A note on the numerical stability of the convection-diffusion equation

Sedat Biringen (*)

ABSTRACT

Stability problems related to some finite-difference representations of the one-dimensional convection-diffusion equation are investigated. Numerical experiments are performed to test the applicability of the restrictive conditions of linear stability as well as to test the effect of an additional boundary condition on the otherwise well-posed Cauchy problem.

1. INTRODUCTION

This paper deals with numerical stability problems related to some finite difference (FD) approximations of the one-dimensional convection-diffusion (CD) equation. The CD equation is employed as a model for initial-boundary value problems that arise frequently in fluid dynamics. Of particular interest are the effects of spatially varying convection velocity, \( u \), and the magnitude of the diffusion constant, \( \alpha \), on the stability of the FD equations. The main objective of the paper is to test the applicability of the restrictive conditions of linear stability obtained from the von Neumann method (see e.g. [1]) when there are very rapid changes in the prescribed distribution of \( u(x) \), e.g. discontinuities. Another problem of interest is the effect on numerical stability of an additional boundary condition on the otherwise well-posed Cauchy problem which is simulated by the CD equation as \( \alpha \rightarrow 0 \). An example of the occurrence of such a problem in fluid dynamics is given in [2].

2. STABILITY ANALYSIS AND THE MODEL EQUATION

In von Neumann's linear stability method, the initial line of errors is expressed as a complex Fourier series

\[
E(x) = \sum_j A_j e^{i\beta_j x}
\]

where \( A_j \) are the amplitude functions and \( \beta_j \) are frequencies. Assuming linear FD equations and constant amplitudes, the error can be written as

\[
E(x, t) = e^{\gamma t} e^{i\beta x}
\]

where \( \gamma = \gamma(\beta) \) is complex in general. The original error, \( e^{i\beta x} \), will not grow in time if

\[
|e^{\gamma t}| < 1
\]

It should be noted that the method is valid for FD equations with constant coefficients. For difference equations with variable coefficients, it is locally applicable; a FD scheme is expected to remain stable if the linear stability condition is satisfied at every point in the flow field.

The CD model equation is derived from the equations of motion for an incompressible fluid used in conjunction with the continuity equation and the definition of vorticity.

In this work the one-dimensional form of the vorticity equation was employed as the model equation. This reads

\[
\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} = \alpha \frac{\partial^2 \omega}{\partial x^2}
\]

where \( \alpha = 1/Re \), \( Re = U_0 L/\nu \), \( U_0 \) and \( L \) are the reference velocity and length scales, respectively, whereas \( \nu \) is the kinematic viscosity.

3. STABILITY CONSIDERATIONS

In this section various FD representations to equation (6) are investigated. Implicit and explicit schemes are employed and the related stability criteria are determined from the linear analysis; details are given in [4]. In the explicit FD representations of the CD equation, the Courant-Friedrichs-Lewy (CFL) condition must be satisfied (see [1]) to ensure convergence of the FD equations as \( \alpha \rightarrow 0 \). The CFL condition, in effect, imposes a restriction on the step size in the marching direction, i.e.
c = u \Delta t \Delta x \leq 1 \quad (5)

where c is the Courant number.

### 3.1. Explicit formula

The CD equation, equation (4) with \( a > 0 \), is parabolic in \( t \). An explicit FD representation of the CD equation can be obtained by a forward-time centered-space differencing (see [3]). This gives

\[
\omega_{m}^{n+1} = \omega_{m}^{n} - \frac{c}{2} u_{m}^{n} (\omega_{m+1}^{n} - \omega_{m-1}^{n}) + d (\omega_{m+1}^{n} - 2 \omega_{m}^{n} + \omega_{m-1}^{n}) \quad (6)
\]

where \( c = (u \Delta t) / \Delta x \) and \( d = (a \Delta t) / \Delta x^2 \). To investigate the linear stability properties, \( \omega = \xi^2 e^{i \beta m \Delta x} \) is substituted into the difference equation. This gives a complex amplification factor, the modulus of which reads

\[
|\xi|^2 = \left(1 + 2d \cos^2 \left(\frac{\beta h}{2}\right)ight) + c^2 \left(1 - \cos^2 \beta h\right) \quad (7)
\]

Hence, stability requires \( |\xi|^2 < 1 \) and \( d < \frac{1}{2} \). According to [5], a more general condition of linear stability for equation (6) is

\[
R_c = \frac{u}{a} \frac{\Delta x}{\Delta t} < 2 \quad \text{(convective stability)} \quad (8a)
\]

\[
d = \frac{a}{\Delta x^2} < 1/2 \quad \text{(diffusive stability)} \quad (8b)
\]

where \( R_c \) is the cell Reynolds number. These conditions implicitly take into account that \( c < 1 \).

### 3.2. Implicit formula

The fully implicit scheme makes use of backward-time centered-space differences (see [3]). For equation (4) this gives

\[
\omega_{m}^{n+1} - \omega_{m}^{n} + \frac{c}{2} u_{m}^{n+1} (\omega_{m+1}^{n+1} - \omega_{m-1}^{n+1}) = d (\omega_{m+1}^{n+1} - 2 \omega_{m}^{n+1} + \omega_{m-1}^{n+1}) \quad (9)
\]

In this case the modulus of the complex amplification factor is given by

\[
|\xi|^2 = \frac{1}{\left(1 + 4d^2 \sin^2 \left(\frac{\beta h}{2}\right)\right)^2 + c^2 \sin^2 \beta h} \quad (10)
\]

From equation (10) it is clear that the stability condition, which requires \(|\xi|^2 < 1\), is unconditionally satisfied.

### 4. NUMERICAL EXPERIMENTS

In all the calculations (details are given in [4]) equal step sizes were taken along the \( x \) and \( t \) coordinates. This resulted in a variable Courant number as a function of \( u(x) \). Tridiagonal coefficient matrices occurring in the implicit FD equations were solved by using the Thomas algorithm (see [3]).

#### 4.1. Case 1 : \( u(x) = ax + b \)

The FD equations given in section (3) were integrated using the following initial conditions

\[
u(x) = a \left(\frac{x}{L}\right) + b; \quad \omega(x, 0) = \xi n u \quad (11)
\]

where \( a \) and \( b \) are constants.

For this initial condition, the convection equation can be integrated to give [6]

\[
\omega = \xi n u - at \quad (12)
\]

The convection velocity, \( u \), was prescribed as

\[
\begin{cases}
-1.6 x + 0.9 & 0 < x < 0.5 \\
1.6 x - 0.7 & 0.5 < x < 1.0
\end{cases}
\]

and the boundary conditions were imposed as \( \omega(0, t) = 1.6t \) and \( \omega(L, t) = -1.6t \). With \( u(x) \) prescribed as in equation (13) and \( \Delta t / \Delta x = 1 \), \( |c| < 1 \) is always satisfied.
oscillations at the boundary $x/L = 1$ whereas the interior solution was smooth (except for $\alpha = 0.0001$ for the implicit solution) although $R_c \gg 2$. We attribute this to the condition that as $\alpha \to 0$ the CD equation assumes a hyperbolic character so that the boundary condition at $x/L = 1$ alters the well posed character of the finite-difference equations. Note that the CD equation with $\alpha = 0$ is hyperbolic in time; the equation for its characteristic lines are $x - u(x)t = \text{const.}$ whereas the characteristic direction, $\lambda$, is $\lambda = dx/dt = u(x)$. But as $x/L \to 0$, $u(x) \to -0.7$ and as $x/L \to 1$, $u(x) \to -0.7$ so that the characteristics are ingoing only when $x/L \to 0$, i.e. the characteristics do not change sign at the boundary $x/L = 1$. The boundary condition imposed at $x/L = 1$ alters the well-posedness of the hyperbolic equation and this, in turn, results in oscillations which propagate into the solution domain. The additional boundary condition at $x/L = 1$ is likely a more serious source of error than the violation of the convective stability restriction obtained from the linear stability analysis.

In figures 3 and 4 results obtained from the implicit scheme are shown. Figure 3 displays instabilities that propagate from the boundary at $x/L$ and figure 4 shows that these instabilities grow in time. The oscillations that appear around the region of velocity discontinuity are static in character and hence could be attributed to the centered convection operator. This supports our previous argument that in the case of small $\alpha$, the CD equation assumes a hyperbolic character and the additional boundary condition at $x/L = 1$ could be a source of numerical instability more stringent than the violation of the convective and diffusive stability requirements.
4.2. Case 2: \(u(x)\) as a step function

The stability properties of the implicit formulation of the CD equation were further investigated by prescribing \(u(x)\) as a step function. The initial distribution of \(\omega\) was calculated from

\[
\omega(0, t) = e^x
\]

where \(x = x/L\).

For this initial condition the convection equation has the exact solution

\[
\omega(X, t) = e^{(X - Ut)}
\]

where \(U = u/L\). The convective velocity is prescribed as

\[
U = \begin{cases} 
0 & \frac{0}{L} < \frac{x}{L} < 0.5 \\
1 & 0.5 < \frac{x}{L} < 1
\end{cases}
\]

where \(k\) is the amount of jump. Exact boundary conditions compatible with (14) and (15) were imposed

\[
\omega(0, t) = e^{-Ut}; \quad \omega(L, t) = e^{(1-Ut)}
\]

A series of calculations were carried out for the implicit formulation of the CD equation with initial and boundary conditions prescribed as in (15)-(17). In all the calculations the value of \(\alpha\) was set equal to 0.001; this gives \(R_c = 29\), which is much greater than the linear stability upper limit, i.e. \(R_c = 2\). In figure 5 results for various values of \(k\) are shown. For \(k = 0.5\), although the convective stability condition is violated, our results give a smooth solution. Increasing values of \(k\), however, result in instabilities which are especially prominent near the boundary \(x/L = 1\). Considering the small value of \(\alpha\) and hence the hyperbolic character of the CD equation we conclude that the boundary condition at \(x/L = 1\) is the main source of error and numerical instability. This becomes more pronounced with the increasing severity of the initial field, i.e. with increasing gradients and greater jumps in the prescribed convection velocity.

The main conclusions of this work are now summarized:

a) For the CD equation with \(\alpha \to 0\), the boundary condition at \(x/L = 1\) alters the well-posed character of the differential equation resulting in boundary instabilities.

b) The unconditionally stable implicit scheme for the CD equation (with center differenced operators) produces instabilities of the static type around regions of large gradients of the convection velocity.

c) The explicit formulation of the CD equation using center-differenced operators maintains its stability for values of \(R_c\) much larger than those permitted by the linear stability analysis.

d) The jump in the convection velocity is observed to have a significant effect on the stability of the implicit formulation of the CD equation.

REFERENCES


