PERGAMON Computers and Mathematics with Applications 40 (2000) 157–164

An International Journal
computers & mathematics
with applications

www.elsevier.nl/locate/camwa

Numerical Conformal Mapping via the Bergman Kernel Using the Generalized Minimum Residual Method

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Abstract—The Bergman kernel function is known to satisfy a certain boundary integral equation of the second kind. For boundaries that possess symmetrical qualities, the integral equation can be transformed into another integral equation that uses only a small part of the original boundary. This paper applies an iterative procedure known as the generalized minimum residual method for the computation of the Riemann mapping function via the Bergman kernel. The complexity of this procedure is $O(n^2)$, where n is the number of collocation points on the boundary of the region. Numerical implementation on some test regions is also presented.

Keywords—Conformal mapping, Integral equation, Bergman kernel, GMRES.

1. INTRODUCTION

Conformal maps are indispensable tools in studying flows, fields, and in solving boundary-value problems. Of special importance is the Riemann mapping function which maps a simply connected domain Ω onto a disk (see, for example, [1]). Closely connected to the Riemann map is the Bergman kernel of Ω. This kernel is also related to some orthonormal polynomials where the orthogonality is defined with respect to an area integral. However, numerical experiments in conformal mapping via this kernel based on orthonormal polynomials show that this method is generally unstable and demands high accuracy for numerical purposes [2–5]. An alternative method of computing the Bergman kernel which totally avoids orthonormalization is based on a certain integral equation of the second kind [6]. For a region Ω whose boundary is smooth, the integral equation's kernel is the real and smooth Neumann kernel. Numerical conformal mapping

0898-1221/00/$ - see front matter Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
PII: S0898-1221(00)00149-8

for symmetric regions via the Bergman kernel is also treated in [6]. Another method of computing the Riemann map is by means of the Szegö kernel which also satisfies a certain integral equation of the second kind [7–9].

In [6] some of the integral equations involved are solved numerically by the Nyström's method. The solution as implemented there takes $O(n^3)$ operation counts by means of the Gaussian elimination. In this paper, we apply an iterative procedure known as the generalized minimum residual method (GMRES) method which reduces this figure to $O(n^2)$.

Organization of this paper is as follows. Section 2 contains a brief review of the Bergman kernel, its relationship with the Riemann mapping function, and a few results of the original work in [6]. Section 3 has the numerical implementation using the GMRES, and Section 4 contains numerical results.

2. THE RIEMANN MAP AND THE BERGMAN KERNEL

Let $\Omega \subset \mathbb{C}$ be any simply connected bounded region whose boundary Γ is assumed to be of class C^2 Jordan curve, i.e., a simple closed curve that is twice continuously differentiable. This means Γ admits a counterclockwise parameterization $z(t)$, $0 \le t \le \beta$, $z(0) = z(\beta)$, $z'(0) = z'(\beta)$, $z''(0) = z''(\beta)$, with $z'(t) = \frac{dz}{dt} \neq 0$, for all t. Thus, the boundary Γ is smooth and possesses a continuously turning tangent. The unit tangent to Γ at the point $z(t)$ will be denoted by $T(z) = z'(t)/|z'(t)|$.

Let $a \in \Omega$ be a fixed point and let R be the Riemann mapping $R : \Omega \to$ Unit Disk normalized at a, that is

$$R(a) = 0, \qquad R'(a) > 0. \tag{2.1}$$

The existence and uniqueness of this map is well known (a proof can be found, for example, in [1]). Since Γ is of class C^2, R' can be extended to a function that is continuous on the closure $\bar{\Omega} = \Omega \cup \Gamma$, such that $R'(z) \neq 0$ for $z \in \Gamma$ (Kellogg's theorem, see, e.g., [10]).

Closely connected to the Riemann mapping function R is the Bergman kernel (briefly, B) of Ω. For definition and basic properties of B, see, e.g., [11–13]. The Bergman kernel $B(z, a)$ is continuous with respect to z on $\bar{\Omega}$, and

$$R'(z) = \sqrt{\frac{\pi}{B(a,a)}} B(z,a), \qquad z \in \bar{\Omega}. \tag{2.2}$$

It can also be shown (see [7]) that the Riemann mapping function can be computed on the boundary Γ without any integration by means of

$$R(z) = \frac{1}{i} T(z) \frac{R'(z)}{|R'(z)|}, \qquad z \in \Gamma. \tag{2.3}$$

Hence, to solve the conformal mapping problem it is sufficient to compute the Bergman kernel. This kernel can be computed as the solution of a second kind integral equation as given in the theorems below.

THEOREM 1. *(See [6].) Let Γ be of class C^2 and $a \in \Omega$ be fixed. The Bergman kernel $B(z, a)$ is the unique continuous solution to the integral equation*

$$B(z,a) + \int_\Gamma M(z,w) B(w,a) |dw| = -\frac{1}{\pi} \frac{\overline{T(z)^2}}{(\bar{z} - \bar{a})^2}, \qquad z \in \Gamma, \tag{2.4}$$

where

$$M(z,w) = \begin{cases} \dfrac{T(w)}{2\pi i} \left[\dfrac{\overline{T(z)^2}}{\bar{w} - \bar{z}} - \dfrac{1}{w - z} \right], & z, w \in \Gamma,\ z \neq w, \\ \dfrac{1}{2\pi} \dfrac{\operatorname{Im}\left[z''(t) \overline{z'(t)} \right]}{|z'(t)|^3}, & z = w \in \Gamma. \end{cases}$$

This integral equation for the Bergman kernel B, however, takes a more pleasant form if we multiply both sides by $T(z)$ and let $\tilde{B}(z,a) = T(z)B(z,a)$, to obtain the following statement.

THEOREM 2. *(See [6].) Let Γ be of class C^2 and $a \in \Omega$ be fixed. The function $\tilde{B}(z,a) = T(z)B(z,a)$ is the unique continuous solution to the integral equation*

$$\tilde{B}(z,a) + \int_\Gamma N(z,w)\tilde{B}(w,a)|dw| = -\frac{1}{\pi}\frac{\overline{T(z)}}{(\bar{z}-\bar{a})^2}, \qquad z \in \Gamma, \tag{2.5}$$

where

$$N(z,w) = \begin{cases} \dfrac{1}{\pi}\operatorname{Im}\left[\dfrac{T(z)}{z-w}\right], & z,w \in \Gamma,\ z \neq w, \\ \dfrac{1}{2\pi}\dfrac{\operatorname{Im}\left[z''(t)\overline{z'(t)}\right]}{|z'(t)|^3}, & z = w \in \Gamma. \end{cases} \tag{2.6}$$

The real kernel N is the familiar Neumann kernel which arises frequently in the integral equations of potential theory and conformal mapping (see, e.g., [11]). The Neumann kernel N is continuous at all points $(z,w) \in \Gamma \times \Gamma$. Also note that $T(z)M(z,w) = T(w)N(z,w)$ for $z,w \in \Gamma$.

If the region Ω has symmetrical qualities, then the integral equation (2.5) for $\tilde{B}$ can be transformed into another integral equation that uses only a small part of the original boundary.

THEOREM 3. *(See [6].) Let Γ be a closed Jordan curve which is invariant under rotations about the origin with an angle $2\pi/m$, where m is a positive integer. Denote the part of Γ with the angular domain $0 \le \arg z \le 2\pi/m$ by Γ_1. Let $\alpha_j := e^{i2\pi j/m}$, for $j = 0,1,\ldots,m-1$. Then, $\tilde{B}(\alpha_j z,0) = \alpha_j \tilde{B}(z,0)$ and*

$$\tilde{B}(z,0) + \int_{\Gamma_1} N_m(z,w)\tilde{B}(w,0)|dw| = -\frac{1}{\pi}\frac{\overline{T(z)}}{\bar{z}^2}, \qquad z \in \Gamma_1, \tag{2.7}$$

where

$$N_m(z,w) = \begin{cases} \dfrac{m}{2\pi i}\left[\dfrac{\overline{wz^{m-2}T(z)}}{\overline{w^m - z^m}} - \dfrac{w^{m-1}T(z)}{w^m - z^m}\right], & z,w \in \Gamma_1,\ z \neq w, \\ N(z,z) + \dfrac{1}{4\pi i}\left[\dfrac{(3-m)\overline{T(z)}}{\bar{z}} - \dfrac{(m-1)T(z)}{z}\right], & z = w \in \Gamma_1, \end{cases} \tag{2.8}$$

where N is the Neumann kernel as defined in (2.6).

Theorem 3 numerically means that it suffices to compute the solution $\tilde{B}$ on the boundary Γ_1, which is just $1/m$ of the original boundary Γ. Note that the new kernel N_m in Theorem 3 is no longer a real kernel. However, for the case $m = 2$ the kernel is real.

Now, consider the case when the boundary Γ is symmetric with m axes of symmetry. Thus, the angle between any two adjacent axes of symmetry is π/m. Let the real axis be one of the axes of symmetry. This type of symmetry is actually a special case of rotational symmetry about the origin with an angle $2\pi/m$. The only added feature here is that the curve Γ_1 is now symmetric with respect to the ray $\theta = \pi/m$. Let G_1 and G_2 denote the curves, respectively, below and above the ray $\theta = \pi/m$. Thus, if $z \in G_2$, then $\alpha_1 \bar{z} \in G_1$.

THEOREM 4. *(See [6].) Let Γ be a closed Jordan curve with m axes of symmetry. Let the real axis be one of the axes of symmetry. Denote the part of Γ with the angular domain $0 \le \arg z \le \pi/m$ by G_1. Then, $\tilde{B}(\bar{z},0) = -\overline{\tilde{B}(z,0)}$ and*

$$\tilde{B}(z,0) + \int_{G_1} N_m(z,w)\tilde{B}(w,0)|dw| - \int_{G_1} N_m(z,\bar{w})\overline{\tilde{B}(w,0)}|dw| = -\frac{1}{\pi}\frac{\overline{T(z)}}{\bar{z}^2}, \qquad z \in \Gamma_1, \tag{2.9}$$

where N_m is defined as in (2.8).

Equation (2.9) numerically means that it suffices to compute the solution $\tilde{B}$ on the boundary G_1, which is just $1/(2m)$ of the original boundary Γ.

3. NUMERICAL IMPLEMENTATION

In this section, we consider the numerical solution of the integral equation (2.9). Suppose Γ admits a counterclockwise parametrization $z(t)$, $0 \le t \le \beta$, $z(0) = z(\beta)$ with $\dot{z}(t) = \frac{dz}{dt} \neq 0$, for all t. Thus, G_1 has the parametrization $z(t)$, $0 \le t \le \gamma$ with $2m\gamma = \beta$ and $z(0) \neq z(\gamma)$. Then (2.9) can be rewritten as

$$\phi(t) + \int_0^\gamma \lambda_m(t,s)\phi(s)\,ds - \int_0^\gamma \nu_m(t,s)\bar{\phi}(s)\,ds = \psi(t), \qquad 0 \le t \le \gamma, \tag{3.1}$$

where

$$\begin{aligned}
\phi(t) &= |z'(t)|\tilde{B}(z(t),0),\\
\lambda_m(t,s) &= |z'(t)|N_m(z(t),z(s)),\\
\nu_m(t,s) &= |z'(t)|N_m\left(z(t),\overline{z(s)}\right),\\
\psi(t) &= -\frac{1}{\pi}\frac{\overline{z'(t)}}{\overline{z(t)^2}}.
\end{aligned}$$

The Nyström algorithm associated with an n-point midpoint rule replaces the integral equation (3.1) with

$$\phi(t_i) + \frac{\gamma}{n}\sum_{k=1}^{n}\lambda_m(t_i,t_k)\phi(t_k) - \frac{\gamma}{n}\sum_{k=1}^{n}\nu_m(t_i,t_k)\bar{\phi}(t_k) = \psi(t_i), \tag{3.2}$$

for $i = 1,\ldots,n$, $t_i = \gamma(2i-1)/(2n)$. This gives rise to a complex system of n linear equations in n unknowns

$$(I+E)\mathbf{x} - F\bar{\mathbf{x}} = \mathbf{y}, \tag{3.3}$$

where $x_i = \phi(t_i)$, $y_i = \psi(t_i)$, $E_{ik} = \gamma\lambda_m(t_i,t_k)/n$, and $F_{ik} = \gamma\nu_m(t_i,t_k)/n$. The complex system (3.3) can be rewritten as a $2n$ by $2n$ real system. To see this, we write $\mathbf{x} = \Re\mathbf{x} + i\Im\mathbf{x}$, $\mathbf{y} = \Re\mathbf{y} + i\Im\mathbf{y}$. Upon substituting and comparing the real and imaginary parts, (3.3) becomes

$$(I+J)\mathbf{u} = \mathbf{p}, \tag{3.4}$$

where

$$J = \begin{bmatrix} E-F & O \\ O & E+F \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} \Re\mathbf{x} \\ \Im\mathbf{x} \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} \Re\mathbf{y} \\ \Im\mathbf{y} \end{bmatrix}.$$

If $m = 2$, then the matrices E and F are real.

After we obtain the computed solution vector $\mathbf{x}$ at the collocation points t_i of the boundary G_1, the solution vector on the entire boundary Γ is given by

$$\left[\mathbf{x}^{*\mathsf{T}}|\alpha_1\mathbf{x}^{*\mathsf{T}}|\cdots|\alpha_{m-1}\mathbf{x}^{*\mathsf{T}}\right]^T, \tag{3.5}$$

which consists of m partitions and where

$$\mathbf{x}^{*\mathsf{T}} = [x_1,\ldots,x_{n+1},-\alpha_1\bar{x}_n,\ldots,-\alpha_1\bar{x}_2] \tag{3.6}$$

represents the solution vector on $\Gamma_1 = G_1 \cup G_2$.

We solve the system (3.4) using an iterative method based on the restarted version of the generalized minimal residual (GMRES) method of Saad and Schultz [14]. The GMRES method is formulated in such a way that it is directly applicable to linear systems whose coefficient matrices are not symmetric and/or positive definite. In [15], the performance of GMRES was found to be quite efficient in a test case involving four different integral equations arising in

potential theory. The fundamental idea of GMRES is to minimize the norm of the computed residual vector, $\mathbf{r}_i$ ($\mathbf{r}_i \equiv \mathbf{p} - (I+J)\mathbf{u}_i$) at each iteration. We describe next the restarted version of GMRES denoted by GMRES(p), where p is some fixed positive integer parameter.
GMRES(p).

(1) *Start.* Choose initial solution $\mathbf{u}_0$ and compute $\mathbf{r}_0 = \mathbf{p} - (I+J)\mathbf{u}_0$ and $\mathbf{v}_1 = \mathbf{r}_0/\|\mathbf{r}_0\|$, where $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$, with $\langle\cdot,\cdot\rangle$ being the usual dot product.

(2) *Iterate.* For $j = 1, 2, \ldots, p$ do

$$h_{i,j} = \langle (I+J)\mathbf{v}_j, \mathbf{v}_i \rangle, \qquad i = 1, 2, \ldots, j,$$

$$\hat{\mathbf{v}}_{j+1} = (I+J)\mathbf{v}_j - \sum_{i=1}^{j} h_{i,j}\mathbf{v}_i,$$

$$h_{j+1,j} = \|\hat{\mathbf{v}}_{j+1}\|, \qquad \text{and}$$

$$\mathbf{v}_{j+1} = \frac{\hat{\mathbf{v}}_{j+1}}{h_{j+1,j}}.$$

(3) *Form the approximate solution* $\mathbf{u}_p = \mathbf{u}_0 + V_p\mathbf{y}_p$, where the vector $\mathbf{y}_p$ minimizes $\|(\|\mathbf{r}_0\|)\mathbf{e}_1 - H_p\mathbf{y}\|$ and where the vector $\mathbf{e}_1$ is the first column of the $(p+1)\times(p+1)$ identity matrix, and

$$V_p = (\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \ldots \quad \mathbf{v}_p\,)$$

has orthogonal columns, and

$$H_p = \begin{pmatrix} h_{11} & h_{12} & \ldots & h_{1p} \\ h_{21} & \ddots & & \vdots \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & h_{p,p-1} & h_{pp} \\ 0 & \ldots & 0 & h_{p+1,p} \end{pmatrix}$$

is a $(p+1)\times p$ upper Hessenberg matrix of full rank p.

(4) *Restart.* Compute $\mathbf{r}_p = \mathbf{p} - (I+J)\mathbf{u}_p$; if satisfied then stop, else compute $\mathbf{u}_0 := \mathbf{u}_p$, $\mathbf{v}_1 := \mathbf{u}_p/\|\mathbf{u}_p\|$ and go to Step 2.

Step 2 of this algorithm is commonly known as the Arnoldi's process which uses the Gram-Schmidt method for computing an l_2-orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ of the Krylov subspace $K_p = \text{span}\{\mathbf{r}_0, (I+J)\mathbf{r}_0, \ldots, (I+J)^{p-1}\mathbf{r}_0\}$. In our numerical implementation we have replaced the Gram-Schmidt algorithm of Step 2 by the modified Gram-Schmidt algorithm [16]. As for Step 3, a common procedure of solving $\mathbf{y}_p$ is to perform the QR factorization of H_p, i.e., $Q_pH_p = R_p$, and then use the upper-triangular matrix R_p to solve a system of the type $R_p\mathbf{y}_p = g_p$. For a detailed description of GMRES(p) algorithm, we refer the reader to [14]. In our numerical examples we have used GMRES(p) with $p = 5$, 10, 15, and 20. It is observed that the process GMRES(p) improved substantially as p increased. For GMRES(20), the number of iterations is at most 2.

After we obtain the computed solutions ϕ at the collocation points t_i, discretization of (3.1) provides us with an interpolation formula

$$\phi(t) = \psi(t) - \frac{\gamma}{n}\sum_{k=1}^{n}\lambda_m(t,t_k)\phi(t_k) + \frac{\gamma}{n}\sum_{k=1}^{n}\nu_m(t,t_k)\bar{\phi}(t_k). \tag{3.7}$$

We start the iteration for solving (3.4) with the right-hand side $\mathbf{p}$ for small n, and then use the interpolation formula (3.7) to provide good starting vectors for large n.

Finally, the boundary correspondence function $\theta(t)$ to any representation $z = z(t)$, $0 \le t \le \beta$, of Γ, defined by

$$R(z(t)) = e^{i\theta(t)}, \tag{3.8}$$

may be computed (without integration) by the formula [6]

$$\theta(t) = \arg(-i\phi(t)). \tag{3.9}$$

4. SOME NUMERICAL RESULTS

In this section, the numerical scheme discussed in Section 3 is applied to several two-fold symmetric regions (i.e., $m = 2$) with the usual normalization $R(0) = 0$, $R'(0) > 0$. The stopping criterion employed in the iterative process of the GMRES is $\|\mathbf{r}^{(i)}\|_\infty < 10^{-13}$. The computer programming was entirely done using the MATHEMATICA package [17] in single precision (16 digit machine precision).

Following [9], we list the sup-norm error $\|\theta(t) - \theta_M(t)\|_\infty$, where $\theta(t)$ is the exact boundary correspondence function and $\theta_M(t)$ is the approximation obtained by means of (3.4) and the interpolation formula (3.7) using M equally spaced points in the parameter interval, most of which are not the original collocation points. In all our experiments, we have chosen $M = 30$. This allows exact comparisons with some of the numerical results given in [8,9]. Tables 1–4 show the numerical results.

EXAMPLE 1. Ellipse ($0 \le \epsilon < 1$, axis ratio $= (1+\epsilon)/(1-\epsilon)$).

$$z(t) = e^{it} + \epsilon e^{-it},$$
$$\theta(t) = t + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{\epsilon^k}{1+\epsilon^{2k}} \sin(2kt).$$

Table 1. Ellipse. Error norm $\|\theta(t) - \theta_M(t)\|_\infty$.

	Axis Ratio					
n	1.2	1.5	2.0	3.0	5.0	10.0
1	1.0(−02)	1.3(−01)	–	–	–	–
2	1.6(−04)	9.6(−03)	2.1(−01)	–	–	–
4	2.1(−08)	3.0(−05)	4.6(−03)	7.1(−01)	–	–
8	8.9(−16)	1.5(−10)	1.4(−06)	5.7(−03)	–	–
16	–	8.9(−16)	6.6(−14)	1.7(−07)	9.6(−02)	–
32	–	–	2.0(−15)	2.5(−15)	4.4(−07)	–
64	–	–	–	–	2.4(−12)	7.3(−02)
128	–	–	–	–	4.1(−13)	7.3(−08)
256	–	–	–	–	–	9.1(−08)

EXAMPLE 2. Inverted ellipse ($0 < p \le 1$).

$$z(t) = \sqrt{1 - (1-p^2)\cos^2(t)}\, e^{it},$$
$$\tan\theta(t) = p^{-1} \tan t.$$

Table 2. Inverted ellipse. Error norm $\|\theta(t) - \theta_M(t)\|_\infty$.

	Values of p				
n	0.8	0.5	0.2	0.1	0.05
1	4.0(−03)	1.1(−01)	–	–	–
2	3.7(−06)	7.7(−03)	–	–	–
4	5.0(−08)	9.6(−04)	–	–	–
8	1.0(−14)	2.3(−06)	1.3(−01)	–	–
16	5.6(−16)	3.6(−12)	5.9(−03)	–	–
32	–	4.2(−15)	4.0(−05)	2.9(−01)	–
64	–	–	1.7(−09)	8.7(−03)	–
128	–	–	4.4(−15)	3.9(−05)	–
256	–	–	–	1.1(−09)	2.0(−02)

EXAMPLE 3. Oval of Cassini ($|z-\alpha|\,|z+\alpha| = 1$, $0 \le \alpha < 1$).

$$z(t) = \left(\alpha^2 \cos 2t + \sqrt{1-\alpha^4 \sin^2 2t}\right)^{1/2} e^{it},$$
$$\theta(t) = t - \frac{1}{2}\arg(\zeta(t)),$$
$$\zeta(t) = \sqrt{1-\alpha^4 \sin^2 2t} + i\alpha^2 \sin 2t.$$

Table 3. Oval of Cassini. Error norm $\|\theta(t) - \theta_M(t)\|_\infty$.

	Values of α						
n	0.2	0.5	0.8	0.9	0.99	0.999	0.9999
1	2.4(−05)	6.5(−03)	–	–	–	–	–
2	4.0(−09)	4.2(−05)	8.0(−03)	4.5(−02)	–	–	–
4	1.3(−15)	6.0(−09)	7.6(−05)	1.8(−03)	–	–	–
8	–	1.6(−15)	1.4(−08)	4.9(−06)	2.2(−02)	1.7(−01)	–
16	–	–	2.2(−15)	6.8(−11)	5.5(−04)	7.1(−02)	–
32	–	–	–	1.8(−15)	5.5(−07)	8.1(−03)	1.1(−01)
64	–	–	–	–	9.2(−13)	8.9(−05)	3.8(−02)
128	–	–	–	–	3.6(−14)	1.7(−08)	2.4(−03)
256	–	–	–	–	–	2.0(−13)	8.9(−06)

EXAMPLE 4. Square.

$$z(t) = \frac{1}{\sqrt{2}}[1-(1-i)t], \qquad 0 < t < 1, \quad \text{(first quadrant)},$$
$$\cos\theta(t) = \left[1 + \frac{1}{2}\left(\frac{1}{\operatorname{cn}(Kt)} - \operatorname{cn}(Kt)\right)^2\right]^{-1}, \qquad K = \frac{1}{4\pi}\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2,$$

where cn is one of the Jacobian elliptic functions and $\Gamma(\cdot)$ is the gamma function. Notice, that for this region, the smoothness condition for the boundary is no longer satisfied. In this example, the tangents at $t = 0$ and $t = 1$ are not even defined. However, since we are employing the n midpoint rule, the tangents at these two endpoints are not needed. Nevertheless, we still get reasonable approximations to the solutions.

Table 4. Square.

n	$\|\theta(t) - \theta_M(t)\|_\infty$
8	7.2(−02)
16	1.9(−02)
32	4.4(−03)
64	1.0(−03)
128	2.6(−04)
256	6.4(−05)

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