Topology and its Applications 35 (1990) 1-19
North-Holland

ON SUBSEQUENTIAL SPACES

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Received 6 November 1987
Revised 5 April 1988, 1 December 1988 and 5 January 1989

Simple generators for the coreflective category of subsequential spaces, one of them countable, are constructed. Every such must have subsequential order ω_1. Subsequentialness is a local property and a countable property, both in a strong sense. A T2-subsequential space may be pseudocompact without being sequential, in contrast to T2-subsequential compact (countably compact, sequentially compact) spaces all being sequential. A compact subsequential space need not be sequential.

AMS (MOS) Subj. Class.: 54D55, 54B30

subsequential spaces   sequential spaces
subsequential order   prime spaces

Introduction

Although sequential spaces have been studied intensively since the mid-sixties, their subspaces, the *subsequential spaces* have been less well investigated. As late as 1976 it was said that "very little is known about these spaces..." [20]. Except for a convergence characterization given by Aniskovic in 1981 [1] and a smattering of other results, referred to below, that assertion is still true. This paper is a modest attempt to add to that small store of knowledge. Recall that the category, SEQ, of sequential spaces is coreflective in TOP, and is simply generated by the compact space, S_1, consisting of a convergent sequence and its limit point [7]. S_1 is of sequential order 1 [4]. Similarly, the category, SSEQ, of subsequential spaces is also a coreflective subcategory of TOP (Corollary 3.3) and is also simply generated (Theorem 4.6). There is, in fact, a countable generator for SSEQ (Theorem 7.3), but, by contrast, no compact generator, nor a generator with order less than ω_1 (Theorem 8.5).

* Supported by a grant from AT&T.

0166-8641/90/$3.50

Section 1 is devoted to examples, preliminary observations, and a review of some known results. In Section 2 we see that while T2-subsequential spaces with strong compactness properties must be sequential, they can be pseudocompact without being sequential. Section 3 contains mostly known results about the category SSEQ. In Section 4 we see that subsequential spaces are determined by their countable subspaces, and use this fact to produce a single generator for SSEQ.

In Section 5 the prime factorization of subsequential spaces is introduced and used to show that subsequentialness is a local property. In Section 6 the subsequential order is defined and a family of "canonical" subsequential prime spaces are constructed. In Section 7 prime spaces are used to produce smaller single generators for SSEQ, one of them countable. In Section 8 we show that, in contrast to sequential spaces, any simple generator of SSEQ must have order ω_1.

Unless specifically mentioned, we assume no separation axioms.

1. Examples and observations

This first section is intended to ease the reader into the subject of subsequential spaces by providing examples of familiar spaces that are and are not subsequential, and by noting easily proven properties that place the subsequential spaces among other, better known classes.

1.1. *Every sequential space is trivially subsequential.*

The simplest example of a subsequential space which is not sequential is due to Arens [2, 8]. We describe this space next.

1.2. Example. S_2^-. The space S_2 is constructed from the convergence sequence S_1 by attaching a convergent sequence to each isolated point of S_1 and using the quotient topology. Calling the limit point of the original sequence the level-zero point, the other points of the original sequence, the level-one points, and the isolated points of S_2, the level-two points, let S_2^- be the subspace of S_2 which omits all level-one points. In S_2^- no sequence of level-two points can converge to the level-zero point. Thus S_2^- is not sequential.

There are, of course, many spaces which are not subsequential. The space $\mathbb{N}_p$ below is one such. (The Hausdorff case has been noted by Michael [9], Arhangel'skiĭ [3], and no doubt others. We outline a proof of this case for the convenience of the reader. We shall see following Corollary 6.3 below that the Hausdorff hypothesis is not needed. This also follows from a result of Aniskovic [1].)

1.3. Example. $\mathbb{N}_p$. The space $\mathbb{N}_p$ consists of the discrete space $\mathbb{N}$ together with one point p from the growth of its Stone–Čech compactification. $\mathbb{N}_p$ is not a subspace of any Hausdorff sequential space.

Proof. If $\mathbb{N}_p$ is contained in a sequential Hausdorff space S, then the closure, $\bar{\mathbb{N}}$, of $\mathbb{N}$ in S contains a sequence $\{s_n\}$ converging to p. Using Hausdorffness and the compactness of the sequence, find disjoint open sets U and V of S containing the odd and even terms of $\{s_n\}$ respectively. Then $U \cap \mathbb{N}$ and $V \cap \mathbb{N}$ are disjoint sets belonging to the ultrafilter p, a contradiction. □

(Kannan has shown [17] that $\mathbb{N}_p$ is not a subspace of a Hausdorff sequential space when the topology of p in the above example is taken to be the meet of the neighborhood filters of countably many points of $\beta\mathbb{N}\backslash\mathbb{N}$. Aniškovic has other results of this type [1].)

It is obvious that subsequential spaces are hereditary.

1.4. *Every subspace of a subsequential space is subsequential.*

The following two observations (1.5 and 1.6) were made by Nogura [23].

Recall that a space of countable tightness [21, 26] is one in which a set is closed whenever it contains the closure of each of its countable subsets. Since every sequential space is countably tight, as are subspaces of countably tight spaces, we conclude that:

1.5. *Every subsequential space is of countable tightness.*

Since Simon produced a sequential space whose square is not countably tight [27], Nogura notes that:

1.6. *The product of two subsequential spaces need not be subsequential.*

A slick categorical proof for infinite products is included following the proof of Proposition 3.2 below.

Since each space is the continuous image of a discrete space, it is clear that:

1.7. *The continuous image of a subsequential space need not be subsequential.*

2. Compactness properties

In this section we consider compactness properties of subsequential spaces. The major new results in this section are Propositions 2.3 and 2.4 which show that pseudocompact subspaces of Hausdorff sequential spaces need not be sequential.

There are compact Hausdorff spaces that are not subsequential.

2.1. *Both $\beta\mathbb{N}$ and the ordinal space, ω_1+1, are not subsequential.*

Proof. Neither has countable tightness. Apply 1.5. □

The examples $\beta\mathbb{N}$ and ω_1+1 suggest that a subsequential compact Hausdorff space must already be sequential. Since each compact (countably compact, sequentially compact) subset of a Hausdorff sequential space is closed [8], and closed subsets of sequential spaces are again sequential [7], every T2-subsequential (i.e., spaces embeddable in Hausdorff sequential spaces) compact space is sequential. This has been previously observed in [20].

However, this result does not hold for subsequential spaces.

2.2. *There are compact (and therefore countably compact, pseudocompact, etc.) subsequential spaces which are not sequential, i.e., a compact (countably compact, pseudocompact, etc.) subspace of a sequential space need not be sequential.*

Proof. Add a new point p to S_2 (as described in Example 1.2 above) whose neighborhoods contain all but finitely many level-two points. Retain the old neighborhoods of the other points. This space is sequential but not Hausdorff. Its subspace $S_2^- \cup \{p\}$ is compact but not sequential. □

We do not know if a compact Hausdorff subspace of a (non-Hausdorff) sequential space need be sequential. However, such a space must have countable tightness and, hence, in the presence of the proper forcing axiom, must be sequential [6]. Since it is consistent that a compact Hausdorff space of countable tightness not be sequential [24], it may be that this question is answerable only in term of an additional set-theoretical axiom.

Question: Must a compact Hausdorff subsequential space be sequential in ZFC?

But pseudocompactness is not enough to insure that a subspace of even a Hausdorff sequential space is sequential.

2.3. Proposition. *There is a T2-subsequential, pseudocompact, Hausdorff space that is not sequential.*

Proof. Begin with Ψ, the space built from $\mathbb{N}$ by choosing a maximal pairwise almost disjoint collection Γ of infinite subsets of $\mathbb{N}$, and adding a point p_C for each $C \in \Gamma$. Thus $\Psi = \mathbb{N} \cup \{p_C \mid C \in \Gamma\}$, where each point $n \in \mathbb{N}$ is isolated in Ψ, and basic neighborhoods of p_C consist of p_C itself and all but finitely many members of C. Thus C is a sequence converging to p_C. Ψ is known to be sequential, Hausdorff, and pseudocompact but not countably compact (and hence not normal). Call points of $\mathbb{N}$ level-zero points and point p_C level-one points of Ψ.

From Ψ construct Ψ_2 in the same manner that Ψ was built from $\mathbb{N}$, i.e., choose a maximal pairwise almost disjoint family Γ_2 of countably infinite subsets of the level-one points of Ψ. For each member C of Γ_2, add a level-two point p_C. Thus

$$\Psi_2 = \Psi \cup \{p_C \mid C \in \Gamma_2\}.$$

Points of Ψ have their original neighborhoods. A basic neighborhood of a level-two point, p_C, consists of p_C itself, all but finitely many level-one points of C, and a basic neighborhood of each of these level-one points. Ψ_2 has all the properties claimed above for Ψ. (Ψ_2 is pseudo-compact since its dense subset, Ψ, is.)

From Ψ_2 construct Ψ_3 in an analogous manner adding level-three points with a sequence of level-two points converging to each. Ψ_3 again has all the properties claimed above for Ψ. Each level-three point p_C in Ψ_3 has its neighborhoods defined by a sequence C of level-two points in the chosen collection C_3.

Now fix some C_0 in Γ_3 and let $X = \Psi_3 \backslash C_0$. X is a subspace of the sequential space Ψ_3. It is Hausdorff, and is pseudocompact since it contains Ψ as a dense subspace. But X is not sequential since $X \backslash \{p_{C_0}\}$ is sequentially closed but not closed. Thus X is the desired space. □

The same central idea can be used to create a Tychonoff (in fact zero-dimensional) example, but one must work a little harder. Having gone through the previous example will make the next easier to follow.

2.4. Proposition. *There is a T2-subsequential, pseudocompact, zero-dimensional space that is not sequential.*

Proof. Let $\mathscr{C}$ denote the Cantor set. Begin with a sequence $\{\mathscr{C}_n\}$ of Cantor sets. Denote the zero point of $\mathscr{C}_n$ by 0_n. From the collection of infinite subsequences of $\{\mathscr{C}_n\}$ choose a maximal, pairwise almost disjoint family $\Phi = \{F_\alpha \mid \alpha < \mathfrak{c}\}$. Thus each F_α is a sequence of Cantor sets $\{\mathscr{C}_{\alpha_n}\}$. For each α let $X_\alpha = F_\alpha \cup \{\mathscr{C}_{\alpha_0}\}$, where each $\mathscr{C}_{\alpha_0}$ is yet another Cantor set with its zero point denoted by 0_{α_0}. Provide each X_α with the topology induced from the product topology on $\mathscr{C} \times S_1$. Thus each X_α inherits from the Euclidean topology of the plane. Our initial space is $X = \bigcup X_\alpha = \bigcup\{\mathscr{C}_n\} \cup \bigcup\{\mathscr{C}_{\alpha_0}\}$ with its topology taken from the X_α.

We must expand X in order that it have a nonsequential subspace. This expansion is analogous to the construction of Ψ_2 in the proof of Proposition 2.3.

Given a particular F_α consisting of Cantor sets $\{\mathscr{C}_{\alpha_i}\}$, choose within each such $\mathscr{C}_{\alpha_i}$ a compact-open subset A_i not containing 0_{α_i}. Further, let the diameter of A_i go to zero as i increases. Finally, restrict your choices so that 0_{α_0} is a limit point of the sequence $\{A_{\alpha_i}\}$. Every neighborhood of 0_{α_0} will contain all but finitely many members of the sequence. From the collection of all possible such sequences choose a maximal pairwise almost disjoint family Γ_α. Do this for each α.

Now let $Y_\alpha = X_\alpha \cup \{p_{\alpha_\kappa}\}$ where the p_{α_κ} are in one-to-one correspondence with the sequences in Γ_α. Points in $Y_\alpha \cap X_\alpha$ retain their X_α neighborhoods except for the point 0_{α_0}. The inherited neighborhoods of this point may now omit the points of finitely many of the sequences in Γ_α. Neighborhoods of each p_{α_κ} will contain that point and all the points of all but finitely many of the compact-open sets in its corresponding sequence from Γ_α.

Finally, let $Y = \bigcup Y_\alpha \cup \{p\}$ with each point except p retaining its Y_α neighborhoods. Neighborhoods of p may omit a finite number of the Y_α. It is tedious, but straightforward, to verify that Y is both Hausdorff and zero-dimensional.

To check that Y is sequential, note that each point, except the 0_{α_0} points and p itself, has a countable basis of neighborhoods. A routine verification shows that if $0_{\alpha_0} \in \bar{A}$, then 0_{α_0} can be reached by at most two sequential closure operations starting from A. The same is true if $p \in \bar{A}$. Thus Y is sequential [4].

Now let $Z = Y \backslash \bigcup \{0_{\alpha_0}\}$. Z is Hausdorff, zero-dimensional and subsequential. It remains to show that Z is pseudocompact but not sequential.

The pseudocompactness of Z is most easily seen from the following characterization: *A Tychonoff space is pseudocompact iff it contains no infinite, locally finite family of pairwise disjoint open sets.* This is simply a restatement of [12, Lemma 9.13, p. 134]. To see that Z is pseudocompact suppose there were such a family $\{U_n\}$ of open sets. If infinitely many U_n belonged to one Cantor set $\mathscr{C}_k$, it certainly cannot be locally finite. If the $\{U_n\}$ meet infinitely many different $\mathscr{C}_k$, they must meet infinitely many of the Cantor sets in some F_α and so again cannot be locally finite. If infinitely many meet some one $X_\alpha \backslash F_\alpha$ yet again, they cannot be locally finite. If they meet infinitely many $X_\alpha \backslash F_\alpha$, then every neighborhood of p meets infinitely many of them. Thus Z is pseudocompact.

That Z is not sequential follows from the fact that no sequence from $\bigcup F_\alpha = \bigcup \mathscr{C}_n$ can converge to p. Let $A = \{0_n\}$, the set of zero points of our original Cantor sets. Sequences from A can converge only to some 0_{α_0}. But these points are precisely those omitted from $Z = Y \backslash \bigcup \{0_{\alpha_0}\}$. Noting that $p \in \bar{A}$ then shows that Z is not sequential and is, hence, the desired space. □

We close this section by noting that, like pseudocompactness, paracompactness is not sufficient to make a subsequential space sequential. There are countable subsequential but not sequential regular spaces (see Example 1.2). Each such, being Lindelöf, is paracompact.

2.5. *A paracompact (σ-compact, Lindelöf) subsequential space need not be sequential.*

3. The category SSEQ

In this section we consider properties of the subcategory SSEQ of TOP. Most of these results have appeared earlier in Kannan's dissertation [14]. They are included here since that dissertation is not readily available (was not available to us). Two preliminaries are needed. First, it is easy to see that SSEQ is additive.

3.1. *The topological sum of subsequential spaces is again subsequential.*

To show that SSEQ is divisible requires a little proof.

3.2. Proposition. *Every quotient of a subsequential space is subsequential.*

Proof. Suppose X is a subspace of a sequential space S, and that $q: X \to Y$ is an onto quotient map. Let P be the partition of X induced by q. Define a partition P' of S by

$$P' = P \cup \{\{s\} \mid s \in S \backslash X\}.$$

P' induces a quotient map $q': S \to S\backslash P'$. A point $y \in Y$ corresponds to a cell $q^{-1}(y)$ in P and hence in P'. Thus we have a one-to-one function $e': Y \to S\backslash P'$ so that $e' \circ q = q' \circ e$, where $e: X \to S$ is the inclusion map. Since q is a quotient map and $q' \circ e$ is continuous, e' is continuous. Suppose U is open in Y. Then $q^{-1}(U)$ is open in X. Hence there is an open set W in S whose intersection with X is $q^{-1}(U)$. But W is saturated with respect to P' and hence $q'(W)$ is open in $S\backslash P'$. But $e'(U) = q'(W) \cup e'(Y)$. Thus e' is an embedding, and Y is subsequential. □

Since Kannan [14] has shown that the only additive, productive, divisible and hereditary subcategory of TOP is TOP itself, SSEQ, enjoying the other three properties, cannot be productive.

Recall that a subcategory CAT is *coreflective* in the containing category TOP if each TOP object X has a coreflection $cat\,X$ in CAT and a morphism $e: cat\,X \to X$ so that any morphism $S \to X$ from a CAT object S factors through e. Monocoreflective means that the morphism e is a monomorphism. In TOP every such morphism e is a bijection [11, 13]. SEQ, the category of sequential spaces, is an example of a coreflective subcategory of TOP. We will denote the sequential coreflection of a space X by sX.

Since additive and divisible subcategories of TOP are monocoreflective [11, 19], we have from 3.1 and Proposition 3.2 that:

3.3. Corollary. *The category,* SSEQ, *of subsequential spaces is a monocoreflective subcategory of* TOP.

We will denote by ssX the subsequential coreflection of a topological space X.

It is easy to discover the subsequential corefelection of the space $\mathbb{N}p$ of Example 1.3. Since p is an ultrafilter, the only spaces mapping continuously and bijectively onto $\mathbb{N}_p$ are $\mathbb{N}$ and $\mathbb{N}_p$ itself. Of these only $\mathbb{N}$ is subsequential (the remark following Corollary 6.3 shows that Hausdorff need not be assumed in Example 1.3).

3.4. Corollary. *The subsequential coreflection, $ss(\mathbb{N}_p)$, of $\mathbb{N}_p$ is the discrete space* $\mathbb{N}$.

From 1.1 and 1.5 we deduce the following corollary, from which it follows that the subsequential coreflection of $\omega_1 + 1$ is ω_1.

3.5. Corollary. *For any space X, the spaces X, cX, ssX, and sX are increasingly finer topologies on the same set, where cX is the countably tight coreflection, and sX the sequential coreflection.*

4. Countable generation

In this section we shall show that subsequential spaces are determined by their countable subsets in a stronger sense than simply being countably tight. Using countable subspaces, the first of three simple generators for the category SSEQ is produced.

4.1. Proposition. *If X is subsequential but not sequential, there is a countable subspace of X which is subsequential but not sequential.*

Proof. If X is subsequential but not sequential, some point $p \in X$ belongs to the closure of a subset A but not to any iterate of its sequential closure. By 1.5 there is a countable subset B of A with $p \in \bar{B}$. Then $B \cup \{p\}$ is the desired countable space. □

One might expect from Proposition 4.1 that a space would be subsequential if each of its countable subsets were. This is not true.

4.2. $\omega_1 + 1$ *is not subsequential (by 2.1) but each of its countable subspaces is sequential.*

Our expectation fails because $\omega_1 + 1$ is not countably tight as the next proposition shows.

4.3. Proposition. *If X is countably tight with each countable subspace subsequential, then X is subsequential.*

Proof. Call (p, A) a nonsequential pair in X if p belongs to the closure of A in X but is not in any iterate of the sequential closure of A. By Proposition 4.1 we need consider only countable A. For each countable nonsequential pair (p, A) choose a sequential space $S(p, A)$ containing $A_p = \{p\} \cup A$. Let S be the topological sum of all the $S(p, A)$. All the inclusion maps $A_p \to X$ map a subset of S into X. The adjunction space formed from S and X via this map is a sequential space containing X (using Proposition 4.1). □

Since the converse of Proposition 4.3 is obvious, the proposition yields a characterization of subsequential spaces.

As an aside, we note an easily proven analogous result for sequential spaces.

4.4. *If X is countably tight with each countable subset sequential, then X is a Fréchet space.*

Let us say that a space is *generated by a family of spaces* if it is a quotient of the topological sum of members of the family. A collection of spaces will be said to be *generated by a family* if each of its members are. A collection generated by a singleton family is said to be *simply generated* with that singleton as its *simple generator*. For example, the category SEQ is simply generated by S_1 [7].

Note that any family of sequential spaces can only generate sequential spaces. Since any compact Hausdorff subsequential space is sequential (see the paragraph following 2.1), SSEQ can have no compact Hausdorff generator as SEQ does.

The proof of Proposition 4.3 essentially shows that each subsequential space is generated by its countable subspaces.

4.5. *Every subsequential space is the quotient of a topological sum of its countable subspaces.*

This was also noted in Kannan's thesis [14].

We use this observation to give our first proof that a single space generates all subsequential spaces.

4.6. Theorem. *The category SSEQ is simply generated. One such generator can be expressed as a topological sum of countable subsequential spaces. Its cardinality is 2^c.*

Proof. Since there are at most 2^c topologies on a countable set, we can choose a small skeleton SK for the category of countable subsequential spaces. The topological sum, SUM, of the family SK is the desired simple generator by 4.5, using the fact that each summand of SUM is itself a quotient of SUM (simply shrink every other space in SUM to a fixed point in the copy of the given summand). □

The space SUM is more numerous than needed. We will construct smaller generators, even a countable generator, in Section 7.

5. Local properties and prime factorization

Call a space with precisely one nonisolated point a prime space. (Hausdorff prime spaces are precisely the nondiscrete door spaces according to Kelley [18].) The space S_2^- of Example 1.2 is such a space. For technical convenience we will also include any singleton space among the prime spaces. Given any space X and a point p in X, denote by X_p the prime space constructed by making each point of X, other than p, isolated, with p retaining its original neighborhoods. Call X_p the *prime factor* of X at p, and call the process of constructing X_p *factoring* X at p. The folklore of topology has it that each space is the quotient of the topolgical sum of all its prime factors. Prime spaces and prime factorizations will play an important role in our subsequent study of subsequential spaces.

Factoring a sequential space may well result in a nonsequential space. For example, factor S_2 at its level-zero point. However the factoring of subsequential spaces is well behaved.

5.1. Proposition. *If p belongs to the subsequential space X, then the prime space X_p is also subsequential.*

Proof. This proof will construct a space reminiscent of the Alexandroff double of a space.

Embed X in a sequential space Y. Now factor X at p forming the prime space X_p. Let Z be the disjoint sum of Y and $X_p\backslash\{p\}$. Each point of Z which comes from X_p is then isolated in Z. Points of X, other than p, will appear twice in Z, once as an element of Y, and once from X_p. Distinguish these by calling them x and x' respectively. Each x' is isolated in Z. A neighborhood in Z of a point y of $Y\backslash X$ consists of $U \cup U'$ where U is a Y-neighborhood of y and $U' = \{x' \in X_p \mid x \in U \cap X\}$. If y belongs to X itself, omit y' from $U \cup U'$ where these are defined as above.

Note that $X_p\backslash\{p\} \cup \{p\}$, as a subspace of Z is homeomorphic to X_p. It remains only to show that Z is sequential, e.g. that each point z in the closure of a subset A of Z can be reached by taking sequential closures starting from A [4]. If z is isolated, it belongs to A and there is nothing to prove. If $z \in \overline{A \cap Y}$, it can be so reached since Y is sequential and retains its topology as a subspace of Z. Otherwise assume that A is a subspace of $X_p\backslash\{p\}$. A neighborhood $U \cup U'$ of z meets A only in U'. Thus $z \in \overline{U' \cap A}$. By the way neighborhoods of z are defined, z is in the closure of B, where $B = \{x \in X \mid x' \in U' \cap A\}$. Thus z is reached from B by successive sequential closures. But successive sequential closures add the same points starting from either B or from $U' \cap A$. □

5.2. Corollary. *A space is subsequential iff each of its prime factors is subsequential.*

Proof. One direction is proved above. The other follows since X is the quotient of the topological sum of all the X_p. □

See [16, p. 39] for a more general version of this result.

Call a space X *subsequential at a point* p if X_p is subsequential.

5.3. Corollary. *A space is subsequential iff it is subsequential at every point.*

5.4. Corollary. *A space is subsequential iff it is locally subsequential* (*e.g. each point has a subsequential neighborhood*).

Proof. Suppose X is locally subsequential and p is a point of X. Take U to be an open subsequential neighborhood of p in X. Then U_p is subsequential. But $X_p = U_p \cup (X\backslash U_p)$ with points of $X\backslash U_p$ isolated. Thus each X_p is subsequential. □

6. Sequential and subsequential order and canonical spaces

In this section we use the notion of a sequential sum of pointed spaces to construct a family of prime spaces of various subsequential complexity (order). The presence of one of these spaces as a subspace gives a lower bound on the subsequential order of the containing space.

Given a countable family, (X_n, x_n), of pointed topological spaces, the *sequential sum* [4] of the family is constructed by first forming their disjoint topological sum, and then forming the adjunction space over the map that sends each x_n to $1/n$ in S_1 (e.g. put S_1 into the topological sum also, and form the quotient obtained by identifying each x_n with the corresponding $1/n$). The point associated with 0 in S_1 will be called the *level-zero point* (*or zero point*) of the sequential sum. Each of the S_n [4] is the sequential sum of a countable collection of copies of S_{n-1}, for $n > 1$. We will need to know that sequential coreflections (see just before Corollary 3.3) distribute through sequential sums.

6.1. Lemma. *The sequential coreflection of a sequential sum of pointed spaces is the sequential sum of the sequential coreflections of the summands.*

Proof. Let $\{X_n, x_n\}$ be a sequence of pointed spaces. Let $s(X_n)$ be the sequential coreflection of X_n, and let $SS(X_n)$ be the sequential sum of the X_n. Then $s(SS(X_n))$ is the unique space such that any map $f: S \to SS(X_n)$, with S sequential, factors through the identity $1: s(SS(X_n)) \to SS(X_n)$, e.g. there is a continuous $f': S \to s(SS(X_n))$, where $f = 1 \circ f'$. We will show that $SS(s(X_n))$ satisfies this condition.

Denote by Y_n the inverse of X_n under f. Each Y_n is open in S and therefore sequential. Denote by Y_0 the inverse image of the level-zero point of $SS(s(X_n))$. Denote by f_n the restriction of f to Y_n, and by $f_{n'}$ the factorization of f_n through $s(X_n)$. Let f'_0 map Y_0 onto the level-zero point of $SS(s(X_n))$. Let f' be the map from S to $SS(s(X_n))$ defined by the f_n. Suppose that the sequence $\{a_n\}$ converges to p in S. If p belongs to some X_n, assume that all the a_n also belong. Then the sequence $\{f'_n(a_n)\}$ converges to $f'_n(p)$ in $s(X_n)$ and hence also in $SS(s(X_n))$. If p belongs to no Y_n, then $f'(p)$ belongs to the zero point of $SS(s(X_n))$. Thus by the continuity of f, we may assume that $\{a_n\}$ maps entirely into the x_n. Hence $\{f'(a_n)\}$ converges to $\{f'(p)\}$ in $SS(s(X_n))$, and the proof is complete since sequential continuity implies continuity. □

Define inductively for each ordinal $\alpha < \omega_1$, a family TS_α of spaces as follows: Let TS_1 contain only the space S_1. For any nonlimit $\alpha > 1$, let TS_α consist of all possible sequential sums with summands belonging to TS_β where $\beta = \alpha - 1$. If α is a limit ordinal, let TS_α consist of all possible sequential sums with each summand X_i belonging to TS_{α_i}, and with $\alpha = \sup\{\alpha_i\}$.

If M is a subset of any space X, denote by M_α the αth sequential closure of the set M.

6.2. Lemma. *If p belongs to M_α but not to M_β for any $\beta < \alpha$, then there is a space S_α in* TS_α *and a mapping $f: S_\alpha \to X$ which sends all isolated points of S_α into M and maps only the zero point of S_α onto p. (When X is Hausdorff, f can be taken to be an injection.)*

Proof. For $\alpha = 1$, the proof is obvious. Suppose the assertion is true for all subsets K of X and all points q of K_β for any $\beta < \alpha$. Then p is the limit of a sequence $\{a_i\}$ where each a_i belongs to some M_{β_i} for $\beta_i < \alpha$. Construct a (pairwise disjoint, if X is Hausdorff) family of open sets $\{U_i\}$ with U_i containing a_i. By the inductive hypothesis, for each i there is a (one-to-one, if X is Hausdorff) mapping $f_i: S_{\beta_i} \to M \cap U_i$ with isolated points of S_{β_i} mapping into M and only the level-zero point of S_{β_i} mapping onto a_i. If α is a nonlimit ordinal, then we may take all $\beta_i = \alpha - 1$. Otherwise, β_i can be taken to be an increasing sequence of ordinals converging to α. In either case, the sequential sum of the family S_{β_i} is an S_α belonging to TS_α, and the maps f_i define the desired map with only the level-zero point of S_α going to p. □

Recall that the sequential order, $so(X)$, of a sequential space X (as defined in [4]) is the least ordinal α such that whenever a member x of X is in the closure of a subset A of X, x is in the αth iterate of the sequential closure of A. For example, the space S_2 defined in Example 1.2 above has sequential order 2. In [4], for each n, a space S_n of sequential order n is defined by induction using the concept of a sequential sum.

Sequential spaces of order α for $0 \leqslant \alpha \leqslant \omega_1$ were also constructed in [4] using sequential sums.

An ordinal invariant for SSEQ, called the subsequential order, will be defined in terms of sequential order. The subsequential order, $sso(X)$, of a subsequential space, X, is the infimum of the $so(S)$ where S is a sequential space containing X as a dense subset.

Since sequential orders cannot exceed ω_1 [14], the same is true for subsequential orders.

For each space X in TS_α, denote by X^-, the prime subspace of X consisting of the zero point and all isolated points. Denote by TS_α^-, the collection of all X^- for X in TS_α.

A subsequential space of order 1 is embeddable in a hereditarily sequential space and is therefore sequential, in fact Fréchet. It is easily seen that S_2^- has subsequential order 2.

6.3. Corollary. *If $p \in M_\alpha \setminus M_\beta$ for any $\beta < \alpha$, then there is a space $S_\alpha^- \in \mathrm{TS}_\alpha^-$ and a mapping (an injection if X is Hausdorff) $f: S_\alpha^- \to X$ which sends all isolated points of S_α^- into M and maps the zero point of S_α^- onto p. Hence, in particular, if X is any subsequential prime space of order $\alpha < \omega_1$, there is a member S_α^- in TS_α^- and a (one-to-one if X is Hausdorff) map as above.*

This corollary yields yet another proof that $\mathbb{N}_p$ cannot be embedded in any sequential space, Hausdorff or not (see Example 1.3 and Corollary 3.4 above).

The corollary will be used to find a countable generator for SSEQ in the next section.

Having defined subsequential order, it is natural to ask about the subsequential order of a sequential space.

Question: Do the sequential and subsequential orders coincide for sequential spaces?

From Lemmas 6.4.2 and 6.4.3 used in the proof of the next theorem, it follows that the answer is yes for all members of the families TS_α.

6.4. Theorem. *If a sequential space Y contains a member of TS_α, then the sequential order of Y is at least α. Thus the subsequential order of any member X of TS_α is α.*

The proof will use internal lemmas. The theorem is obviously true for $\alpha = 1$ or $\alpha = 2$. In the lemmas that follow, we assume that $\alpha > 2$.

6.4.1. Lemma. *If S_α is a member of TS_α, then every open neighborhood of the zero point, 0, of S_α contains an open set X which is a member of TS_α with the same zero point.*

Denote the level-zero point of S_α by 0. Denote by X_i the summands of S_α, and by $\{x_i\}$ their level-zero points. We will also assume that S_α is densely embedded in Y.

6.4.2. Lemma. *If $\{a_i\}$ is any sequence in Y but not S_α, if each a_i is the limit of a sequence a_{ij} from the union of the X_i, and no a_i is in the closure of any single X_j, then 0 is not a cluster point of $\{a_i\}$.* In fact, there is a *Y-neighborhood, W, of 0 with each a_i in the closure of $S_\alpha \setminus W$.*

Proof. Since the sequence $\{x_i\}$ of level-zero points of the summands converges to 0, we may assume that no x_k occurs among the $\{a_{ij}\}$. Without loss of generality we may also assume that for any $k > 1$, the sequence $\{a_{kj}\}$ misses all the summands X_i for $i < k$. Under these assumptions it follows that the double sequence $\{a_{ij}\}$ meets each X_i in at most finitely many points. Let U_k be the complement of $\{a_{ij}\}$ in X_k. U_k is open in X_k and contains x_k. Let U be the union of the U_k with 0 included. U is then an open neighborhood of 0 in S_α. Hence some open W in Y meets S_α in U, and thus contains 0, but misses the double sequence $\{a_{ij}\}$. Hence W cannot contain any a_i. □

6.4.3. Lemma. *If a point b of $Y \setminus S_\alpha$ can be reached from the isolated points M of some X_j in less than α iterates of the sequential closure, then there is an X_j-neighborhood U' of 0 with b in the closure of $M \setminus U'$. Equivalently, there is a Y-neighborhood of 0 that misses b.*

Proof. We will prove the lemma by induction on α. For $\alpha = 1$ the assertion is meaningless. For $\alpha = 2$, $X_j = S_2$. If b can be reached in one sequential closure, then

b is the limit of a sequence b_i of isolated points. If the b_i climb infinitely high in a single column of S_2, let U' be the neighborhood of 0 consisting of all higher columns. If not, the b_i must intersect infinitely many columns. Removing one intersection point from each such column constructs the desired X_j-neighborhood U'.

Suppose the lemma true for all $\beta < \alpha$. If $\alpha = \beta + 1$, X_j is the sequential sum of a family $\{X_i\}$ with each X_i in TS_β. If b can be reached from M (the union of the isolated points of the X_i) via β sequential closures, then b is the limit of a sequence $\{b_i\}$ where each b_i can be reached from M in less than β sequential closures. Let B_1 be the set of those b_i that can be reached from a single summand X_{ji} of X_j Let $B_2 = \{b_i\} \setminus B_1$. If B_1 has only finitely many points, we may suppose it to be empty. Lemma 6.4.2 now gives the desired result. If B_1 is infinite, use the induction assumption infinitely many times to construct open sets U_i in X_{ji} missing those points (only finitely many—use the intersection of the neighborhoods provided by the assumption) reachable from X_{ji}. These U_i, together with 0 and those X_j not containing some U_i, constitute an X_j-neighborhood of 0 with b in the closure of its complement.

If α is a limit ordinal, a similar argument works. We omit the details. □

6.4.4. Lemma. *If $\{a_i\}$ is a countable subset of $Y \setminus S_\alpha$ from which 0 can be reached in l iterations of sequential closure, and if each a_i can be reached from the isolated points of S_α in no more than m iterations, where $m + l < \alpha$, then there is a countable subset b_i of $Y \setminus S_\alpha$ with no b_i in the closure of a single summand X_j of S_α, such that 0 can be reached in n iterations from b_i, and each b_i can be reached from the isolated points of S_α in no more than t iterations, with $t < m$ and $t + n < \alpha$.*

Proof. By Lemma 6.4.3 we need consider only that subset of $\{a_i\}$ whose points are not in the closure of a single X_j. By Lemma 6.4.2 we have that $m > 1$. Hence each a_i is the limit of a sequence $\{a_{ij}\}$ in $Y \setminus S_\alpha$ where each $\{a_{ij}\}$ can be reached from the isolated points of S_α in less than m iterations. The union of the $\{a_{ij}\}$ is again a countable subset of $Y \setminus S_\alpha$ from which 0 can be reached in $1 + l$ iterations. Let B be the subset of the union of the $\{a_{ij}\}$ whose elements are not in the closure of a single summand. By Lemma 6.4.3, B is the required subset. □

Proof of Theorem 6.4. If 0 can be reached from the isolated points M of S_α in less than α iterations, there is some sequence $\{a_i\}$ in $Y \setminus S_\alpha$ which converges to 0 and with each a_i reached from M in m iterations where $m + 1 < \alpha$. Applying Lemma 6.4.4 finitely many times, one arrives at a contradiction of Lemma 6.4.2. □

7. Generation by prime spaces

In this section we use prime spaces to construct simple generators for SSEQ. For any S_α in TS_α, let T_α denote the isolated points of S_α together with its level-zero point. Denote by TSS_α the family of all T_α with S_α in TS_α.

7.1. Theorem. *There is a family of cardinality* $\mathfrak{c}$ *of prime subsequential spaces which generates* SSEQ. *In fact, the family* TSS_α, *for each* $\alpha < \omega_1$, *generates all subsequential spaces of order* α.

This result follows easily from the following two key lemmas.

7.1.1. Lemma. *If* p *belongs to* M_α *but not to* M_β *for any* $\beta < \alpha$, *then there is a space* T_α *in* TSS_α *and a mapping* $f: T_\alpha \to X$ *which sends all isolated points of* T_α *into* M *and maps only the zero point of* T_α *onto* p.

Proof. This follows easily from Lemma 6.2. □

7.1.2. Lemma. *Suppose* X *is a topological space so that for each* x *belonging to the closure of a subset* A *of* X *but not to* A, *there is a map* f_x *from a prime space* (P_x, p_x) *into* X *taking* p_x *to* x *and the rest of* P_x *into* A. *If the ranges of all the* f_x *cover* X, *then the sum*, q, *of all such* f_x *is a quotient map from the sum of all the* (P_x, p_x) *onto* X.

Proof. Take a subset A of X such that $q^{-1}(A)$ is closed. If $x \in \bar{A} \setminus A$, some summand (P_x, p_x) maps into $A \cup \{x\}$ with p_x mapping to x. But $q^{-1}(A) \cap P_x$ is closed and, hence, must contain p_x, contradicting $x \notin A$. □

Proof of Theorem 7.1. Given subsequential space X, apply Lemma 7.1.1 for each applicable subset $A \subset X$ and point $x \in \bar{A}$, generating a family of mappings from spaces in TSS_α into X. By Lemma 7.1.2, the map from the topological sum of the domains of the maps from the family is a quotient map. The proof of Theorem 7.1 is now complete. □

This theorem yields immediately a second simple generator for SSEQ.

7.2. Corollary. *The topological sum of all the* T_α *in all the* TSS_α *is a space of cardinality* $\mathfrak{c}$ *that generates* SSEQ.

This corollary is an improvement of Theorem 4.6. However it can be improved further as follows.

7.3. Theorem. *There is a countable space* PS *with subsequential order* ω_1 *that generates* SSEQ.

Proof. The construction of the space PS is based on that of the classical space Ψ of Isbell [12]. Start with the natural numbers $\mathbb{N}$. Choose a maximal pairwise almost disjoint family, Γ, of countably infinite subsets of $\mathbb{N}$. Γ will have cardinality $\mathfrak{c}$. Index the members of Γ with the ordinals less than $\mathfrak{c}$, forming the family $\{G_\alpha \mid \alpha < \mathfrak{c}\}$. The space PS will consist of $\mathbb{N}$ and a new point p. Each point of $\mathbb{N}$ will be isolated. The neighborhoods of p will be such that for each $\alpha < \mathfrak{c}$, $G_\alpha \cup \{p\}$ will be homeomorphic to some $R_\alpha = T_\beta$ belonging to TSS_β. (This is possible since the cardinality of all the T_α is also $\mathfrak{c}$.) One way to achieve this is to map the topological sum of all the such R_α onto PS by mapping each R_α onto $G_\alpha \cup \{p\}$ bijectively with the level-zero point of R_α going to p. The quotient topology induced on $\mathbb{N} \cup \{p\}$ by this mapping will produce PS, which is subsequential.

Since the map $q_\alpha : \mathrm{PS} \to G_\alpha \cup \{p\}$ defined by collapsing all points of $\mathrm{PS} \setminus G_\alpha$ to p is a quotient map, PS generates each T_α. By Theorem 7.1, PS therefore generates SSEQ, and the proof is complete. □

8. Minimal subsequential order of a generator

In this section we show that the order of the simple generator constructed in Theorem 7.3 cannot be improved.

8.1. Lemma. *If S_α is a member of TS_α and if the zero point, 0, of S_α is a cluster point of a subset M of isolated points of S_α, then the union of the successive sequential closures of M is homeomorphic to some member of TS_α.*

Proof. The lemma is clearly true for $\alpha = 1$ or $\alpha = 2$. Suppose it is true for all $\beta < \alpha$, and that the 0 point of S_α (a member of TS_α) is a cluster point of a subset M of isolated points of S_α. Since S_α is sequential there is a sequence $\{b_i\}$ in S_α converging to 0, with each b_i a cluster point of M. By the inductive hypothesis, there is for each i a member X_i of TS_{β_i} with b_i its zero point and $X_i \cap M$ consisting of isolated points. The topological sum of the X_i is then a member of TS_α, has 0 as its zero point, and is the union of the successive sequential closures of M in S_α. □

8.2. Lemma. *If T_α is a member of TSS_α and if the zero point, 0, of T_α is a cluster point of a subset M of T_α, then $M \cup \{0\}$ is homeomorphic to some member of TSS_α.*

Proof. This follows immediately from Lemma 8.1 and the definitions. □

A mapping $f: X \to Y$, where X and Y are prime spaces is called a *prime map* if the zero point of X is the unique point of X mapping onto the zero point of Y. Note that the composition of prime maps are again prime maps.

8.3. Lemma. *If $\alpha \geq \beta$, and X and Y are members of TSS_α and TSS_β respectively, then there is a prime map from X onto Y.*

Proof. For finite $n \geq 2$, to map S_n^- onto S_{n-1}^- via a prime map, recall that S_n^- is a subspace of the sequential sum of copies of S_{n-1}^-. Call the summands X_i. Map each X_i onto S_{n-1}^- via the identity map, and send the zero point of S_n^- to the zero point of S_{n-1}^-. Thus the lemma holds for finite ordinals, since the identity suffices for S_n^- onto S_n^-.

Now assume the lemma holds for all ordinals less than α.

If α is a limit ordinal, then a member X of TSS_α is a subspace of the sequential sum of summands X_{α_i}, where the α_i are nondecreasing, and X_{α_i} belongs to TSS_{α_i}, and the α_i converge to α. Let Y also belong to TSS_α, with Y a subspace of the sequential sum of Y_{β_j} and the same conditions holding. Since the sequences $\{\alpha_i\}$ and $\{\beta_j\}$ both converge to α, there is for each j some first α_j among the α_i that is strictly greater than β_j. By the inductive hypothesis, map each X_{α_i} with $\alpha_j - 1 < \alpha_i \leq \alpha_j$ onto B_j.

If $\beta < \alpha$ choose some $\alpha_i > \beta$. By our inductive hypothesis there is a prime map $f: X_{\alpha_i} \to Y$.

Similarly, there are prime maps $f_j: X_{\alpha_j} \to X_{\alpha_i}$ for all $j > i$. Using the compositions of f and the f_j and mapping all X_{α_k} for $k < i$ to an isolated point in Y, one can construct the desired prime map.

The proof of the nonlimit case is an easier version of the above. □

8.4. Lemma. *If $\alpha > \beta$, and X and Y are members of* TSS_α *and* TSS_β *respectively, then there is no prime map from Y onto X.*

Proof. The lemma is obviously true for $\beta = 1$ by compactness.

Assume the lemma is true for all ordinals $\leq \alpha$ (with $\alpha > 1$). We will show it true for $\alpha + 1$. If X belongs to $\mathrm{TSS}_{\alpha+1}$, then X is a subset of the topological sum of spaces X_i in TSS_α consisting of 0 and the isolated points. Suppose Y belongs to TSS_α. Then Y is, in the same way, a subset of the sequential sum of the Y_i, with Y_i belonging to some TSS_{β_i} with $\beta_i < \alpha$. If f is a prime map from Y to X, let $f_1 = f|Y_1$. Let g_1 be the extension of f_1 that takes the zero point, y_1, of Y_1 to the level-zero point, 0, of X. g_1 cannot be continuous at 0, since its composition with a prime map from X onto X_1 (guaranteed by Lemma 8.3) would contradict our hypothesis. Hence there is an open neighborhood W_1 of 0 whose inverse is not open in Y_1. This implies that there are open neighborhoods W_{1_i} of the zero points, x_i, of X_i such that $f^{-1}(X \setminus \bigcup W_{1_i} \cap Y_1)$ clusters at y_1. In the same way, for any n and for all $i \geq n$ there are open sets W_{n_i}, decreasing as n increases, behaving similarly at y_n of Y_N. Let W consist of 0 and the union of the diagonal sets of the W_{n_i}. Then W is an open neighborhood of 0 in X whose inverse is not open in Y since $f^{-1}(X \setminus W)$ clusters at every y_i of Y_i.

Now suppose α is a limit ordinal and that the lemma is true for all ordinals $\beta < \alpha$. If $\beta < \alpha$, then $\beta + 1 < \alpha$. Suppose $Y \in \mathrm{TSS}_\beta$, $X \in \mathrm{TSS}_\alpha$ and $Z \in \mathrm{TSS}_\beta + 1$. If there is a prime map, $f: Y \to X$, then there is, by Lemma 8.3, a prime map from Y to Z contradicting the result of the last paragraph. This completes the proof. □

8.5. Theorem. *No space with subsequential order less than ω_1 generates* SSEQ. *In particular, no T_α with $\alpha < \omega_1$ generates* SSEQ.

Proof. Suppose X is some subsequential space of order $\beta < \omega_1$ that generates SSEQ. For some $\alpha > \beta$ choose a prime space $T_\alpha \in \mathrm{TSS}_\alpha$ of order α. Then there is a quotient map $q: \bigoplus X_i \to T_\alpha$ from a topological sum of copies of X to T_α. Since $\{0\}$ is not open in T_α, $q^{-1}(0)$ meets some X_i, say X_0, in a nonopen set. Therefore, there is some point $p \in X_0$ and a set $M \subset X_0$ with $p \in \bar{M} \setminus M$ and M disjoint from $q^{-1}(0)$. $M \cup \{0\}$ is a prime space.

Let $q_0 = q | M \cup \{p\}$. q_0 is a prime map. Using Lemma 7.1.1, find some $T_\gamma \in \mathrm{TSS}_\gamma$ with $\gamma \leqslant \beta$ and a prime map f from T_γ onto $M \cup \{p\}$. Then $f \circ q_0$ is a prime map from T_γ onto $q_0(M \cup \{p\})$ which belongs to TSS_α by Lemma 8.2. This contradicts Lemma 8.4. □

8.6. Theorem. *Any space* $X \in \mathrm{TSS}_\alpha \setminus \mathrm{TSS}_\beta$ *generates all subsequential spaces of order* α, *where* $\beta < \alpha < \omega_1$.

The proof is essentially contained in the proof of Theorem 8.5 above.

The authors are indebted to the referee for pointing out that the example in Proposition 2.3 fails to be regular, and also for saving the authors from an embarrassing error.

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