation and all end and curved boundary conditions of the cylinder are satisfied exactly. Two different solutions were obtained corresponding to the real or complex characteristic roots of the governing equation, depending on the combination of the elastic material constants. The solution by Watanabe [Watanabe, S., 1996. Elastic analysis of axi-symmetric finite cylinder constrained radial displacement on the loading end. Structural Engineering/Earthquake Engineering JSC 13, 175s–185s] for isotropic cylinders under compression test can be recovered as a special case. Our numerical results show that both the non-uniform stress distribution and the difference between the apparent and the true Young's moduli of the cylinder are very sensitive to the anisotropy of Young's moduli, Poisson's ratios and shear moduli. A more distinct bulging shape of the cylinder is expected when anisotropy in shear modulus is strong, the cylinder is relatively short, and the end constraint is large. The bulging shape, however, does not depend strongly on anisotropy of either Poisson's ratio or Young's modulus.

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1. Introduction

Uniaxial and triaxial compressions of solid circular cylinders of finite length are among the most popular methods in obtaining the elastic moduli and the compressive strength of materials, including soil, rock, concrete, ice, biomaterials, composites, and ceramics. It has, however, long been recognized that friction exists between two end surfaces of the cylinder and the loading platens. Consequently, the stress distribution within a cylinder under compression is normally non-uniform.

There were numerous experimental studies investigating the effect of friction on the moduli and the strength of solid cylinders. Experimental studies on soils include the works by Bishop and Green (1965), Olson et al. (1964), Rowe and Barden (1964), Kirkpatrick and Belshaw (1968), and Hettler and Vardoulakis (1984); those on rocks include Hudson et al. (1971), Labuz and Bridell (1993), Hallbauer et al. (1973), and Vardoulakis et al. (1998); and those on concrete include Read and Hegemier (1984), Torrenti et al. (1993), Starodubsky et al. (1994), and Choi and Shah (1998).

Filon (1902) analyzed theoretically the non-uniform elastic stress distribution within a finite cylinder when the radial displacement of the two end surfaces is perfectly constrained. Since then the solution of non-uniform stress distribution has been improved by many researchers, including Pickett (1944), Balla (1960a,b), Peng (1971, 1973), Al-Chalabi (1972), Brady (1971a,b), Moghe and Neff (1971), Al-Chalabi and Huang (1974), Al-Chalabi et al. (1974), and Watanabe (1996). For more general cases of non-axisymmetric boundary conditions, Chau and Wei (2000) provided a general solution framework for finite isotropic cylinders.

Among previous theoretical studies, Filon (1902), Moore (1966), Edelman (1949), Moghe and Neff (1971), Nayak (1974) and Chau (1997) considered the correction factor for estimating the “true Young's modulus” which is needed to multiply to the “apparent Young's modulus” (which can be obtained by assuming a uniform stress distribution within a cylinder). Experimental studies by Gent and Lindley (1959) on highly elastic rubber blocks show that the apparent Young's modulus may differ significantly from the true Young's modulus, depending on the shape or aspect ratio of the blocks. Such correction factor is very useful in the interpretation of the moduli and strength of solid cylinders (e.g. Kotte et al., 1969; Hawkes and Mellor, 1970; Nayak, 1974).
All of the previous studies are for isotropic solid circular cylinders. Many natural and man-made materials are, however, inherently anisotropic, and many kinds of materials can be modeled adequately by transversely isotropy, such as ice (Lliboutry, 1993), rock (Nova, 1980; Cazacu and Cristescu, 1999), fiber-reinforced composite (Theocaris, 1995), cartilage (Bursac et al., 1999), and semiconductors (Wei, 2007, 2008). Therefore, in this study we investigate the non-uniform stress distribution within a transversely isotropic solid circular cylinder subjected to compression with end constraint induced by friction.

Since the main objective of this paper is to provide an analytical solution for transversely isotropic cylinders, theoretical analyses for these cylinders are also reviewed briefly here. For infinitely long transversely isotropic cylinders, Kasano et al. (1980, 1982) have considered the stress distribution due to both ring loads and two diametral point loads. The orthogonality of the “end problem” of transversely isotropic cylinders was considered by Byrnes and Archer (1975). For finite transversely isotropic cylinders, Chau (1994) considered the free vibrations problem, while Chau (1992, 1993, 1995) investigated the instabilities of pressure-sensitive cylinders. Applying Elliot’s (1948) stress function, Mitra (1965) considered the axisymmetric deformation of a finite transversely isotropic cylinder subjected to arbitrary traction on the end surfaces and displacement on the curved surface. The axisymmetric problems involving more general boundary conditions were considered by Vendhan and Archer (1978) and by Grigorenko and Kryukov (1998). However, there is no formulation for solving finite and transversely isotropic cylinders under compression test with end friction.

In this study, the stress function of Lekhnitskii (1963) will be employed. A new solution form of Fourier–Bessel expansions will be proposed in order to satisfy the boundary conditions of both end and curved boundaries exactly. Emphasis, here, is on how anisotropy of the solid affects the stress distributions within transversely isotropic cylinders under compression test with end friction. The results of this study should provide interplay between anisotropy and friction on the non-uniform stress distribution within the solid cylinder subjected to both unconfined and confined compression test (i.e. uniaxial compression and the conventional triaxial compression). In addition, the stress analysis of finite cylinders has also been found useful in the analysis of fibre-reinforced composites (e.g. Smith and Spencer, 1970; Wu et al., 2000).

2. Hooke’s law for transversely isotropic solids

Consider a homogeneous and transversely isotropic cylinder of diameter D (or 2R) and length H (or 2h) with the two end surfaces parallel to the plane of isotropy. For the cylindrical coordinate system (r,θ,z) shown in Fig. 1, the generalized Hooke’s law for transversely isotropic solids can be written as

\[
\begin{align*}
\tau_{rr} &= \alpha_{11}\sigma_{rr} + \alpha_{12}\sigma_{r\theta} + \alpha_{13}\sigma_{rz}, \\
\tau_{r\theta} &= \alpha_{21}\sigma_{rr} + \alpha_{22}\sigma_{r\theta} + \alpha_{23}\sigma_{rz}, \\
\tau_{rz} &= \alpha_{31}\sigma_{rr} + \alpha_{32}\sigma_{r\theta} + \alpha_{33}\sigma_{rz},
\end{align*}
\]

\[
\gamma_{11}' = \alpha_{61}\sigma_{rr},
\]

where

\[
\begin{align*}
\alpha_{11} &= \frac{1}{E_1}, & \alpha_{12} &= \frac{\nu_{12}}{E_1}, & \alpha_{13} &= \frac{-1}{E_1}, & \alpha_{33} &= \frac{1}{E_1}, \\
\alpha_{21} &= \frac{1}{E_2}, & \alpha_{61} &= \frac{2}{E_2}, & \alpha_{31} &= \frac{\nu_{13}}{E_1}, & \alpha_{32} &= \frac{1}{E_2},
\end{align*}
\]

The stress tensor is denoted by \( \sigma \), and the normal and shear strains by \( \varepsilon \) and \( \gamma \), respectively. Physically, \( E_1 \) and \( E_2 \) are the Young’s moduli governing axial deformations in the planes of isotropy (i.e. any plane parallel to the two end surfaces) and along direction perpendicular to it (i.e. parallel to the z-axis), respectively. The Poisson’s ratios \( \nu_1 \) and \( \nu_2 \) characterize transverse reductions in the plane of isotropy under tension in the same plane and under tension along the z-axis, respectively. The shear moduli for the plane of isotropy and for planes parallel to the z-axis are denoted by \( G_1 \) and \( G_2 \), respectively.

3. Finite cylinders under compression with end radial constraint induced by friction

Friction inevitably exists between the loading platens and the two end surfaces of the cylinder in usual compression tests. The end surfaces are, therefore, constrained from free expansion induced by Poisson effect. In fact, numerous experimental studies show that friction does exist in compression of concrete (Torrenti et al., 1993; Starodubsky et al., 1994; Choi and Shah, 1998), soil (Hettler and Vardoulakis, 1984; Drescher and Vardoulakis, 1982) and rock (Labuz and Bridell, 1993; Vardoulakis et al., 1998). Consequently, the deformation of the cylinder is not uniform, and a typical deformed shape of a solid cylinder under compression is illustrated in Fig. 1. (Note that a more detailed analysis on the deformed shape will be given in Section 8.2.3.) The boundary conditions for a finite solid circular cylinder under a confined compression test (or the conventional triaxial test) can be written as

\[
\begin{align*}
\sigma_{rr} &= p_0, & \text{on } r &= R, \\
\sigma_{zz} &= 0, & \text{on } r &= R, \\
\partial w \bigg/ \partial r &= 0, & \text{on } z &= \pm h, \\
u &= f(r), & \text{on } z &= \pm h, \\
\int_0^R 2\pi r \sigma_{zz} \, dr &= P, & \text{on } z &= \pm h,
\end{align*}
\]

where \( P \) is the total load acting on the loading platens. Physically, these boundary conditions imply that the cylinder is subjected to an axial compression of magnitude \( P \) with confining stress of \( p_0 \) and with no end rotation. In rotation, boundary condition Eq. (5) ensures that the loading platens to remain horizontal at all time. Boundary condition Eq. (6) describes the constraint on the radial displacement induced by end friction. By adopting the approach by Chau (1997), Watanabe (1996) and Wei (2007), we assume that \( f(r) = \frac{\pi h P}{2R^2} \) in Eq. (6), which implies a uniform radial strain on the two end surfaces. The factor \( \alpha \) represents the degree of constraint on the radial displacement on the end surfaces. If friction is negligible, the end surface is free to expand and \( \alpha = 1 \); if the radial displacement on the end surfaces is completely constrained, no slip occurs between the cylinder and loading platens and \( \alpha = 0 \). In usual compression tests, partial slip may often occur and \( 0 < \alpha < 1 \), depending on the contact condition between the end surfaces and the loading platens. In addition, it is possible that slippage may not be uniform or it may not be proportional to the radial distance from the center. For example, based upon the result of finite element analysis, Watanabe (1996) proposed another type of end boundary displacement as \( f(r) = \frac{\pi h P}{2R^2} \left[ \frac{2(1-\nu)}{(1-\alpha)^2} \right] \). The method of solution to be discussed next will be applicable for any general form of \( f(r) \). Note that boundary condition Eq. (6) is not precisely a frictional condition, although it simulates frictional constraint and allows for the possibility of considering cases intermediate between full and zero lateral constraint.

4. Stress function for axisymmetric deformation of transversely isotropic solids

As shown by Lekhnitskii (1963), the displacement components \( u \) and \( w \) can be expressed by a single function \( \phi \) for the case of axisymmetric deformation as
where \( \rho = |r|; \ \lambda_s \) is the s-th root of \( J_{2s}(\lambda_s) = 0; \ \gamma_1 = \lambda_s \kappa \) and \( \zeta_n = n\pi/\kappa; \ \kappa \) is a geometrical shape ratio defined as \( \kappa = h/k; \ A_n, B_n, C_n \) and \( D_n \) are unknown coefficients to be determined from boundary conditions; \( p_1, p_2, q_1 \) and \( q_2 \) are constants depending on material properties; and \( J_n(x) \) and \( I_n(x) \) are Bessel functions and modified Bessel function of the first kind of zero order, respectively (Abramowitz and Stegun, 1965).

Substitution of Eq. (13) into Eq. (11) yields

\[
q_{12} = \frac{[c + e] \pm \sqrt{(c + e)^2 - 4d}}{2d}, \quad p_{12} = q_{12} \sqrt{d}
\]  

(14)

It can be proved that \( p_{12} = q_{12} = 1 \) for isotropic solids, and \( p_{12} \) and \( q_{12} \) are either real \((p_{12} \neq q_{12} \neq 1)\) or complex conjugate pair, but not pure imaginary for transversely isotropic solids. The elastic constants for magnesium, some particular species of Douglas fir, spruce and pine have been found leading to real \( q_{12} \), while those of zinc and some semiconductors lead to complex \( q_{12} \) (Byrnes and Archer, 1975). In addition, it is worthy to note that \( E_{ij}/E_i > 1 \), or \( v_{ij} ; v_{ij} < 1 \) or \( G_i; G_i > 1 \) usually leads to real \( q_{12} \), while the other values may lead to complex \( q_{12} \).

In particular, if \( q_{12} \) are real \((p_{12} \neq q_{12} \neq 1)\), we propose the general expression for the stress function \( \varphi \) as

\[
\varphi = -R^2 q_0 \left\{ A_0 \frac{\kappa^2 \eta^2}{6} + C_4 \frac{\kappa \eta^2}{2} + \sum_{n=1}^{\infty} \frac{\sin(\pi \eta)}{\pi} \left\{ A_n \varphi_n \left( p_{12} \varphi_n \right) \right. \\
+ B_n \varphi_n \left( p_{22} \varphi_n \right) \right\} + \sum_{n=1}^{\infty} \left\{ J_n(\lambda_{1s}) \left\{ C_n \sinh(q_1 \gamma_{1n}) + D_n \sinh(q_2 \gamma_{1n}) \right\} \right. \\
+ \left. \sum_{n=1}^{\infty} J_n(\lambda_{2s}) \left\{ C_n \cosh(q_1 \gamma_{2n}) + D_n \cosh(q_2 \gamma_{2n}) \right\} \right\}
\]  

(15)

where \( q_0 \) is the mean normal stress on the end surfaces defined as \( q_0 = P/\pi R^2 \), and \( A_0, C_4, A_n, B_n, C_n \) and \( D_n \) are real unknown constants to be determined. Note that additional terms corresponding to \( A_0 \) and \( C_4 \) have been added and they will lead to uniform normal stresses. Similar to the case of isotropic cylinders, both of the infinite series in Eq. (15) are required to satisfy the boundary conditions given in Eqs. (3)–(7).

If \( q_{12} \) are complex, say \( q_{12} = q_{0} \pm q_{0} i \), the general expression for the stress function \( \varphi \) can be recast as

\[
\varphi = -R^2 q_0 \left\{ A_0 \frac{\kappa^2 \eta^2}{6} + C_4 \frac{\kappa \eta^2}{2} + \sum_{n=1}^{\infty} \frac{\sin(\pi \eta)}{\pi} \left\{ A_n \left( \frac{\varphi_n \left( p_{12} \varphi_n \right)}{\lambda_{1s}} \right) \right. \\
+ B_n \left( \frac{\varphi_n \left( p_{22} \varphi_n \right)}{\lambda_{2s}} \right) \right\} + \sum_{n=1}^{\infty} \left\{ J_n(\lambda_{1s}) \left\{ C_n \sinh(q_1 \gamma_{1n}) \cos(q_2 \gamma_{1n}) + D_n \cosh(q_0 \gamma_{1n}) \sin(q_0 \gamma_{1n}) \right\} \right. \\
+ \left. \sum_{n=1}^{\infty} J_n(\lambda_{2s}) \left\{ C_n \cosh(q_1 \gamma_{2n}) \sin(q_2 \gamma_{2n}) + D_n \sinh(q_0 \gamma_{2n}) \cos(q_0 \gamma_{2n}) \right\} \right\}
\]  

(16)

where \( A_0, C_4, A_n, B_n, C_n \) and \( D_n \) are real unknown coefficients to be determined by boundary conditions Eqs. (3)–(7).

It should be emphasized whether Eq. (15) or Eq. (16) is used, depending solely on the specific type of the transversely isotropic solids.
solid through the calculation of the roots \(q_{1,2}\) according to Eq. (14). To distinguish the two different kinds of solutions for transversely isotropic solids, we call those solutions corresponding to real \(q_{1,2}\) as Solution I and those to complex \(q_{1,2}\) as Solution II.

Moreover, it is straightforward to show that both Eq. (15) and (16) satisfy governing equation Eq. (11) automatically. Before the boundary conditions Eqs. (3)–(7) are considered, stresses and displacements will first be expressed in terms of the unknown constants in the next section.

### 6. Expressions for stress and displacement components

#### 6.1. Solution I: solution corresponding to real \(q_{1,2}\)

In the case of real \(q_{1,2}\), substitution of Eq. (15) into Eqs. (8), (9) and the relation between the strain and displacement components, then into Eq. (1) yields the following expressions for the stress and displacement:

\[
\begin{align*}
\sigma_{rr}/q_0 &= A_0e + (a + b)C_0 + \sum_{n=1}^{\infty} \cos(n\pi\eta) |A_n| \Pi_1(p_1, \rho) + B_n \Pi_2(p_2, \rho)|
\end{align*}
\]

(17)

\[
\begin{align*}
\sigma_{rr}/q_0 &= \sum_{n=1}^{\infty} \sin(n\pi\eta) |A_n| \Pi_2(p_1, \rho) + B_n \Pi_2(p_2, \rho)|
\end{align*}
\]

(18)

\[
\begin{align*}
\frac{u}{q_0}R &= (1 - b)(a_{11} - a_{12}) \left\{ \left[ A_0p_{11}(p_1, \rho) + \sum_{n=1}^{\infty} \cos(n\pi\eta) |A_n| p_{11}(p_1, \rho) \right] 
\right. \\
&+ B_0p_{12}(p_2, \rho) \left. + \sum_{n=1}^{\infty} \frac{I_1(\lambda_n \rho)}{\lambda_n} \left[ C_n |q_{1}, q_{1}, \eta| + D_n q_{3} \cos(q_{1}, q_{1}, \eta) \right] \right\}
\end{align*}
\]

(19)

\[
\begin{align*}
\frac{w}{q_0}R &= -2a_{44}C_0 + A_0(a_{33}d - 2a_{13}e)X_0 \eta + \sum_{n=1}^{\infty} \sin(n\pi\eta) |A_n| \Pi_3(p_1, \rho)
\]

(20)

where

\[
\begin{align*}
\Pi_1(x, \rho) &= (ax^2 - e) |l_0(x_{\rho, \rho})| + (b - a)x |l_1(x_{\rho, \rho})| \\
\Pi_2(x, \rho) &= (ex - x^3) |l_0(x_{\rho, \rho})| \\
\Pi_3(x, \rho) &= [-aq_0x^2 + (a_{33}x - 2a_{13}e) |l_0(x_{\rho, \rho})| \\
\Gamma(\rho) &= -aq_0(\lambda_{\rho, \rho}) + (a - b) \frac{I_1(\lambda_{\rho, \rho})}{\lambda_{\rho, \rho}}
\end{align*}
\]

(21)–(24)

#### 7. Determination of unknown coefficients

##### 7.1. Solution I: all roots are real

The boundary condition \(\sigma_{rr} = 0\) on the curved surface \(\rho = 1\) (or \(r = R\)) leads to

\[
\begin{align*}
A_0 &= E_0\Pi_2(2, 1), \quad B_0 = -E_0\Pi_1(2, 1)
\end{align*}
\]

(31)

where \(E_0\) is an extra constant introduced to simplify the later presentation and it will be fixed later such that the subsequent formulas can be expressed in a more efficient manner.

The boundary condition \(\partial w/\partial r = 0\) on the two end surfaces \(\eta = \pm 1\) (i.e., \(z = \pm h\)) leads to

\[
\begin{align*}
C_1 &= F_1(a_{44} - (a_{33}d - 2a_{13}e)q_0^2) \sin(q_1q_1) \\
D_1 &= -F_1(a_{44} - (a_{33}d - 2a_{13}e)q_0^2) \sin(q_1q_1)
\end{align*}
\]

(32)

where \(F_1\) is another constant introduced to simplify the subsequent presentation.
The radial stress $\sigma_{rr}$ on the curved surface $r = 1$ (i.e. $r = R$) can be obtained by setting $\rho = 1$ in Eq. (17):

$$\sigma_{rr}/q_0 = A_0 e + (b + 1)C_0 + \sum_{n=1}^{\infty} \cos(n\pi\eta)[A_n \Pi_{1}(p_1, 1) + B_n \Pi_{1}(p_2, 1)]$$

$$+ \sum_{n=1}^{\infty} \left[ C_n q_{1}^1 \Gamma(1) + e^{q_{1}^2} \Gamma(1) + \cos(q_{1} 1) \eta \right]$$

$$+ D_n [q_{2}^1 \Gamma(1) + e^{q_{2}^2} \Gamma(1) + \cos(q_{2} 1) \eta]$$

(33)

By applying a Fourier expansion for the hyperbolic cosine in Eq. (33) and then expressing the result in terms of the constants $E_m$ and $F_n$, we have

$$\sigma_{rr}/q_0 = A_0 e + (b + 1)C_0 + \sum_{n=1}^{\infty} F_n Q_{m0}/2$$

$$+ \sum_{n=1}^{\infty} \left[ E_0 A_n + \sum_{n=1}^{\infty} F_n Q_{m0} \right] \cos(n\pi\eta)$$

(34)

where

$$Q_m = J_0(1) \left\{ [a_{44} - (a_{33}d - 2a_{13}e)] q_{2}^2 \right\} \left\{ \cos(q_{1} 1) \right\}$$

$$- a_{44} - (a_{33}d - 2a_{13}e) q_{2}^2 \left\{ \cos(q_{1} 1) \right\}$$

(35)

$$G_m(x) = \frac{2(-1)^n x^n \sin(q_{1} x)}{y^{2} + (n\pi)^2}$$

(36)

$$\Lambda_n = \Pi_{1}(p_2, 1) \Pi_{1}(p_1, 1) - \Pi_{1}(p_1, 1) \Pi_{1}(p_2, 1)$$

(37)

The boundary condition $\sigma_{rr} = 0$ on $\rho = 1$ can now be applied and the following relations between $A_n$ and $C_n$ and between $E_m$ and $F_n$ are obtained:

$$A_0 e + (b + 1)C_0 + \sum_{n=1}^{\infty} F_n Q_{m0}/2 = P_0 \frac{Q_0}{q_0}$$

(38)

$$E_0 A_n + \sum_{n=1}^{\infty} F_n Q_{m0} = 0$$

(39)

Substitution of Eqs. (31) and (32) into Eq. (19) and set $\eta = \pm 1$ yield the following expression for the radial displacement on the two end surfaces (i.e. on $z = \pm h$) as

$$u \frac{q_0 R}{q_0} = (1 - b)(a_{11} - a_{12}) \left\{ C_0 (1 + \sum_{n=1}^{\infty} \cos(n\pi\eta) A_n \Pi_{1}(p_1, 1) + B_n \Pi_{1}(p_2, 1) \right\}$$

$$- \Pi_{1}(p_2, 1) \Pi_{1}(p_1, 1) + \sum_{n=1}^{\infty} \frac{J_1(1) \Pi_{1}(p_1, 1)}{\Lambda_n} \right\}$$

(40)

where

$$\Omega_n = -a_{44} - (a_{33}d - 2a_{13}e) q_{2}^2 q_{1} \sin(q_{1} 1) \cos(q_{1} 1)$$

$$- a_{44} - (a_{33}d - 2a_{13}e) q_{2}^2 q_{2} \cos(q_{1} 1) \sin(q_{1} 1)$$

(41)

To apply the end boundary condition Eq. (6), we first expand Eq. (40) into a Fourier–Bessel series as

$$u \frac{q_0 R}{q_0} = (1 - b)(a_{11} - a_{12}) \sum_{n=1}^{\infty} \frac{J_1(1) \Pi_{1}(p_1, 1)}{\Lambda_n} - \frac{2C_0}{J_0(1)} + F_0 \Omega_n + \sum_{n=1}^{\infty} E_n R_n$$

(42)

where

$$R_n = \frac{(-1)^n}{\pi} \left\{ (p_2 - p_1) q_{1}^2 q_{2}^1 H_m(p_1) - (p_1 - p_2) q_{1}^2 q_{2}^1 H_m(p_2) \right\}$$

$$- \frac{2(-1)^n x^n \sin(q_{1} x)}{y^{2} + (n\pi)^2}$$

(43)

To match the boundary condition Eq. (6) with the radial displacement given in Eq. (42), (6) is also expanded into a Fourier–Bessel series as

$$u(\rho) = \sum_{n=1}^{\infty} a_n J_1(\lambda_n \rho)$$

(45)

where $a_n$ can be obtained as

$$a_n = - \frac{2R(1 - v_1) E_p p_0 - v_1 E_p p_0}{2E_1 E_2 J_0(1)}$$

(46)

$$\frac{1}{\pi} (1 - v_1) E_p p_0 - \frac{1}{E_{r} q_0 - v_1}$$

(47)

Finally, by comparing the coefficients of Eqs. (45) and (42), we have

$$\sigma_{zz}/q_0 = -A_0 d - 2C_0 + \sum_{n=1}^{\infty} \cos(n\pi\eta)[A_n \Pi_{1}(p_1, 1) + B_n \Pi_{1}(p_2, 1)]$$

$$+ \sum_{n=1}^{\infty} \left\{ C_n q_{1} \Gamma(1) + e^{q_{1}^2} \Gamma(1) + \cos(q_{1} 1) \eta \right\}$$

$$+ D_n [q_{2}^1 \Gamma(1) + e^{q_{2}^2} \Gamma(1) + \cos(q_{2} 1) \eta]$$

(48)

Substitution of Eq. (48) into Eq. (7) with $\eta = \pm 1$ leads to

$$-2C_0 - dA_0 + \sum_{n=1}^{\infty} E_n X_n = 1$$

(49)

where

$$X_n = \frac{2(-1)^n}{\pi} \left\{ (d - c p_1^2) (p_2 - p_1) q_{1}^2 q_{2}^1 H_m(p_1) \right\}$$

$$\frac{1}{p_1} I_1(q_{1} p_{1}^2)$$

(50)

To determine the unknown coefficients $A_0$, $C_0$, $E_m$, and $F_n$, the coupled system of equations, Eqs. 38, 39, 47 and 49, has to be solved simultaneously. For numerical implementation, we can truncate the infinite series in these equations and retain only the first $n$ and $s$ terms. Then, there will be $(s + n + 2)$ equations for the $(s^2 + n^2 + 2)$ unknown coefficients of $A_0$, $C_0$, $E_m$, and $F_n$. Finally, $A_0$, $B_0$, $C_0$, and $D_0$ can be obtained by substitution of $F_n$ and $E_m$ into Eqs. (31) and (32). Once these coefficients are determined, the stress and displacement fields inside the cylinder can be evaluated according to Eqs. (17)-(20).

7.2. Solution II: all roots are complex

By adopting the procedure used in obtaining Solution I, the unknown coefficients $A_0$, $B_0$, $C_0$, and $D_0$ for Solution II can be determined. Without going into the details, we simply report the following formulas:

$$\bar{A}_0 = E \cdot \bar{A}_0, \bar{B}_0, \bar{C}_0, \bar{D}_0$$

(51)

$$\bar{T}_1 = F \cdot \bar{T}_1, \bar{R}_1, \bar{S}_1$$

(52)

where

$$\bar{R}_1 = \bar{R}_1(q_6, q_6), \bar{S}_1 = \bar{S}_1(q_6, q_6)$$

(53)

$$\bar{R}_2(q_6, q_6) = a_6 a_6 + 2a_6 a_6 \sin(q_6 1) \cos(q_6 1)$$

$$+ \sin(q_6 1) \cos(q_6 1)$$

(54)
The present solution with that by Watanabe (1996) for relative axial displacement $w/w_0$ of loading end in isotropic case.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>Present solution</th>
<th>Solution by Watanabe (1996)</th>
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<tbody>
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<td></td>
<td>Analytic solution</td>
<td>FEM</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>$z$</td>
<td>$x$</td>
</tr>
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<tr>
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</tbody>
</table>

Note that $\Gamma_0$ can be obtained by interchanging the terms "cosh$q_i\gamma_i$" and "sinh$q_i\gamma_i$" with each other in Eq. (65), and replacing "$-$" by "$+$" in front of the second bracket in the numerator of Eq. (65).

The unknown coefficients $A_0$, $C_0$, $A_n$, and $C_n$ can be first solved from system of equations of Eqs. (55)-(58). Then, $A_n$, $B_n$, $C_n$, and $D_n$ can be obtained by substitution of $F_n$ and $E_n$ into Eqs. (51) and (52). Once these coefficients are solved, the stress and displacement fields inside a transversely isotropic cylinder corresponding to Solution II can be evaluated using Eqs. (25)-(28).

8. Numerical results and discussions

8.1. Isotropic cylinders under compression

8.1.1. Comparison of the isotropic limit to Watanabe (1996) solution

As mentioned in Section 1, non-uniform deformation of a finite isotropic cylinder under confined compression has been investigated by Watanabe (1996). Therefore, the isotropic limit of the present solution for finite transversely isotropic cylinders will be compared to that by Watanabe (1996). In particular, we will consider the limiting case of both Solutions I and II to Watanabe's (1996) solution for isotropic cylinders. As an example, we consider the unconfined compression (i.e. $p_0 = 0$) of an isotropic cylinder with $\nu = 0.2$, and at the same time we also consider the compression of two "nearly-isotropic cylinders", taken as the limiting cases from both Solutions I and II. For real $q_{1,2}$ or Solution I, we set $E_1 = E_2 = E$, $\nu_1 = 0.2$, and $G_1 = G_2$; for complex $q_{1,2}$ or Solution II, we consider the limiting case of $E_1 = E_2 = E$, $\nu_2 = 0.2$, $\nu_1 = 0.001$, and $G_1 = G_2$. Table 1 compares the normalized axial displacement at the end surface ($w/w_0$) obtained by the limiting cases of Solutions I and II, and by Watanabe (1996), where $w_0$ is the axial displacement for the case of smooth contact (i.e. no end friction).

8.1.2. The correction factor for Young's modulus for isotropic cylinders

For isotropic cylinders compressed between two rough end platens, Chau (1997) derived an approximate solution for the ratio of the true Young's modulus $E$ and the apparent Young's modulus $\bar{E}$ (which is obtained by assuming a uniform stress within the isotropic cylinder). The exact solution for the
The parameters used are, however, obtained by Watanabe (1998). Although the details are not given here, we have been able to reproduce Fig. 4 of Watanabe (1998) by considering the isotropic limits of both Solutions I and II. Thus, the validity of the present solution is again illustrated.

8.2. Transversely isotropic cylinders under compression

The main contribution of this paper is in deriving an anisotropic counterpart of the analytical solution of isotropic cylinders under compression obtained by Watanabe (1996). Before parametric studies on the degree of anisotropy are carried out, three anisotropic factors are introduced:

\[ E_L = \frac{E}{1 - \nu_l \nu_m}, \quad E_T = \frac{E}{1 - \nu_l \nu_m}, \quad G_T = \frac{G}{1 - \nu_l \nu_m} \]  

They characterize the effects of the anisotropy in Young’s modulus, Poisson’s ratio, and shear modulus of the solids, respectively.

8.2.1. Stress patterns within the transversely isotropic cylinders

One application of the non-uniform stress field predicted by the present theory is the interpretation of the plastic or inelastic zone (as well as the failure pattern) of compressed cylinders (e.g. Filon, 1902; Balla, 1960a,b; Hawkes and Mellor, 1970; Kotte et al., 1969). Although strictly speaking elastic stress distribution ceases to apply once inelastic deformation occurs within the cylinder, previous studies show that the predicted failure mode, in general, agrees well with those of experiments if the cylinder is brittle, such as concrete and rock (Filon, 1902; Kotte et al., 1969; Momber, 2000). Therefore, elastic stress analysis does provide useful insight and meaningful estimation of the actual stress distribution before the brittle cylinder breaks under compression.

There are several ways to present contour plot for the non-uniform stress distribution within a cylinder. One can plot all normal and shear stresses by one (e.g. Peng, 1971; D’Appolonia and Newmark, 1951; Al-Chalabi and Huang, 1974), the principal stresses (e.g. Filon, 1902; Kotte et al., 1969), or a measure of equivalent shear stress (e.g. Balla, 1960a,b; Hawkes and Mellor, 1970). In this study, the normalized stress \( \tau_0/q_0 \) is defined as

\[ \frac{\tau_0}{q_0} = \frac{1}{q_0} \sqrt{\frac{1}{6} \left[ (\sigma_{rr} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{tt})^2 + (\sigma_{tt} - \sigma_{rr})^2 \right] + \sigma_{rr}^2} = \omega \]  

will be adopted. Note that the octahedral shear stress \( \tau_o \) is related to \( \tau_0 \) by \( \tau_0 = \sqrt{2/3} \tau_o \).

Fig. 2a shows the contour of \( \tau_0/q_0 \) in a quarter of the meridian plane for \( E_L = E_T = E, \nu_l = 0.2, \nu_T = 0.99999, G_T = G_l \) and for \( x = 0.0, \) and \( \kappa = 2.0 \). As mentioned by Balla (1960a), the shadow areas can be understood as a kind of plastic or inelastic domain, and the actual size of the inelastic zone depends on the threshold stress level for yielding or cracking. Alternatively, Hawkes and Mellor (1970) proposed to plot an equivalent stress measure which is more relevant to friction type of the geomaterials:

\[ C(\sigma_1, \sigma_3, \mu) = \frac{\sigma_1}{2} \left[ (\mu^2 + 1)^{1/2} - \mu \right] - \frac{\sigma_3}{2} \left[ (\mu^2 + 1)^{1/2} + \mu \right] \]  

where \( \mu \) is the coefficient of internal friction of the material, and \( \sigma_1 \) and \( \sigma_3 \) are the maximum and minimum principal stresses, respectively. When Eq. (71) is interpreted using the Mohr-Coulomb failure criterion (a criterion commonly used for geomaterials), \( C \) can be set to the intrinsic shear strength \( S_0 \). When it is interpreted using the McClintock and Walsh (1963) cracking criterion, \( C \) can be set to \( 2\sigma_r \), where \( \sigma_r \) is the uniaxial tensile strength of the solid (Hawkes and Mellor, 1970). Note that the McClintock-Walsh criterion is for wing crack initiation from an inclined frictional shear crack under axial compression.

Fig. 2b plots \( C \) for \( \mu = 0.7 \) and for one quarter of the meridian plane, the rest being symmetrical. The other parameters used are the same as those for Fig. 2a. Despite the differences in definitions, the patterns of inelastic deformation interpreted from these plots are indeed similar. In the following discussions, we will focus on the \( C \)-plot of Hawkes and Mellor (1970) and use Fig. 2b as the isotropic basis to investigate the effect of the degree of anisotropy on the pattern and the magnitude of the inelastic zones. Note that \( C = 0.264 \) has been arbitrarily set as the threshold for inelastic deformation. The actual value of \( C \) for a specific frictional geomaterial can be obtained by the uniaxial tensile strength test.

Fig. 3 plots the contours of \( C \) for various values of \( \chi_E \) for \( \chi_E = \chi_C = 1.0, \nu_l = 0.20, \kappa = 2.0, \phi = 0.0, \) and \( \mu = 0.7 \). Note that \( \mu = 0.7 \) is chosen for frictional geomaterials, such as rocks. The present solution can be applied to other brittle materials by choosing suitable values for \( \mu \). The two figures for \( \chi_E \geq 1 \) are obtained from Solution I, while the other two figures for \( \chi_E < 1 \) are obtained from Solution II. Fig. 3 shows that a smaller anisotropic factor \( \chi_E \) will result in a larger inelastic zone. For isotropic solids, as sug-

![Fig. 2. Contour plots of non-uniform stress distribution in a quarter of the meridian plane: (a) normalized stress \( \tau_0/q_0 \) defined in Eq. (70); (b) contour of \( C \) defined in Eq. (71). The parameters used are \( E_L = E_T = E, \nu_l = 0.2, \nu_T = 0.99999, G_T = G_L, \mu = 0.7, \phi = 0.0 \) and \( \kappa = 2.0 \).]
suggested by Filon (1902), the contour plot of \( \chi_E = 1 \) indicates that two conical fragments will fall off from the loading platens; and this observation generally agrees with those of experiments for concrete and isotropic rock (Kotte et al., 1969). But, for the case of highly anisotropic cylinders (e.g. \( \chi_E = 2.0 \)), such conclusion will no longer be valid. Therefore, for anisotropic cylinders, such as composite cylinders, the failure pattern may differ significantly from that of isotropic materials.

Fig. 3. Contour plots of \( C \) for various anisotropic factors \( \chi_E \) for \( \chi_r = \chi_{G} = 1.0, \nu_t = 0.20, \kappa = 2.0, \alpha = 0.0, \) and \( \mu = 0.7 \).

Fig. 4. Contour plots of \( C \) for various anisotropic factors \( \chi_V \) for \( \chi_E = \chi_{G} = 1.0, \nu_t = 0.20, \kappa = 2.0, \alpha = 0.0, \) and \( \mu = 0.7 \).
Fig. 4 plots the contours of C for various values of \( \chi_v \). The other parameters are the same as those used in Fig. 3. The two figures for \( \chi_v \leq 1 \) are obtained from Solution I, while the other two figures for \( \chi_v > 1 \) are obtained from Solution II. In general, the size of the inelastic zone increases with \( \chi_v \). Hence, the degree of anisotropy in Poisson’s ratio may lead to a larger inelastic zone in the cylinder.

Fig. 5 plots the contours of C for various values of \( \chi_c \). The other parameters are the same as those used in Fig. 3. The three figures for \( \chi_c \geq 1 \) are obtained from Solution I, while the other figure for \( \chi_c < 1 \) is obtained from Solution II. Contrary to the conclusion obtained in Fig. 4, the size of the inelastic zone actually decreases with increase of \( \chi_c \).

In all previous calculations, we have used the standard specimen geometry \( \kappa = 2 \) recommended by the standard testing methods either for rock, soil or concrete (ASTM, 1979; ISRM, 1979; Neville, 1995; Lambe and Whitman, 1979; Goodman, 1989). To examine the geometric effect, Fig. 6 plots the contours of C for various values of slenderness ratio \( \kappa \) for \( \chi_v = 0.5, \chi_v = \chi_c = 1.0, \nu_T = 0.20, \alpha = 0.0 \). From the first glance, it seems peculiar to see that the stress distribution in short cylinders (with a smaller \( \kappa \)) is more uniform than that of longer cylinder (with a larger \( \kappa \)). But for relatively short cylinders (say \( \kappa = 0.5 \)) restrained by end surfaces, lateral bulging of the cylinder is likely to be limited. Consequently, the degree of non-uniformity in stress distribution is also limited.

So far, we have assumed a non-slip end surface (\( \alpha = 0 \)) or the friction between the loading platen and the cylinder is high enough to prevent slippage. To illustrate the effect of partial slip, Fig. 7 plots the contours of C for various friction factors \( \alpha \) for \( \chi_v = 0.5, \chi_v = \chi_c = 1.0, \nu_T = 0.20, \kappa = 2.0 \). As expected, the stress distribution becomes more uniform as \( \alpha \) increases. However, as shown in Fig. 7 the inelastic zone at the corner remains at \( \alpha = 0.8 \) even when the inelastic zone close to the center completely disappears. This indicates that stress singularity exist at \( \eta = \rho = 1 \), as remarked earlier.

All of the previous calculations are for uniaxial compression test. To examine the effect of confining stress on the stress distribution, Fig. 8 plots the contours of C for confining pressure \( p_0 = 0, 0.1 \rho_0 \) for \( \chi_v = 0.5, \chi_v = \chi_c = 1.0, \nu_T = 0.20, \kappa = 2.0, \alpha = 0.0 \). As expected, the confining stress inhibits the free expansion of the cylinder away from the end surface and, thus, reduces the degree of non-uniformity of the stress distribution.

To examine the normal stress distribution on the end surface, Fig. 9 plots \( \sigma_{zz}/\rho_0 \) versus \( r/R \) on the end surface \( \eta = \pm 1 \) for various values of \( \chi_v \). The calculations are for various values of \( \chi_v \) with \( \chi_v = \chi_c = 1, \nu_T = 0.20, \alpha = 0.0 \). It was observed that the normal stress distribution on the loading end surfaces is more uniform when \( \chi_v \geq 1 \), comparing to the case of \( \chi_v < 1 \).

8.2.2. Correction factor for Young’s modulus

Uniaxial compression on circular cylinders is one of the most popular ways to obtain the Young’s modulus of a solid. For isotropic solids, if no friction is developed between the end surfaces and the loading platens, the Young’s modulus can be measured exactly from a uniaxial compression test. However, due to friction on the end surfaces, the apparent Young’s modulus obtained from an usual uniaxial compression test is always larger than the true one of a solid (Chau, 1997). The difference between the apparent Young’s modulus and the true one, in general, decreases with length, but increases with Poisson’s ratio of the cylinder. In fact, another application of the elastic stress analysis is to provide the correction factor in finding the true Young’s modulus (e.g. Filon, 1902; Nayak, 1974; Chau, 1997, 1999; Watanabe, 1998).

Such a correction factor, however, has never been considered for transversely isotropic cylinders under compression. Therefore, the correction factor for Young’s modulus \( E_E/E_0 \) (where \( E_0 \) is the apparent Young’s modulus and can be obtained by assuming a uniform stress within the transversely isotropic cylinder) will be considered here. Note that the ratio \( E_E/E_0 \) equals to \( \omega_{zz}/\omega_0 \) which is the ratio of the vertical displacement on the end surface (i.e. \( \eta = 1 \)) with frictional effect to that without frictional effect.

Fig. 10 shows the variations of \( E_E/E_0 \) for various anisotropic factors \( \chi_v, \chi_v \) and \( \chi_c \). The calculations are for \( \nu_T = 0.20, \alpha = 0.0 \).
The solid lines are obtained from Solution I, while the dotted lines are obtained from Solution II. In the isotropic limit (i.e. $\chi_e = 1$), our results are the same as those by Watanabe (1998). Fig. 10 clearly shows that the correction factor deviates farther away from unity when $\chi_e$ and $\chi_c$ decrease or when $\chi_e$ increases. Similar to the case of isotropic cylinders (Chau, 1997; Watanabe, 1998), $E_i/E_a$ in general decreases with $R/h$. That is, a smaller ratio of $R/h$ usually leads to a smaller difference between the apparent Young's modulus and...
the true one. Therefore, it seems that the values of $h/R$ of 2 to 3 used for testing isotropic specimens can also be used for transversely isotropic materials. We also observe that Solution I always leads to a larger $EL/EL$, while Solution II always leads to a smaller $EL/EL$, comparing to those in the isotropic case.

8.2.3. Radial displacement on the curved surface

One of the easiest ways to examine the validity of the analysis is to observe the deformed shape of the cylinders under loading. Therefore, Fig. 11a–c plots the radial displacement $u/u_0$ on the curved surface along $z/h$ for various values of $\chi_C$ for $\kappa = 0.5$, 1.0, and 2.0, respectively, where $u_0$ is the radial displacement for the case without end friction. The other parameters used are $\alpha = 0$, $\chi_E = \chi_f = 1.0$, and $\nu_f = 0.20$. Fig. 11 shows that the bulging shape is more apparent when $\chi_C$ increases if $\kappa = 0.5$; and the bulging mode is more apparent when $\chi_C$ decreases if $\kappa = 1$ and 2. However, if $\kappa < 1$, the radial displacement $u$ is usually smaller than $u_0$; and if $\kappa > 1$, the radial displacement $u$ is usually comparable to $u_0$ in the mid-portion of the cylinder. As expected, Fig. 11 suggests that bulging phenomenon is more apparent in short cylinders. We want to emphasize that the bulging mode plots here is a result of end friction and should not be confused with those bulging caused by material nonlinearity (e.g. Chau and Rudnicki, 1990; Chau, 1992). Fig. 12 shows the effect of friction factor $\alpha$ on the radial displacement on the curved surface. As expected, a uniform displacement is approached as $\alpha \to 1$. We also have plotted the deformed shape as a function of $\chi_E$ and $\nu_E$, but such anisotropy is found insignificant as long as $\chi_E = 1.0$. Therefore, when the end constraint is significant, anisotropy in shear modulus is more instrumental to the development of bulging shape of the cylinder than anisotropy in Poisson’s ratio and Young’s modulus.

9. Conclusions

This paper derives an exact solution for the non-uniform stress and displacement distributions in a finite transversely isotropic cylinder under uniaxial or confined compression with...
radial constraint on the end surfaces induced by frictional contact. The main contribution of the present work is in identifying the complete and appropriate solution form for Lekhnitskii's (1963) stress function that can satisfy both end and curved boundary conditions simultaneously. The newly proposed solution form can also be used to solve other mixed boundary axisymmetric problems of finite transversely isotropic cylinders. The FEM and analytical solutions by Watanabe (1996) for isotropic cylinders under compression test can be recovered as a special case by considering the limiting cases of both Solutions I and II (see Table 1).

To examine the stress distribution, contour plots of an equivalent shear stress for friction type materials were plotted for various anisotropic parameters and slenderness ratio of the cylinder. In general, our numerical results show that a smaller value in the anisotropy factors $\kappa$ for Young's modulus (i.e. $E_x/E_y$), Poisson's ratio (i.e. $\nu_y/\nu_z$), and shear modulus (i.e. $G_x/G_y$) will result in a more non-uniform stress distribution. The stress distribution for short cylinders (with a smaller $\kappa = h/R$, where $h$ and $R$ are the half length and radius of the cylinder) is found more uniform than that of cylinders (with a smaller slenderness ratio). The stress distribution for short cylinders is found more uniform than that of long cylinders (with a larger slenderness ratio). The stress distribution for short cylinders is found more uniform than that of long cylinders (with a larger slenderness ratio).

Fig. 11. Radial displacement $u/u_0$ profile on the curved surface for various friction factors $\varepsilon$. The other parameters used are: $\kappa = 0.5$, $x = 0.1$, $z = 1.0$, and $v_\tau = 0.20$.

Fig. 12. Radial displacement $u/u_0$ profile on the curved surface for various anisotropic factors $\chi_j$. for (a) $\kappa = 0.5$; (b) $\kappa = 1.0$; (c) $\kappa = 2.0$. The other parameters used are: $\chi_x = \chi_y = 1.0$, $v_\tau = 0.20$, and $\varepsilon = 0.0$.

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References


