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Graph Minors.
VIII. A Kuratowski Theorem for General Surfaces

NEIL ROBERTSON

Department of Mathematics, Ohio State University,
231 West 18th Ave., Columbus, Ohio 43210

AND

P. D. SEYMOUR*

Bellcore, 445 South Street, Morristown, New Jersey 07960

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We prove that for any infinite set of graphs of bounded genus, some member of the set is isomorphic to a minor of another. As a consequence, for any surface Σ there is a finite list of graphs, such that a general graph may be drawn in Σ if an only if it topologically contains none of the graphs in the list.

1. INTRODUCTION

Kuratowski's famous theorem of 1930 [6] asserts that a graph is planar if and only if it does not topologically contain K_5 or $K_{3,3}$. (Graphs are finite, and may have loops or multiple edges. A graph G *topologically contains* H if we may obtain a graph isomorphic to H from some subgraph of G by suppressing some divalent vertices.) There arose in the 1930's the proposal to find parallels to this theorem which apply to other surfaces.

If Σ is a surface, let $T(\Sigma)$ denote the class of all graphs which cannot be drawn in Σ and which are minimal with this property under topological containment. If Σ is the plane or sphere then the members of $T(\Sigma)$ are precisely the graphs isomorphic to K_5 or $K_{3,3}$—this is another way to state Kuratowski's theorem. In general, a graph can be drawn in Σ if and only if it topologically contains no members of $T(\Sigma)$. It remains then to determine $T(\Sigma)$ explicitly.

This appears to be very difficult, and there was little progress on the

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problem until 1979, when Archdeacon, Glover, Huneke, and Wang [1, 3] determined $T(\Sigma)$ when Σ is the projective plane. They found that in that case $T(\Sigma)$ has 103 members, up to isomorphism.

One can show that for higher surfaces Σ, $T(\Sigma)$ has a large number of non-isomorphic members, and it is not clear even that it has only finitely many. The question of the finiteness of $T(\Sigma)$ up to isomorphism was raised by Erdős in the 1930's, but again there was little progress until recently. In 1980 Archdeacon and Huneke [2] proved finiteness for all non-orientable surfaces Σ. The orientable cases have remained open (except for the torus, for which R. Bodendiek and K. Wagner have recently announced a proof of finiteness). In this paper we complete the solution to Erdős' problem—we show that for any surface Σ, $T(\Sigma)$ is indeed finite up to isomorphism.

This is proved as a consequence of a more general "well-quasi-ordering" result about graph minors. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by edge-contraction. The "minor" relation is different from topological containment—if H is topologically contained in G then it is isomorphic to a minor of G, but the converse is not necessarily true, as is easily seen. However, Erdős' problem may be reformulated in terms of minors; for, up to isomorphism, $T(\Sigma)$ is finite if and only if the class of minor-minimal graphs which cannot be drawn in Σ is finite. (We prove this fact in Section 2.)

We approach the "minor" formulation of Erdős' problem by means of an idea of Wagner which dates from the 1960's. Let us choose one representative from each isomorphism class of the class of minor-minimal graphs which cannot be drawn in Σ, and let the class of graphs we obtain be A. Now A has the desirable property that it is an *antichain*, that is, no member of A is isomorphic to a minor of another. (This follows from the fact that the members of A are minor-minimal with a certain property, viz. not being drawable in Σ.) Wagner made the conjecture that *every* antichain of graphs is finite.

Although we do not prove Wagner's conjecture here, we shall prove the following special case which is strong enough to settle Erdős' problem. (We hope to prove Wagner's conjecture in full in a later paper.)

(1.1) *For any number g, every antichain of graphs all of genus $\leqslant g$ is finite.*

[The *genus* of a graph is the minimum genus (orientable or non-orientable) of the surfaces in which it can be drawn.]

This settles Erdős' problem; for if Σ is orientable with genus g, it is easy to see that all members of A have genus at most $g+1$, and so by (1.1), A is finite. (The non-orientable case is similar.)

Now for any number g there is a finite list of surfaces such that every graph with genus $\leqslant g$ can be drawn on some surface in the list. Thus to prove (1.1) it suffices to prove the following.

(1.2) *For any surface Σ, every antichain whose members can all be drawn in Σ is finite.*

The key to our proof of (1.2) is to work with a slightly strengthened form, concerning graphs on Σ with some bounded number of vertices from each graph chosen as roots. It is quite easy, using a lemma proved in [11], to show that the rooted form of (1.2) is true in general if it is true when Σ is the sphere and all the roots lie on two distinguished regions (the so-called *cylinder* case). With rather more difficulty and the aid of Higman's well-quasi-ordering theorem for finite sequences, we are able to reduce the cylinder case to the case when Σ is the sphere and all the roots lie on one distinguished region (the *disc* case). This very special case is the most difficult of all; to settle it we need the concepts of tree-width from [7, 9] and part of the theory of patchworks developed in [8]. These steps are performed more or less in reverse order.

Surfaces in this paper have (possibly empty) boundary. While the presence of the boundary has no effect on the embedding capabilities of the surface, it enables us to work only with graphs drawn on a surface where the root vertices are the ones that lie on the boundary, and this is convenient for technical and notational reasons.

2. Erdős' Problem in Minors

Here we show that Erdős' problem can be reduced to showing that for any surface Σ, the class of minor-minimal graphs which cannot be drawn in Σ is finite up to isomorphism. In fact, a more general statement is true. Let P be a property of graphs such that if G has P and H is isomorphic to a minor of G then H has P—we call such a property P a *hereditary* property. For example, if Σ is a surface, the property "G can be drawn in Σ" is hereditary. Let $T(P)$ denote the class of all graphs, minimal under topological containment, without property P, and let $S(P)$ denote the class of all graphs, minimal under minor containment, without property P. The main result of this section is the following.

(2.1) *For any hereditary property P, $T(P)$ is finite up to isomorphism if and only if $S(P)$ is finite up to isomorphism.*

To obtain from this the reformulation of Erdős' problem in terms of minors, we simply take P to be the property of being drawable in Σ. To prove (2.1) we need a lemma. Let e be an edge of a graph G_1 with distinct ends, both with valency at least three. (The *valency* of a vertex is the number of edges incident with it, counting loops twice.) Let G_2 be the graph obtained from G_1 by contracting e. Then we say that G_1 is obtained from G_2 by *splitting* a vertex. We say that G_1 is obtained from G_2 by *splitting vertices* if there is a sequence

$$G_2 = H_0, H_1, H_2, ..., H_k = G_1$$

of graphs where for $1 \leqslant i \leqslant k$, H_i is obtained from H_{i-1} by splitting a vertex. The lemma we need is the following.

(2.2) *If H is a minor of G, there is a graph H' which can be obtained from H by splitting vertices and which is topologically contained in G.*

Proof. Let G' be a minimal subgraph of G which has H as a minor. Then evidently H is obtained from G' by contracting some edges of G', say those in $E_1 \subseteq E(G')$. Now if $e \in E_1$, the graph obtained from G' by deleting e and contracting all other edges of E_1 does not have H as a minor, while the one obtained by contracting all edges of E_1 does. It follows that e is not a loop of the graph obtained from G' by contracting all edges of $E_1 - \{e\}$. Hence, E_1 is the edge set of a subforest of G'. For a similar reason, no edge in E_1 is incident with a vertex with valency one in G'.

Choose a sequence $e_1, ..., e_n$ of distinct edges of E_1 with n maximum, such that for $1 \leqslant i \leqslant n$ there is a vertex of G' of valency two incident both with e_i and with some edge distinct from $e_1, ..., e_i$. Let H' be the graph obtained from G' by contracting $e_1, e_2, ..., e_n$. We claim that H' satisfies the theorem. Certainly (from the definition of $e_1, ..., e_n$) H' is topologically contained in G' and hence in G; it remains to show that H' can be obtained from H by splitting vertices.

Put $E_2 = E(H') \cap E_1$. Then E_2 is the edge-set of a subforest of H', and H may be obtained from H' by contracting the edges in E_2. Now no edge $e \in E_2$ is incident with a vertex of H' with valency 2, because we could then set $e_{n+1} = e$, contrary to the maximality of n. Thus every edge of E_2 has both ends with valency $\geqslant 3$ in H'. Let $E_2 = \{f_1, ..., f_k\}$, and for $0 \leqslant i \leqslant k$ let F_i be $\{f_j : i < j \leqslant k\}$. Let H_i be the graph obtained from H' by contracting all edges in F_i. Then $H_0 = H$ and $H_k = H'$, and for $1 \leqslant i \leqslant k$ H_{i-1} is obtained from H_i by contracting f_i. The ends of f_i in H_i are distinct, since E_2 is the edge-set of a subforest of H'. We claim that both ends of f_i have valency $\geqslant 3$ in H_i. For let u be an end of f_i in H_i, and let v be the corresponding end of f_i in H'. Let T be the component of the forest $(V(H'), F_i)$ with $v \in V(T)$. Now every edge of H' not in $E(T)$ but incident with a vertex of T is incident with u in H_i. If $|V(T)| = 1$ it follows that u has valency $\geqslant 3$ in H_i, since v has valency $\geqslant 3$ in H'. If $|V(T)| \geqslant 2$, there is a vertex w of T, with valency 1 in T, different from v. Then w has valency $\geqslant 3$ in H', and so is incident with at least two edges of H' not in $E(T)$. It follows that u has valency $\geqslant 3$ in H_i. We deduce that H_i is obtained from H_{i-1} by splitting a vertex, and so H' is obtained from H by splitting vertices, as required. ▮

For any graph H, let $Z(H)$ denote the class of all graphs which can be obtained from H by splitting vertices.

(2.3) *For any graph H, $Z(H)$ is finite up to isomorphism.*

Proof. For any graph G, define $N(G)$ to be $2|E(G)| - 3|V'|$ where V' is the set of vertices of G with valency $\geqslant 3$. It is easy to see that $N(G) \geqslant 0$, and that if G is obtained from H by splitting a vertex then $N(G) < N(H)$. The result follows. ▮

Proof of (2.1). If H is minor-minimal without property P, then it is certainly minimal without P under topological containment, and so $S(P) \subseteq T(P)$. Suppose now that $G \in T(P)$. We know that G does not have property P, and so it has a minor $H \in S(P)$. By (2.1), there is a graph $H' \in Z(H)$ which is topologically contained in G. But H' does not have P, since H is a minor of H', and yet $G \in T(P)$; thus $G = H'$. If follows that $G \in Z(H)$. Hence

$$S(P) \subseteq T(P) \subseteq \bigcup (Z(H) : H \in S(P)).$$

But by (2.2), $Z(H)$ is finite up to isomorphism for each graph H; and the result follows. ▮

3. Well-Quasi-Orders

We state in this section some basic results about well-quasi-orders which we shall need for our argument. For proofs, see [4].

A *quasi-order* $\mathbf{Q} = (Q, \leqslant)$ is a class Q together with a transitive, reflexive relation $\leqslant$. (It becomes a partial order if we also require $\leqslant$ to be antisymmetric.) A quasi-order $(Q, \leqslant)$ is a *well-quasi-order* if for every countable sequence $q_1, q_2, \ldots$ of members of Q there exist $j > i \geqslant 1$ such that $q_i \leqslant q_j$.

(3.1) *If $\mathbf{Q} = (Q, \leqslant)$ is a well-quasi-order then every countable sequence $q_1, q_2, \ldots$ of members of Q has a countable subsequence $q_{i_1}, q_{i_2}, \ldots$ (where $i_1 < i_2 < \cdots$) such that $q_{i_1} \leqslant q_{i_2} \leqslant \cdots$.*

If $\mathbf{Q}_1 = (Q_1, \leqslant_1)$ and $\mathbf{Q}_2 = (Q_2, \leqslant_2)$ are quasi-orders, we define $\mathbf{Q}_1 \times \mathbf{Q}_2 = (Q_1 \times Q_2, \leqslant)$, where $(p_1, p_2) \leqslant (q_1, q_2)$ if $p_1 \leqslant_1 q_1$ and $p_2 \leqslant_2 q_2$.

(3.2) *If* $\mathbf{Q}_2$ *and* $\mathbf{Q}_2$ *are well-quasi-orders, so is* $\mathbf{Q}_1 \times \mathbf{Q}_2$.

If $\mathbf{Q} = (Q, \leqslant)$ is a quasi-order, $\mathbf{Q}^{<\omega} = (Q^{<\omega}, \leqslant)$ is defined as follows. $Q^{<\omega}$ is the class of all finite sequences of members of Q. If $(p_1, ..., p_r)$, $(q_1, ..., q_s) \in Q^{<\omega}$, we say that $(p_1, ..., p_r) \leqslant (q_1, ..., q_s)$ if there exist integers

$$1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant s$$

such that $p_j \leqslant q_{i_j}$ $(1 \leqslant j \leqslant r)$. The following is due to Higman [4].

(3.3) *If* $\mathbf{Q}$ *is a well-quasi-order then so is* $\mathbf{Q}^{<\omega}$.

4. Grafts

In the course of the paper we shall define a "simulation" of X in Y, for several different kinds of objects X and Y. "X is simulated in Y" will always mean "there is a simulation of X in Y."

Let G, H be graphs. It is easy to see that H is isomorphic to a minor of G if and only if there is a function σ such that for each edge e of H, $\sigma(e)$ is an edge of G, and for every vertex v of H, $\sigma(v)$ is a non-null connected subgraph of G, with the following properties:

(S1) $\sigma(e) \neq \sigma(e')$ for distinct $e, e' \in E(H)$

(S2) $\sigma(v)$ and $\sigma(v')$ are vertex-disjoint for distinct $v, v' \in V(H)$

(S3) $\sigma(e)$ is not an edge of $\sigma(v)$, for $e \in E(H)$ and $v \in V(H)$

(S4) if e is a loop of H incident with $v \in V(H)$ then $\sigma(e)$ is incident only with vertices of $\sigma(v)$

(S5) if $e \in E(H)$ has distinct ends $v_1, v_2 \in V(H)$ then $\sigma(e)$ has one end in $\sigma(v_1)$ and the other in $\sigma(v_2)$.

We call such a function σ a *simulation of* H *in* G.

A *graft* is a pair (G, T) where G is a graph and $T \subseteq V(G)$. We call $|T|$ the *index* of the graft. If (G, T), (H, U) are grafts, a *simulation of* (H, U) *in* (G, T) is a simulation σ of H in G with the following additional property:

(S6) $|T| = |U|$, and for each $u \in U$, some vertex of $\sigma(u)$ is in T. (This vertex is necessarily unique.)

Let $\mathscr{A}$ be a class of grafts. The relation "is simulated in" is a quasi-order of $\mathscr{A}$. If it is a well-quasi-order we say that $\mathscr{A}$ is *well-rooted.* If there is a maximum index of the members of $\mathscr{A}$, we call this the *index* of $\mathscr{A}$. (Conventionally, the empty set has index zero.)

(4.1) *If* $\mathscr{A}$ *is well-rooted then it has an index.*

Proof. If $\mathscr{A}$ has no index, then there is a countable sequence (G_1, T_1), $(G_2, T_2), \ldots$ of members of $\mathscr{A}$ such that $|T_1| < |T_2| < \cdots$. But then for $1 \leqslant i < i'$, (G_i, T_i) is not simulated in $(G_{i'}, T_{i'})$, because (S6) is not satisfied. Hence $\mathscr{A}$ is not well-rooted. ∎

We mention that we believe that the converse to (4.1) also holds, and hope to publish a proof of this in a later paper. This would clearly imply Wagner's conjecture.

(4.2) *If $\mathscr{A}_1, \mathscr{A}_2$ are well-rooted then so is $\mathscr{A}_1 \cup \mathscr{A}_2$.*

Proof. Any countable sequence of elements of $\mathscr{A}_1 \cup \mathscr{A}_2$ has a countable subsequence either with all entries in $\mathscr{A}_1$ or with all entries in $\mathscr{A}_2$. The result follows. ∎

Grafts are one way to work with multiply-rooted graphs. But sometimes we need to distinguish the roots, and it is more convenient to work with k-tuples of vertices rather than subsets. A *rooted graph* then is a pair (G, τ), where G is a graph and τ is a finite sequence of vertices of G. The *index* of a rooted graph is the length of τ. If τ has length k and $1 \leqslant i \leqslant k$, $\tau(i)$ denotes the ith term of τ. If (G, τ) and (H, ν) are rooted graphs, a *simulation* of (H, ν) in (G, τ) is a simulation σ of H in G with the following additional property:

(S7) (G, τ) and (H, ν) have the same index k say, and for $1 \leqslant i \leqslant k$, $\tau(i)$ is a vertex of $\sigma(\nu(i))$; and for $1 \leqslant i < j \leqslant k$, $\tau(i) = \tau(j)$ if and only if $\nu(i) = \nu(j)$.

If (G, τ) is a rooted graph of index k, its *underlying graft* is (G, T), where $T = \{\tau(1), \ldots, \tau(k)\}$. It is easy to see that if (H, ν) is simulated in (G, τ) then (H, U) is simulated in (G, T), where (H, U) and (G, T) are the respective underlying grafts.

Again, the relation "is simulated in" is a quasi-order on any class of rooted graphs; and again if it is a well-quasi-order we say the class of rooted graphs is *well-rooted.* We need the following lemma.

(4.3) *Let $\mathscr{A}$ be a well-rooted class of grafts, and let $\mathscr{B}$ be a class of rooted graphs of bounded index, such that for each member of $\mathscr{B}$, its underlying graft is in $\mathscr{A}$. Then $\mathscr{B}$ is well-rooted.*

Proof. It is enough to prove that for every number k, the class of members of $\mathscr{B}$ with index k is well-rooted, since $\mathscr{B}$ is the union of only finitely many such classes.

Thus we may assume that all members of $\mathscr{B}$ have index k. Let (G_1, τ_1), $(G_2, \tau_2), \ldots$ be a countable sequence of members of $\mathscr{B}$. For each i, let

$(G_i, T_i) \in \mathscr{A}$ be the underlying graft of (G_i, τ_i). Since $\mathscr{A}$ is well-rooted, there is by (3.1) a countable sequence of integers $0 < i_1 < i_2 < \cdots$ such that for $j \geqslant 1$, (G_{i_j}, T_{i_j}) is simulated in $(G_{i_{j+1}}, T_{i_{j+1}})$.

We wish to show that for some $i' > i \geqslant 1$, (G_i, τ_i) is simulated in $(G_{i'}, \tau_{i'})$. If this is true for the subsequence (G_{i_j}, τ_{i_j}) $(j = 1, 2, ...)$ then it is true for the original sequence. Thus, for simplicity of notation, we replace the original sequence by the subsequence. Hence (G_i, T_i) is simulated in (G_{i+1}, T_{i+1}), for $i = 1, 2, ...$ Let σ_i be a simulation of (G_i, T_i) in (G_{i+1}, T_{i+1}) and for each $v \in T_{i+1}$, let $\sigma_i^{-1}(v)$ be the element $u \in T_i$ such that v is a vertex of $\sigma_i(u)$.

For $i = 1, 2, ...$ and for each $v \in T_i$, let $\pi_i(v)$ be

$$\sigma_1^{-1}(\ldots \sigma_{i-2}^{-1}(\sigma_{i-1}^{-1}(v)) \ldots).$$

Then $\pi_i(v) \in T_1$. Since T_1 is finite, there exist $i' > i \geqslant 1$ such that

$$\pi_{i'}((\tau_{i'}(j)) = \pi_i(\tau_i(j)) \qquad (1 \leqslant j \leqslant k).$$

But then

$$\tau_i(j) = \sigma_i^{-1}(\sigma_{i+1}^{-1} \ldots (\sigma_{i'-1}^{-1}(\tau_{i'}(j))) \ldots) \qquad (1 \leqslant j \leqslant k)$$

and so the composition in the natural sense of $\sigma_i, \sigma_{i+1}, ..., \sigma_{i'}$ is a simulation of (G_i, τ_i) in $(G_{i'}, \tau_{i'})$ as required. ∎

5. Drawings of Grafts

A *surface* is a compact 2-manifold with (possibly empty) boundary. We denote the boundary of a surface Σ by $bd\,\Sigma$, and each component of $bd\,\Sigma$ is called a *cuff* of Σ. We denote the closure of a subset $Z \subseteq \Sigma$ by $\bar{Z}$. If X is a topological space, an *X-arc* in Σ is a subset of Σ homeomorphic to X. In particular, an *O-arc* is a subset of Σ homeomorphic to a circle. (Every cuff of Σ is thus an O-arc.) We denote by $\Sigma(a, b, c)$ the surface obtained from the sphere by adding a handles and b cross-caps, and removing the interiors of c pairwise disjoint closed discs. Every connected surface is homeomorphic to $\Sigma(a, b, c)$ for some choice of a, b, c. A *drawing* Γ is a pair $(U(\Gamma), V(\Gamma))$, where $U(\Gamma) \subseteq \Sigma$ is closed and $V(\Gamma) \subseteq U(\Gamma)$ is finite, such that

(i) $U(\Gamma) - V(\Gamma)$ has only finitely many components, called *edges*

(ii) for each edge e, $(e, \bar{e})$ is homeomorphic either to $((0, 1), [0, 1])$ or to $(S^1 - \{x\}, S^1)$ where $x \in S^1$

(iii) for each edge e, $e \cap bd\,\Sigma = \varnothing$.

Let Γ be a drawing in Σ, and let (G, T) be a graft. Γ is said to be a *drawing of* (G, T) (in Σ) if there is a bijection α from $V(G)$ to $V(\Gamma)$ and a bijection β from $E(G)$ to the set of edges of Γ, such that

(i) for $v \in V(G)$ and $e \in E(G)$, v is incident with e if and only if $\alpha(v) \in \overline{\beta(e)}$

(ii) for $v \in V(G)$, $\alpha(v) \in bd\,\Sigma$ if and only if $v \in T$.

In this situation, we shall say $\alpha(v)$ *represents* $v \in V(G)$. If (G, T) has a drawing in Σ, we say that Σ *embeds* (G, T). If $\mathscr{A}$ is a set of grafts (respectively, rooted graphs), we say that Σ *embeds* $\mathscr{A}$ if Σ embeds (the underlying graft of) each member of $\mathscr{A}$.

If Γ is a drawing in Σ, we shall often use graph-theoretic terminology for Γ, speaking of the vertices, paths, circuits, etc., of Γ in the natural way, when we expect no confusion to arise. If Γ is a drawing in Σ and $X \subseteq \Sigma$, we say that X is *Γ-normal* if $X \cap U(\Gamma) \subseteq V(\Gamma)$. If $Y \subseteq \Sigma$ is closed and no edge of Γ meets both Y and $\Sigma - Y$, then $(U(\Gamma) \cap Y, V(\Gamma) \cap Y)$ is a drawing which we denote by $\Gamma \cap Y$.

6. Foundations on a Disc

We can now state our main theorem, and begin its proof. The theorem is the following.

(6.1) *If $\mathscr{A}$ is a class of graphs with bounded index, and some surface Σ embeds $\mathscr{A}$, then $\mathscr{A}$ is well-rooted.*

This result clearly implies (1.2), which is its special case when $\mathscr{A}$ has index zero. The first and most difficult part of the proof is to show that (6.1) is true when Σ is the disc $\Sigma(0, 0, 1)$, and this we shall do in the next seven sections.

Let Σ be a disc. For each O-arc $F \subseteq \Sigma$, exactly one component of its complement does not intersect $bd\,\Sigma$ and points of it we say are *inside* F. If $X \subseteq \Sigma$, we say X is *inside* F if x is inside F for all $x \in X$. We say F *encloses* X if $X - F$ is inside F. If Γ is a drawing in Σ, and C is a circuit of Γ, then the union of closures of its edges is an O-arc; and we shall often denote this O-arc by C. We define $\rho(C)$ to be the maximum number of mutually disjoint paths of Γ between C and $bd\,\Sigma$. By a form of Menger's theorem, $\rho(C)$ equals the minimum of $|U(\Gamma) \cap F|$ taken over all Γ-normal O-arcs F with $C - F$ inside F. A *C-ring* is such an O-arc F which attains the minimum, that is, with $|V(\Gamma) \cap F| = \rho(C)$. We next study C-rings, and to do so we need the following lemmas.

(6.2) *Let Σ be a sphere and let F_1, F_2 be O-arcs of Σ with $|F_1 \cap F_2| \geqslant 2$. Let $z \in \Sigma - (F_1 \cup F_2)$. Then there is an O-arc $F_3 \subseteq F_1 \cup F_2$ such that the component of $\Sigma - F_3$ which contains z contains no point of $F_1 \cup F_2$.*

The proof of this is straightforward point-set topology, and we omit it.

(6.3) *Let Σ be a sphere and let F_1, F_2 be O-arcs in Σ. Let $z_1, z_2 \in \Sigma - (F_1 \cup F_2)$, and be in distinct components of $\Sigma - F_i$ $(i = 1, 2)$. Then there are O-arcs F_3, F_4 in Σ such that*

(i) $F_3 \cup F_4 \subseteq F_1 \cup F_2$, $F_3 \cap F_4 \subseteq F_1 \cap F_2$

(ii) *the component of $\Sigma - F_3$ which contains z_1 contains no point of $F_1 \cup F_2$*

(iii) *the component of $\Sigma - F_4$ which contains z_2 contains no point of $F_1 \cup F_2$.*

Proof. If $|F_1 \cap F_2| \leqslant 1$ this is obvious. If $|F_1 \cap F_2| \geqslant 2$ then (6.2) applies. Let F_3, F_4 be the boundaries of the components of $\Sigma - (F_1 \cup F_2)$ containing z_1, z_2, respectively. Then (ii) and (iii) are satisfied and $F_3 \cup F_4 \subseteq F_1 \cup F_2$. It remains to show that $F_3 \cap F_4 \subseteq F_1 \cap F_2$. Let $x \in F_3 \cap F_4$. Then x is a limit point both of the component of $\Sigma - (F_1 \cup F_2)$ containing z_1 and of the component containing z_2. It follows that $x \in F_1 \cap F_2$ as required. ∎

(6.4) *Let Γ be a drawing in a disc Σ, and let C_1, C_2 be circuits of Γ. Let F_i be a C_i-ring $(i = 1, 2)$, such that some point of C_1 is inside F_2, and no point of F_2 is inside C_1. Then there is a C_1-ring enclosed by both F_1 and F_2.*

Proof. Extend Σ to a sphere Σ', and let z_2 be a point of $\Sigma' - \Sigma$. Let z_1 be a point of Σ inside C_1. Now some point of C_1 is inside F_2, and so z_1 is inside F_2, since no point of F_2 is inside C_1. Hence z_1 and z_2 are in different components of both $\Sigma' - F_1$ and $\Sigma' - F_2$. By (6.3) there are O-arcs $F_3, F_4 \subseteq F_1 \cup F_2$, with $F_3 \cap F_4 \subseteq F_1 \cap F_2$, such that

$$D_1 \cap (F_1 \cup F_2) = \varnothing = D_2 \cap (F_1 \cup F_2),$$

where D_1 is the component of $\Sigma' - F_3$ containing z_1, and D_2 is the component of $\Sigma' - F_4$ containing z_2. Now no point of D_1 is in F_1, and so D_1 is a subset of a component of $\Sigma' - F_1$. Similarly, D_2 is a subset of a component of $\Sigma' - F_2$. It follows that $D_1 \subseteq \Sigma$, for no point of $\Sigma' - \Sigma$ is in the same component of $\Sigma' - F_1$ as z_1. We deduce that z_1 is inside F_3, and so F_3 encloses C_1. Since no point of $F_1 \cup F_2$ is inside F_3, and z_1 is inside F_1, F_2, and F_3, it follows that F_1 encloses F_3, and F_2 encloses F_3. On the other hand, $z_2 \in D_2$ and so $\Sigma' - \Sigma \subseteq D_2$; hence $C_2 \cap D_2 = \varnothing$, since no point of $\Sigma' - \Sigma$ is in the same component of $\Sigma' - F_2$ as a point of C_2. Since $C_2 \cap D_2 = \emptyset$ and $\Sigma' - \Sigma \subseteq D_2$, it follows that F_4 encloses C_2. But

$$|V(\Gamma) \cap F_i| = \rho(C_i) \qquad (i = 1, 2)$$

and so

$$|V(\Gamma) \cap F_1 \cap F_2| + |V(\Gamma) \cap (F_1 \cup F_2)| = \rho(C_1) + \rho(C_2).$$

Hence

$$|V(\Gamma) \cap F_3 \cap F_4| + |V(\Gamma) \cap (F_3 \cup F_4)| \leqslant \rho(C_1) + \rho(C_2),$$

that is,

$$|V(\Gamma) \cap F_3| + |V(\Gamma) \cap F_4| \leqslant \rho(C_1) + \rho(C_2).$$

However,

$$|V(\Gamma) \cap F_3| \geqslant \rho(C_1), \qquad |V(\Gamma) \cap F_4| \geqslant \rho(C_2)$$

by definition of $\rho(C_1)$, $\rho(C_2)$, and so equality holds throughout. We deduce that F_3 is a C_1-ring, as required. ∎

(6.5) *Let Γ be a drawing in a disc Σ, and let C_1, C_2 be circuits of Γ. Let F_i be a C_i-ring ($i = 1, 2$), such that F_1 and C_2 bound disjoint open discs, and F_2 and C_1 bound disjoint open discs. Then there is a C_1-ring F_3 and a C_2-ring F_4 such that F_1 encloses F_3, F_2 encloses F_4, and F_3 and F_4 bound disjoint open discs.*

Proof. Extend Σ to a sphere Σ', and let z_i be a point of Σ inside C_i ($i = 1, 2$). By (6.3), there are O-arcs $F_3, F_4 \subseteq F_1 \cup F_2$, with $F_3 \cap F_4 \subseteq F_1 \cap F_2$, such that

$$D_1 \cap (F_1 \cup F_2) = \emptyset = D_2 \cap (F_1 \cup F_2),$$

where D_1 is the component of $\Sigma' - F_3$ containing z_1, and D_2 is the component of $\Sigma' - F_4$ containing z_2. Arguing as in (6.4), we deduce that $D_1, D_2 \subseteq \Sigma$, that D_i is inside F_i ($i = 1, 2$), and that $D_1 \cap D_2 = \emptyset$. Hence F_3 encloses C_1, F_4 encloses C_2, F_1 encloses F_3, and F_2 encloses F_4. Counting as in (6.4), we deduce that F_3 is a C_1-ring and F_4 is a C_2-ring. The result follows. ∎

Let Γ be a drawing in a disc Σ. If F is a Γ-normal O-arc, we define $\Gamma(F)$ to be the set of all edges of Γ inside F. If C is a circuit of Γ and F is a C-ring, we say F is a *minimal* C-ring if there is no C-ring F' with $\Gamma(F') \subset \Gamma(F)$.

(6.6) *Let Γ be a drawing in a disc Σ, and let C be a circuit of Γ. If F_1, F_2 are minimal C-rings then $\Gamma(F_1) = \Gamma(F_2)$.*

Proof. By (6.4) there is a C-ring F_3 enclosed by F_1 and by F_2. Hence $\Gamma(F_3) \subseteq \Gamma(\Gamma_1) \cap \Gamma(F_2)$, and the result follows from the minimality of F_1, F_2. ▌

With Γ, Σ, C as in (6.6), we define $\Delta(C)$ to be the common value of $\Gamma(F)$ taken over all minimal C-rings F.

Again, let Γ be a drawing in a disc Σ. If $r, s \geqslant 0$ are integers, an *(r, s)-nest* in Γ is a sequence $(C_1, ..., C_s)$ of circuits of Γ, such that

(i) for $1 \leqslant i < i' \leqslant s$, C_i is inside $C_{i'}$ (and hence C_i and $C_{i'}$ are vertex-disjoint)

(ii) there are r mutually vertex-disjoint paths of Γ between C_1 and C_s.

Let $k = |V(\Gamma) \cap bd\,\Sigma|$. A circuit C of Γ is a *boss* if $\rho(C) < k$ and there is a $((1 + \rho(C)), k)$-nest $(C_1, ..., C_k)$ with $C_1 = C$. The main result of this section is the following.

(6.7) *Let Γ be a drawing in a disc Σ, and let C_1, C_2 be bosses. Then either $\Delta(C_1) \subseteq \Delta(C_2)$, or $\Delta(C_2) \subseteq \Delta(C_1)$, or $\Delta(C_1) \cap \Delta(C_2) = \varnothing$.*

Proof. Let F_i be a minimal C_i-ring, and let $(C_i^1, ..., C_i^k)$ be a $((\rho(C_i) + 1), k)$-nest with $C_i^1 = C_i$ $(i = 1, 2)$. Now $|F_1 \cap V(G)| = \rho(C_1)$ and there are $\rho(C_1) + 1$ mutually disjoint paths of Γ between C_1^1 and C_1^k. Thus F_1 does not meet all these paths. But F_1 encloses C_1', and so some vertex of C_1^k is inside F_1.

Suppose that some point of F_2 is inside C_1. Now

$$|(F_1 \cup F_2) \cap V(G)| \leqslant \rho(C_1) + \rho(C_2) < 2k$$

and so $F_1 \cup F_2$ meets some C_1^j $(1 \leqslant j \leqslant k)$ in at most one point. But then with that value of j, C_1^j encloses F_2 (because some point of F_2 is inside C_1^1), and F_1 encloses C_1^j (because some point of C_1^k is inside F_1). It follows that F_1 enclose F_2, and so $\Delta(C_2) \subseteq \Delta(C_1)$ as required.

We may assume then that no point of F_2 is inside C_1, and similarly that no point of F_1 is inside C_2. Suppose that some point of C_1 is inside F_2. Then by (6.4) there is a C_1-ring F_3 enclosed by both F_1 and F_2. From the minimality of F_1, it follows that $\Gamma(F_1) = \Gamma(F_3)$. But $\Gamma(F_3) \subseteq \Gamma(F_2)$, and so $\Gamma(F_1) \subseteq \Gamma(\Gamma_2)$, as required.

We may assume then that no point of C_1 is inside F_2, and similarly that no point of C_2 is inside F_1. Hence the hypotheses of (6.5) hold, and so there is a C_1-ring F_3 and a C_2-ring F_4, bounding disjoint open discs, with

F_3 enclosed by F_1 and F_4 enclosed by F_2. Then $\Delta(C_1) = \Gamma(F_3)$, by the minimality of F_1, and similarly $\Delta(C_2) = \Gamma(F_4)$. But $\Gamma(F_3) \cap \Gamma(F_4) = \emptyset$, and the result follows. ∎

Let Γ be a drawing in a disc Σ, and let $k = |V(\Gamma) \cap bd\,\Sigma|$. By (6.7), we may choose bosses $C_1, ..., C_t$ of Γ such that

(i) $\Delta(C_1), ..., \Delta(C_t)$ are mutually disjoint, and

(ii) for every boss C, $\Delta(C) \subseteq \Delta(C_i)$ for some i $(1 \leqslant i \leqslant t)$.

For $1 \leqslant i \leqslant t$, let F_i be a minimal C_i-ring. By repeated applications of (6.5), we find that we may choose $F_1, ..., F_t$ so that they bound mutually disjoint open discs. Then we may arrange (by shrinking these discs slightly) that distinct F_i's intersect only in $V(\Gamma)$. Thus, there is a set $\{F_1, ..., F_t\}$, where F_i is a C_i-ring $(1 \leqslant i \leqslant t)$, such that $F_1, ..., F_t$ bound mutually disjoint open discs, and such that $F_i \cap F_{i'} \subseteq V(\Gamma)$ for $1 \leqslant i < i' \leqslant t$. We call the set $\{F_1, ..., F_t\}$ a *foundation* for Γ.

7. Tree-Width

A *hypertree* is a triple $(V, T, \mathscr{F})$ where V is a finite set, T is a tree, and $\mathscr{F} = (X_t : t \in V(T))$ is a family of subsets of V, such that

(i) $\bigcup(X_t : t \in V(T) = V$

(ii) for $t, t', t'' \in V(T)$, if t' lies on the path of T between t and t'' then $X_t \cap X_{t''} \subseteq X_{t'}$.

A graph G is said to have *tree-width* w if $w \geqslant 0$ is minimum such that there is a hypertree $(V, T, \mathscr{F})$ with $V = V(G)$, and with

(i) $|X_t| \leqslant w+1$ for each $t \in V(T)$, and

(ii) for each edge e of G some X_t contains both ends of e.

We define the *tree-width* of a drawing of a graft (G, S) to be the tree-width of G. For more about tree-width, see [7, 8, 9]. The following is proved in [7].

(7.1) *Let $r, s \geqslant 0$ be integers. Then there is an integer w such that every drawing in a disc with no (r, s)-nest has tree-width $\leqslant w$.*

8. Centred O-Arcs

Again, let Γ be a drawing in a disc Σ, and let F be a Γ-normal O-arc. Let $|F \cap V(\Gamma)| = r$. We say that F is *centred* if Γ has a (r, s)-nest $(C_1, ..., C_s)$

where $s=\lceil \frac{1}{2}r \rceil$ such that each C_i is inside F and such that there are r mutually disjoint paths of Γ between C_1 and F. The following may be proved by a slight adaptation of the proof of [7, Theorem (4.1)] the details of which we omit.

(8.1) *If F is centred, there is a (r, s)-nest $(C_1, ..., C_s)$ (where r, s are as above) such that each C_i is inside F, and there are r mutually disjoint paths $P_1, ..., P_r$ of Γ between C_1 and F, such that the intersection of each P_i and each C_j is a path.*

If $C_1, ..., C_s$ and $P_1, ..., P_r$ are as in (8.1), the subdrawing $C_1 \cup \cdots \cup C_s \cup P_1 \cup \cdots \cup P_r$ is called a *sleeve* for F.

(8.2) *If C is a boss of Γ then every C-ring is centred.*

Proof. Let $(C_1, ..., C_k)$ be a $((1+\rho(C)), k)$-nest with $C_1 = C$, where $k = |V(\Gamma) \cap bd\, \Sigma|$. Let F be a C-ring. There are $1+\rho(C)$ mutually disjoint paths of Γ between C_1 and C_k, but $|F \cap V(\Gamma)| = \rho(C)$. Hence there is a path of Γ between C_1 and C_k which does not meet F. It follows that some vertex of C_k is inside F.

Let $1 \leqslant j \leqslant k$, and suppose F meets C_j. Then for $j < j' \leqslant k$, $|F \cap C_{j'}| \geqslant 2$, and so

$$|F \cap V(\Gamma)| \geqslant 2(k-j)+1.$$

Hence $2(k-j)+1 \leqslant \rho(C)$, and so $j \geqslant k - \frac{1}{2}(\rho(C)-1)$. But $\rho(C) < k$ since C is a boss, and so

$$\lceil \tfrac{1}{2}\rho(C) \rceil < k - \tfrac{1}{2}(\rho(C)-1).$$

Thus $C_1, ..., C_{\lceil \rho(C)/2 \rceil}$ are all inside F. Moreover there are $\rho(C)$ mutually disjoint paths between C and $bd\, \Sigma$ by definition of $\rho(C)$, and hence between C_1 and F, as required. ∎

We shall require the following lemmas concerning centred O-arcs in Section 11. The *ends* of a $[0, 1]$-arc are defined in the natural way.

(8.3) *Let F be a Γ-normal centred O-arc. Let I be a Γ-normal $[0, 1]$-arc with ends in F and with all its other points inside F. Then*

$$|I \cap V(\Gamma)| \geqslant \min(|F_1 \cap V(\Gamma)|, |F_2 \cap V(\Gamma)|),$$

where F_1, F_2 are the closures of the two components into which the ends of I divide F.

Proof. Let $|F \cap V(\Gamma)| = r$, and let $(C_1, ..., C_{\lceil r/2 \rceil})$ be the nest in a sleeve for F. For $i = 1, 2$, put $|F_i \cap V(\Gamma)| = r_i$. Then $r_1 + r_2 = r + n$ where

$n = |I \cap F \cap V(\Gamma)|$. We may hence assume that $|I \cap V(\Gamma)| < \frac{1}{2}r + \frac{1}{2}n$, for otherwise the theorem is true. We assume $r > 0$, for otherwise the result is trivial.

If I meets C_1 then it has at least two points in common with $C_2, ..., C_{\lceil r/2 \rceil}$ and so

$$|I \cap V(\Gamma)| \geqslant 2(\lceil \tfrac{1}{2}r \rceil - 1) + 1 + n.$$

Thus

$$2\lceil \tfrac{1}{2}r \rceil - 1 + n < \tfrac{1}{2}r + \tfrac{1}{2}n$$

a contradiction. Hence I does not meet C_1.

But I meets every path of Γ from F_1 to F_2 which is enclosed by F; and so for some i ($i = 1$ or 2) I meets every path from F_i to C_1 which is enclosed by F. There are at least r_i such paths, mutually disjoint, and so $|I \cap V(\Gamma)| \geqslant r_i$ as required. ∎

(8.4) *Let F be a Γ-normal centred O-arc and let Ξ be a drawing of a forest in Σ, such that $U(\Xi) \cap F \subseteq V(\Gamma)$, and $U(\Xi)$ is enclosed by F. Then Γ has a subdrawing Ξ', which is a drawing of a forest, which $U(\Xi')$ enclosed by F, such that for all $u, v \in V(\Gamma) \cap F$, there is a component of Ξ containing both u and v if and only if there is a component of Ξ' containing them both.*

This follows from (8.3) and [10, Theorem (3.6)].

9. Boundary-Linked Nests

Let Γ be a drawing in a disc Σ, with $|V(\Gamma) \cap bd\,\Sigma| = k$. Let $r, s \geqslant 0$ be integers. An (r, s)-nest $(C_1, ..., C_s)$ is *boundary-linked* if there are k mutually disjoint paths of Γ between C_1 and $bd\,\Sigma$.

Let $\{F_1, ..., F_t\}$ be a foundation for Γ. Then by (8.2), each F_i is centred; let Γ_i be a sleeve for F_i ($1 \leqslant i \leqslant t$). Let Γ_0 be the subdrawing of Γ consisting of those vertices and edges inside none of $F_1, ..., F_t$. Let Γ^* be $\Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_t$. We call Γ^* a *truncation* of Γ (*induced* by $\{F_1, ..., F_t\}$).

(9.1) *Let Γ^* be a truncation of Γ, and let r, s be integers with $r, s \geqslant k$. Every (r, s)-nest of Γ^* is boundary-linked in Γ.*

Proof. Let $\{F_1, ..., F_t\}$ be a foundation inducing the truncation Γ^*, and let $\Gamma_1, ..., \Gamma_t, \Gamma_0$ be as above. Let $(C_1, ..., C_s)$ be an (r, s)-nest of Γ^*. Suppose it is not boundary-linked in Γ. Then $\rho(C_1) < k$. But $r \geqslant k$, and $s \geqslant k$, and so C_1 is a boss. Hence F_i encloses C_1 for some i ($1 \leqslant i \leqslant t$), say $i = 1$. There are r mutually disjoint paths of Γ between C_1 and C_s, and so one of them does not meet F_1, since $|F_1 \cap V(\Gamma)| = \rho(C_1) < k \leqslant r$. Hence some vertex of C_s is inside F_1. As in the proof of (8.2), F_1 does not meet any of $C_1, ..., C_{\lceil k/2 \rceil}$ and so $(C_1, ..., C_{\lceil k/2 \rceil})$ is a $(k, \lceil \frac{1}{2}k \rceil)$-nest in Γ_1. But it is easy to see that Γ_1 has no such nest (because for example, its cyclomatic number is too small). This contradiction implies that $(C_1, ..., C_s)$ is boundary-linked, as required. ∎

(9.2) *If $r, s \geqslant k$ there is a number N such that for any drawing Γ with $|V(\Gamma) \cap bd\,\Sigma| = k$, if Γ has no boundary-linked (r, s)-nest, then every truncation Γ^* has tree-width $\leqslant N$.*

This follows from (9.1) and (7.1).

We shall require the following, which is essentially [11, Theorem (9.4)].

(9.3) *Let (H, U) be a graft which embeds in a disc Σ, with $|U| = k$. Then there are integers $r, s \geqslant k$ such that for every graft (G, T) with $|T| = k$, if (G, T) has a drawing in Σ with a boundary-linked (r, s)-nest, then (H, U) is simulated in (G, T).*

From (9.2) and (9.3) we deduce the following.

(9.4) *Let (H, U) be a graft which embeds in a disc Σ with $|U| = k$. Then there is a number w such that for every graft (G, T) with $|T| = k$, if (H, U) is not simulated in (G, T) then every truncation of every drawing of (G, T) in Σ has tree-width $\leqslant w$.*

10. Paintings

Let Σ be a disc. A *painting* Γ in Σ is a pair $(U(\Gamma), N(\Gamma))$ where $U(\Gamma) \subseteq \Sigma$ is closed and $N(\Gamma) \subseteq U(\Gamma)$ is finite, such that

(i) $U(\Gamma) - N(\Gamma)$ has only finitely many components, called *cells*

(ii) for each cell e, $(\bar{e}, e)$ is homeomorphic to $(\Sigma, \Sigma - \{x_1, ..., x_k\})$ where $k = |\bar{e} - e|$ and $x_1, ..., x_k \in bd\,\Sigma$ are distinct

(iii) for each cell e, $e \cap bd\,\Sigma = \varnothing$.

A painting is roughly a generalization of a drawing. For if Γ is a drawing with no loops, we may "thicken" each edge of Γ slightly to obtain a painting in which $|\bar{e} - e| = 2$ for each cell e. If Γ is a drawing with loops such that each loop bounds a region, we may perform an analogous construction; we thicken each non-loop edge, and we fill in the region bounded by each loop. Thus each loop of the drawing corresponds to a cell e of the painting with $|\bar{e} - e| = 1$. Let us say that a drawing is *cellular* if each loop bounds a region.

A painting $\Gamma = (U, N)$ is said to have *tree-width* w if $w \geqslant 0$ is minimum such that there is a hypertree $(V, T, \mathscr{F})$, where $\mathscr{F} = (X_t : t \in V(T))$, with

(i) $|X_t| \leqslant w+1$ for each $t \in V(T)$, and

(ii) for each cell e of Γ, $\bar{e} - e \subseteq X_t$ for some $t \in V(T)$.

If X is a finite set, an *ordering* of X is a bijection from $\{1, 2, ..., |X|\}$ to X. Let $\mathbf{Q} = (Q, \leqslant)$ be a quasi-order. A $\mathbf{Q}$-*portrait* in a disc Σ is a quintuple $(U, N, \alpha, \mu, \phi)$, where (U, N) is a painting, α is an ordering of $U \cap bd\,\Sigma$, μ_e is an ordering of $\bar{e} - e$ for each cell e of (U, N), and ϕ is a function from the set of cells of (U, N) to Q. Its *index* is $|U \cap bd(\Sigma)|$.

Let $(U, N, \alpha, \mu, \phi)$, $(U', N', \alpha', \mu', \phi')$ be $\mathbf{Q}$-portraits. A *simulation* of the first in the second is a pair (Ξ, σ), where Ξ is a drawing of a forest in Σ with vertex set N', and $\sigma(v)$ is a component of Ξ for each $v \in N$, and $\sigma(e)$ is a cell of (U', N') for each cell e of (U, N), such that

(i) every edge of Ξ is a subset of a cell of (U', N')

(ii) for distinct $v_1, v_2 \in N$, $\sigma(v_1)$ and $\sigma(v_2)$ are distinct

(iii) for distinct cells e_1, e_2 of (U, N), $\sigma(e_1)$ and $\sigma(e_2)$ are distinct

(iv) for $v \in N$ and each cell e of (U, N), no edge of $\sigma(v)$ intersects $\sigma(e)$

(v) for each cell e of (U, N), $|\bar{e} - e| = |\overline{\sigma(e)} - \sigma(e)|$

(vi) for each cell e of (U, N) and each integer n with $1 \leqslant n \leqslant |\bar{e} - e|$, $\mu'_{\sigma(e)}(n)$ is a vertex of $\sigma(\mu_e(n))$

(vii) $|U \cap bd\,\Sigma| = |U' \cap bd\,\Sigma|$, and for $1 \leqslant n \leqslant |U \cap bd\,\Sigma|$, $\alpha'(n)$ is a vertex of $\sigma(\alpha(n))$

(viii) for each cell e of (U, N), $\phi(e) \leqslant \phi'(\sigma(e))$.

The *tree-width* of a $\mathbf{Q}$-portrait $(U, N, \alpha, \mu, \phi)$ equals the tree-width of (U, N). The following is proved in [8, Theorem (9.2)].

(10.1) *Let* $\mathbf{Q}$ *be a well-quasi-order, and let* $k, w \geqslant 0$ *be integers. Let* $P_1, P_2, ...$ *be a countable sequence of* $\mathbf{Q}$*-portraits all with index* k *and tree-width* $\leqslant w$. *Then there exist* $j > i \geqslant 1$ *such that* P_i *is simulated in* P_j.

[We remark that the definition of tree-width in [8] differs slightly from our definition here; but the two quantities differ by at most k, and hence the tree-width in the sense of [8] is at most $w + k$.]

11. Paintings from Foundations

Let Σ be a disc and $k \geqslant 0$ an integer. Let $\mathscr{A}(k)$ denote the class of all rooted graphs (G, τ) with index $\leqslant k$ whose underlyng grafts have drawings in Σ and where all the terms in the sequence τ are distinct. Let $*$ be some new element. We define a quasi-order $\mathbf{Q}_k = (\mathscr{A}(k) \cup \{*\}, \leqslant)$ as follows. For $q_1, q_2 \in \mathscr{A}(k) \cup \{*\}$,

$$\begin{aligned} q_1 &\leqslant q_2 && \text{if } q_1, q_2 \in \mathscr{A}(k) \text{ and } q_1 \text{ is simulated in } q_2 \\ q_1 &\leqslant q_2 && \text{if } q_1 = q_2 = * \\ q_1 &\nleqslant q_2 && \text{otherwise.} \end{aligned}$$

Let $(G, \tau) \in \mathscr{A}(k)$, with index exactly k, and let (G, T) be its underlying graft. Then (G, T) embeds in Σ, and so it has a drawing which is cellular, as is easily seen. Let Γ be such a drawing. Let $\{F_1, ..., F_t\}$ be a foundation for Γ. For $1 \leqslant i \leqslant t$, let α_i be an ordering of $V(\Gamma) \cap F_i$ and let ψ_i be the corresponding ordering of the corresponding subset of $V(G)$. Let π be the ordering of $V(\Gamma) \cap bd\,\Sigma$ where for $1 \leqslant n \leqslant k$, $\pi(n)$ is the point of Σ representing the nth term of τ. For $1 \leqslant i \leqslant t$, let Σ_i be the union of F_i and the set of points inside F_i, and let

$$\Sigma_0 = \overline{\Sigma - (\Sigma_1 \cup \cdots \cup \Sigma_t)}$$

For $0 \leqslant i \leqslant t$, let $\Gamma_i = \Gamma \cap \Sigma_i$, and let G_i be the subgraph of G corresponding to Γ_i. Then for each $i \geqslant 1$, $(G_i, \psi_i) \in \mathscr{A}(k-1)$. For each edge e of Γ_0, let $D(e)$ be a disc with $e \subseteq D(e) \subseteq \Sigma$, such that

$$\begin{aligned} D(e) \cap U(\Gamma) &= \bar{e} \\ D(e) \cap \Sigma_i &\subseteq \bar{e} && (1 \leqslant i \leqslant t) \\ D(e) \cap D(e') &\subseteq \bar{e} \cap \bar{e}' && \text{for distinct edges } e, e' \text{ of } \Gamma_0. \end{aligned}$$

(This is the procedure of "thickening" edges discussed earlier—it is possible since Γ is cellular.) Let

$$U' = \Sigma_1 \cup \cdots \cup \Sigma_t \cup \bigcup (D(e) : e \in E(\Gamma_0)),$$

where $E(\Gamma_0)$ denotes the set of edges of Γ_0; and let $N' = V(\Gamma_0)$. Then (U', N') is a painting. The closures of its cells are the sets $\Sigma_1, ..., \Sigma_t$ and the sets $D(e)$ $(e \in E(\Gamma_0))$. For each cell f of (U', N'), we define $\mu_f = \alpha_i$ and $\phi(f) = (G_i, \psi_i)$ if $\bar{f} = \Sigma_i$ for some i, $1 \leqslant i \leqslant t$; and we define μ_f to be some ordering of $\bar{f} - f$, and $\phi(f) = *$, if $\bar{f} = D(e)$ for some $e \in E(\Gamma_0)$. Then (U', N', π, μ, ϕ) is a $\mathbf{Q}_{k-1}$-portrait. We call it a $\mathbf{Q}_{k-1}$-portrait *derived* from (G, τ) (via Γ).

(11.1) *With notation as above, the tree-width of (U', V') is no greater than the tree-width of any corresponding truncation of Γ.*

This is unfortunately not quite a consequence of [7, Theorem (3.8)]. However, it is a consequence of the strengthening of that theorem obtained by replacing the last sentence of its statement by the following. "If G has a tree-decomposition $(T, (Y_t: t \in V(T)))$ of width $\leqslant w$, then H has a tree-decomposition $(T, (X_t: t \in V(T)))$ of width $\leqslant w$ such that $Y_t \cap V(H) \subseteq X_t$ for every $t \in V(T)$, and $\{v_1, ..., v_r\} \subseteq X_{t_0}$ for some $t_0 \in V(T)$." The proof given in [7] serves as well to prove this strengthened form. (We omit the full statement of the result of [7], because to make that intelligible would require a disproportionate number of new definitions.) (11.1) follows from this by applying it to each G_i in turn.

(11.2) *Let (G_1, τ_1), $(G_2, \tau_2) \in \mathscr{A}(k)$, both with index k. Let P_i be a $\mathbf{Q}_{k-1}$-portrait derived from (G_i, τ_i) $(i=1, 2)$. If P_1 is simulated in P_2 then (G_1, τ_1) is simulated in (G_2, τ_2).*

Proof. Let Γ_1, Γ_2 be respective drawings. By (8.4) the drawing Ξ of a forest involved in the simulation relation for the portraits can be chosen to be a subdrawing of Γ_2, since each O-arc in the foundation used for Γ_2 is centred, by (8.2). The rest of the verification of (11.2) is lengthy but straightforward, and is left to the reader. (We point out that the "missing" parts of Γ_1, Γ_2, the insides of the foundation O-arcs, have not been forgotten; the simulation relation works correctly on them because of condition (vii) in the definition of $\mathbf{Q}$-portrait simulation.) ∎

12. Conclusion of the Disc Case

Now we complete the proof of (6.1) when Σ is a disc, as follows. We must show that the class of all grafts with index $\leqslant k$ which embed in a disc Σ is well-rooted. We proceed by induction on k, and assume the result is true for all smaller values of k. By (4.2) and our inductive hypothesis, it suffices to prove that the class of all grafts with index exactly k which embed in Σ is well-rooted.

Let (G_1, T_1), (G_2, T_2), ... be a countable sequence of grafts with index k which embed in Σ. For $j \geqslant 1$, let Γ_j be a drawing of (G_j, T_j) in Σ. We may assume that for $j \geqslant 2$, (G_1, T_1) is not simulated in (G_j, T_j); and so by (9.4) there is a number w such that for all $j \geqslant 2$, every truncation of Γ_j has tree-width $\leqslant w$. For $j \geqslant 2$ let τ_j be some ordering of T_j, and let P_j be a $\mathbf{Q}_{k-1}$-portrait derived from (G_j, τ_j) via Γ_j. By (11.1), P_j has tree-width no greater than w, for $j=2, 3, ...$ By our inductive hypothesis and (4.3), $\mathbf{Q}_{k-1}$ is a well-quasi-order, and so by (10.1) there exist $j' > j \geqslant 2$ such that P_j is simulated in $P_{j'}$. By (11.2), (G_j, τ_j) is simulated in $(G_{j'}, \tau_{j'})$, and so (G_j, T_j) is simulated in $(G_{j'}, T_{j'})$, as required. ∎

13. Vertex Identification

We have now finished the most difficult part of the paper. We shall have no further need for many of the preceding definitions; in particular, tree-width, paintings and foundations will not be used any more in this paper.

We have established that certain classes of grafts are well-rooted. Our method for more complicated surfaces is based on making new well-rooted classes by piecing together old ones in certain ways; and the next two sections explain these constructions.

Let (G, T) be a graft, and let $X \subseteq V(G)$ be such that no edge of G has one end in X and the other in $V(G) - X$. Let G_1, G_2 be the restrictions of G to X and to $V(G) - X$, respectively, and let $T_1 = T \cap X$, $T_2 = T - X$. Then (G_1, T_1), (G_2, T_2) are grafts, and we write

$$(G, T) = (G_1, T_1) \oplus (G_2, T_2).$$

Let $\mathscr{A}$, $\mathscr{B}$ be classes of grafts. We define $\mathscr{A} \oplus \mathscr{B}$ to be the class of all grafts of the form $A \oplus B$, where $A \in \mathscr{A}$ and $B \in \mathscr{B}$.

(13.1) *If $\mathscr{A}, \mathscr{B}$ are well-rooted then so is $\mathscr{A} \oplus \mathscr{B}$.*

Proof. Let $A_1 \oplus B_1, A_2 \oplus B_2, \ldots$ be a countable sequence of members of $\mathscr{A} \oplus \mathscr{B}$, where $A_i \in \mathscr{A}$ and $B_i \in \mathscr{B}$ $(i = 1, 2, \ldots)$. By (3.2) there exist $i' > i \geqslant 1$ such that A_i is simulated in $A_{i'}$ and B_i is simulated in $B_{i'}$. But then $A_i \oplus B_i$ is simulated in $A_{i'} \oplus B_{i'}$, as required. ∎

Let (G, T) be a graft and let $t_1, t_2 \in T$ be distinct. Let G' be obtained from G by identifying t_1 and t_2 forming a new vertex t say; and let T' be one of

$$T - \{t_1, t_2\}, (T - \{t_1, t_2\}) \cup \{t\}.$$

Then (G', T') is a graft, and we say it is obtained from (G, T) by a *vertex identification.*

(13.2) *If $\mathscr{A}$ is well-rooted, then the class of all grafts which can be obtained from a member of $\mathscr{A}$ by a vertex identification is well-rooted.*

Proof. Let (H_1, U_1), $(H_2, U_2), \ldots$ be a countable sequence of grafts, such that for $i = 1, 2, \ldots$ there exists $(G_i, T_i) \in \mathscr{A}$ such that (H_i, U_i) is obtained from (G_i, T_i) by a vertex identification. For $i = 1, 2, \ldots$ let t_{i_1}, t_{i_2} be the two vertices of G_i which are identified to form H_i, and let u_i be the vertex of H_i formed by identifying t_{i_1} and t_{i_2}. By replacing our sequence by a suitable countable subsequence, we may assume that either $u_i \in U_i$ for $i = 1, 2, \ldots$, or $u_i \notin U_i$ for $i = 1, 2, \ldots$. Let τ_i be an ordering of T_i with

$\tau_i(1)=t_{i_1}$, $\tau_i(2)=t_{i_2}$ $(i=1, 2, ...)$. By (4.3), there exist $j>i\geqslant 1$ such that the rooted graph (G_i, τ_i) is simulated in (G_j, τ_j). But then (H_i, U_i) is simulated in (H_j, U_j), as required. ∎

If $\mathscr{A}$ is a class of grafts, let $W^0(\mathscr{A})=\mathscr{A}$; and inductively, for $i=1, 2, ...$ let $W^i(\mathscr{A})$ be the class of all grafts which can be obtained from a member of $W^{i-1}(\mathscr{A})$ by a vertex identification. Let

$$W(\mathscr{A})=W^0(\mathscr{A})\cup W^1(\mathscr{A})\cup \dots.$$

(13.3) *If $\mathscr{A}$ is well-rooted then so is $W(\mathscr{A})$.*

Proof. Since $\mathscr{A}$ is well-rooted, it has an index k by (4.1). We proceed by induction on k. If $k\leqslant 1$ then $W(\mathscr{A})=\mathscr{A}$ and the result is true. We assume then that $k\geqslant 2$. Now $W^1(\mathscr{A})$ has index $\leqslant k-1$ as is easily seen, and by (13.2) $W^1(\mathscr{A})$ is well-rooted. Hence $W(W^1(\mathscr{A}))$ is well-rooted, by our inductive hypothesis. By (4.2), $\mathscr{A}\cup W(W^1(\mathscr{A}))$ is well-rooted; but $W(\mathscr{A})=\mathscr{A}\cup W(W^1(\mathscr{A}))$ and the result follows. ∎

14. Chains, Trains, and Concatenations

As well as the rather easy constructions of the last section, we need a more complicated one, which we call *concatenation.* Here, roughly, we paste together grafts in series, overlapping each with its predecessor. Let $\mathscr{A}$ be a class of grafts and let (G, T) be a graft. A *path-decomposition of* (G, T) *over* $\mathscr{A}$ is a sequence (G_i, Y_{i-1}, Y_i) $(1\leqslant i\leqslant r)$ for some $r\geqslant 1$, such that

(i) $(G_i, Y_{i-1}\cup Y_i)\in\mathscr{A}$ for $1\leqslant i\leqslant r$

(ii) G_i is a subgraph of G for $1\leqslant i\leqslant r$, $G_1\cup\cdots\cup G_r=G$, and $G_1, ..., G_r$ are mutually edge-disjoint

(iii) for $1\leqslant h\leqslant i\leqslant j\leqslant r$, $V(G_h)\cap V(G_j)\subseteq V(G_i)$ (this condition says, roughly, that each graph in the sequence is pasted onto its predecessor)

(iv) for $1\leqslant i<r$, $V(G_i)\cap V(G_{i+1})=Y_i$

(v) $T=Y_0\cup Y_r$.

If $k\geqslant 0$ is an integer, we say that (G, T) is a *k-chain* over $\mathscr{A}$ if it has a path-decomposition (G_i, Y_{i-1}, Y_i) $(1\leqslant i\leqslant r)$ such that $|Y_i|=k$ $(0\leqslant i\leqslant r)$, and there are k mutually vertex-disjoint paths of G between Y_0 and Y_r. We begin with the following lemma which establishes some properties of k-chains.

(14.1) *Let (G, T) be a k-chain, and let (G_i, Y_{i-1}, Y_i) $(1\leqslant i\leqslant r)$ be a path-decomposition with $|Y_i|=k$ $(0\leqslant i\leqslant r)$. Let $P_1, ..., P_k$ be mutually disjoint paths of G from Y_0 to Y_r. Then*

(i) *for* $1 \leqslant i \leqslant r$, *there is no edge of* G *with one end in* $\bigcup_{j<i} V(G_j)$, *the other in* $\bigcup_{j>i} V(G_j)$, *and neither end in* $V(G_i)$

(ii) *if* $1 \leqslant h \leqslant i \leqslant j \leqslant r$, *every path of* G *from* $V(G_h)$ *to* $V(G_j)$ *meets* $V(G_i)$

(iii) *for* $0 \leqslant i \leqslant r$ *and* $1 \leqslant p \leqslant k$, $|Y_i \cap V(P_p)| = 1$

(iv) *for* $1 \leqslant i \leqslant r$ *and* $1 \leqslant p \leqslant k$ *the intersection of* P_p *and* G_i *is a path*

(v) *for* $1 \leqslant p \leqslant k$, P_p *is obtained from the paths* $P_p \cap G_i$ $(i = 1, 2, ..., r)$ *by concatenating them in order.*

Proof. Suppose e is an edge as in (i). Choose h with $1 \leqslant h \leqslant r$ such that $e \in E(G_h)$. By symmetry, we may assume $h \leqslant i$. Let v be the end of e with $v \in \bigcup_{j>i} V(G_j)$, and choose $j > i$ with $v \in V(G_j)$. Then

$$v \in V(G_h) \cap V(G_j) \subseteq V(G_i),$$

a contradiction. Thus there is no such edge, and (i) holds. From this, (ii) follows immediately.

To prove (iii) we show first that for $0 \leqslant i \leqslant r$ and $1 \leqslant p \leqslant k$, $Y_i \cap V(P_p) \neq \emptyset$. This is true if $i = 0$ or $i = r$, since P_p is a path from Y_0 to Y_r. If $0 < i < r$, then P_p meets $V(G_i)$ by (ii), since it meets $V(G_1)$ and $V(G_r)$; and the last vertex of P_p in $V(G_i)$ is also in $V(G_{i+1})$, by (ii), and hence is in Y_i. Thus $Y_i \cap V(P_p) \neq \emptyset$. But since $|Y_i| = k$ and $P_1, ..., P_k$ are mutually vertex-disjoint, (iii) follows.

Now let $1 \leqslant p \leqslant k$ and $1 \leqslant i \leqslant r$. Any vertex of P_p in $V(G_i)$ which is incident with an edge not in G_i is in $Y_{i-1} \cup Y_i$, and so by (iii) there are at most two such vertices; and the same argument shows that if there are two such vertices, the subpath of P_p between them is a subgraph of G_i. Thus (iv) holds, and (v) follows from (iv) and (ii). ∎

(14.2) *If* $\mathscr{A}$ *is well-rooted and* $k \geqslant 0$ *is an integer, the class of all* k*-chains over* $\mathscr{A}$ *is well-rooted.*

Proof. It suffices to prove that for every k' the class of all k-chains over $\mathscr{A}$ with index k' is well-rooted. Let (G^1, T^1), (G^2, T^2), ... be a sequence of k-chains over $\mathscr{A}$, all with the same index, and for $i = 1, 2, ...$ let $(G^i_j, Y^i_{j-1}, Y^i_j)$ $(1 \leqslant j \leqslant r_i)$ be the corresponding path-decomposition. For $i = 1, 2, ...$ let $P^i_1, ..., P^i_k$ be mutually vertex-disjoint paths of G^i from Y^i_0 to $Y^i_{r_i}$.

Let $i \geqslant 1$ be an integer. For $j = 1, 2, ..., r_i$, and for $p = 1, ..., k$, let $\tau^i_j(p)$ be the first vertex of P^i_p in $V(G^i_j)$ (which is necessarily in Y^i_{j-1}), and let $\tau^i_j(p+k)$ be the last vertex of P^i_p in $V(G^i_j)$ (which is necessarily in Y^i_j). Then (G^i_j, τ^i_j) is a rooted graph. Since $|Y^i_{j-1}| = k$ and $P^i_1, ..., P^i_k$ are mutually vertex-disjoint, it follows that $\{\tau^i_j(1), ..., \tau^i_j(k)\} = Y^i_{j-1}$, and similarly that

$\{\tau_j^i(k+1), ..., \tau_j^i(2k)\} = Y_j^i$. Thus $(G_j^i, \tau_j^i) \in \mathscr{B}$, where $\mathscr{B}$ is the class of all rooted graphs of index $2k$ with underlying graft in $\mathscr{A}$. By (4.3), $\mathscr{B}$ is well-rooted, and so by (3.3) there exist $i > h \geqslant 1$ and $j(1), ..., j(r_h)$ such that

$$1 \leqslant j(1) < j(2) < \cdots < j(r_h) \leqslant r_i,$$

and such that for $1 \leqslant n \leqslant r_h$, (G_n^h, τ_n^h) is simulated in $(G_{j(n)}^i, \tau_{j(n)}^i)$. For $1 \leqslant n \leqslant r_h$, let σ_n be the corresponding simulation. Let Q be the subgraph of G^i formed by the union of the $\sigma_n(v)$ for $1 \leqslant n \leqslant r_h$ and $v \in V(G_n^h)$. Let

$$Y = \bigcup (Y_{j(n)-1} \cup Y_{j(n)}^i : 1 \leqslant n \leqslant r_h).$$

Then $Y \subseteq V(Q)$. Let R be the intersection of $P_1^i \cup \cdots \cup P_k^i$ with

$$\bigcup (G_j^i : j \in \{1, ..., r_i\} - \{j(1), ..., j(r_h)\}).$$

Let G be the union of Q and R. For a component K of G, we say K *realizes* $v \in V(G^h)$ if $V(K) \cap V(\sigma_n(v)) \neq \emptyset$ for some n with $1 \leqslant n \leqslant r_h$.

(1) *Every component of G realizes some vertex of $V(G^h)$.*

For if K is a component of G and $V(K) \cap V(Q) = \emptyset$, then K is a component of R, and so K is a subpath of P_p^i for some p with $1 \leqslant p \leqslant k$. Now P_p^i meets $V(Q)$ and so $K \neq P_p^i$, and some end v of K is an internal vertex of P_p^i. But then $v \in V(Q)$ from the definitions of Q and R, and hence $V(K) \cap V(Q) \neq \emptyset$, a contradiction. Thus there is no such component K, as required.

(2) *If $V(\sigma_m(u)) \cap V(\sigma_n(v)) \neq \emptyset$ then $u = v$.*

If $m = n$ the result is clear, and we assume $m < n$ without loss of generality. Choose $x \in V(\sigma_m(u)) \cap V(\sigma_n(v))$. Then

$$x \in V(G_{j(m)}^i) \cap V(G_{j(n)}^i) \subseteq V(G_{j(m)+1}^i)$$

and so $x = \tau_{j(m)}^i(p+k)$ for some p with $1 \leqslant p \leqslant k$, where $x \in V(P_p)$. Similarly $x = \tau_{j(n)}^i(p)$. Since σ_m is a simulation of (G_m^h, τ_m^h) in $(G_{j(m)}^i, \tau_{j(m)}^i)$, it follows that $u = \tau_m^h(p+k)$, and similarly that $v = \tau_n^h(p)$. But for $m < l < n$,

$$\tau_{j(l)}^i(p) = \tau_{j(l)}^i(p+k) = x$$

and so $\tau_l^h(p) = \tau_l^h(p+k)$, since σ_l is a simulation. But

$$\tau_l^h(p+k) = \tau_{l+1}^h(p) \qquad (m \leqslant l \leqslant n-1)$$

and we deduce that $u = v$, as required.

(3) *Every component of G realizes a unique vertex of $V(G^h)$.*

By (1) and (2), it suffices to show that if $x \neq y$ and P is a path of G between $x \in V(\sigma_m(u))$ and $y \in V(\sigma_n(v))$ then $u = v$; and it clearly suffices to prove this when no internal vertex of P is in $V(Q)$. We assume $m \leqslant n$ without loss of generality, and we proceed by induction on $n - m$. Suppose that $e \in E(Q)$ for some edge e of P. Then both ends of e are in $V(Q)$, and so $E(P) = \{e\}$. Choose l with $1 \leqslant l \leqslant r_h$ and $w \in V(G_l^h)$ such that $e \in E(\sigma_l(w))$. By (2), $w = u$ and $w = v$, and so $u = v$, as required. We may assume then that every edge of P is in $E(R)$. Since $E(P) \neq \emptyset$ and $E(P) \subseteq E(R)$, P is a subpath of P_p for some p with $1 \leqslant p \leqslant k$. By (14.1), no vertex of P except x is in $V(G^i_{j(m)})$ (and so $m \neq n$), and similarly, no vertex of P except y is in $V(G^i_{j(n)})$. In particular, $x = \tau^i_{j(m)}(p+k)$ and $y = \tau^i_{j(n)}(p)$. If $n \geqslant m + 2$ then by (14.1) some vertex of P is in $V(G^i_{j(m+1)})$, and the last such vertex is in $Y \subseteq V(Q)$. Since no internal vertex of P is in $V(Q)$, we deduce that one of x, y is in $V(G^i_{j(m+1)})$, and the result follows from (2) and our inductive hypothesis. We may assume then that $n = m + 1$. But $x = \tau^i_{j(m)}(p+k)$ and so $u = \tau^h_m(p+k)$. Similarly, $v = \tau^h_n(p)$. But $\tau^h_m(p+k) = \tau^h_n(p)$ since $n = m + 1$, and so $u = v$, as required.

(4) *Every vertex of $V(G^h)$ is realized by a unique component of G.*

Certainly every vertex of $V(G^h)$ is realized by some component of G. Let $v \in V(G^h)$, and let $x \in V(\sigma_m(v))$, $y \in V(\sigma_n(v))$. We wish to prove that x and y are in the same component of G. If $m = n$ this is true since $\sigma_m(v)$ is connected, and so we may assume $n > m$. Suppose that $n = m + 1$. Then $v \in V(G^h_m) \cap V(G^h_n)$, and so $v \in Y^h_m$. Choose p with $1 \leqslant p \leqslant k$ such that $v = \tau^h_m(p+k) = \tau^h_n(p)$. Then x is the same component of G as $\tau^i_{j(m)}(p+k)$, and y is in the same component of G as $\tau^i_{j(n)}(p)$. But $\tau^i_{j(n)}(p+k)$ and $\tau^i_{j(n)}(p)$ are in the same component of R and hence of G, by (14.1), since $n = m + 1$. The result is therefore true if $n = m + 1$. Now suppose $n > m + 1$. Then $v \in V(G^h_l)$ for $m < l < n$, and we may choose $z_l \in V(G^i_{k(l)})$ $(m \leqslant l \leqslant n)$ such that $z_l \in V(\sigma_l(v))$, and $z_m = x$, $z_n = y$. But for $m \leqslant l < n$, z_l and z_{l+1} are in the same component of G, by our observation above. Hence x and y are in the same component of G, as required.

Now to complete the proof of (14.2), we define σ by

(i) if $v \in V(G^h)$, $\sigma(v)$ is the component of G which realizes v

(ii) if $e \in E(G^h)$, $\sigma(e)$ is $\sigma_n(e)$, where $1 \leqslant n \leqslant r_h$ and e is an edge of G^h_n.

It is easy to verify that σ is a simulation of (G^h, T^h) in (G^i, T^i), as required. This completes the proof of (14.2). ∎

Now we need a mild generalization of k-chains. We say that (G, T) is a *k-train* over $\mathscr{A}$ if it has a path-decomposition (G_i, Y_{i-1}, Y_i) $(1 \leqslant i \leqslant r)$ over $\mathscr{A}$ such that

(i) for $1 \leqslant i < r$, $|Y_i| = k$

(ii) if $r \geqslant 3$ there are k mutually vertex-disjoint paths of G between Y_1 and Y_{r-1}.

If $\mathscr{A}$ is a class of grafts, we denote by $\mathscr{A}^*$ the class of all grafts isomorphic (in the natural sense) to a member of $\mathscr{A}$. Evidently if $\mathscr{A}$ is well-rooted then so is $\mathscr{A}^*$.

(14.3) *Every k-train over $\mathscr{A}$ belongs to $W(\mathscr{A}^* \oplus \mathscr{B}^* \oplus \mathscr{A}^*) \cup W(\mathscr{A}^* \oplus \mathscr{A}^*) \cup \mathscr{A}$ where $\mathscr{B}$ is the class of all k-chains over $\mathscr{A}$.*

Proof. Let (G, T) be a k-train over $\mathscr{A}$, and let (G_i, Y_{i-1}, Y_i) $(1 \leqslant i \leqslant r)$ be the corresponding path-decomposition. If $r = 1$ then $(G, T) \in \mathscr{A}$. If $r = 2$ then $(G, T) \in W(\mathscr{A}^* \oplus \mathscr{A}^*)$. If $r \geqslant 3$ then

$$(G, T) \in W(\mathscr{A}^* \oplus \mathscr{B}^* \oplus \mathscr{A}^*).$$

In each case the theorem is true, as required. ∎

If $\mathscr{A}$ is a class of grafts, $L^k(\mathscr{A})$ denotes the class of k-trains over $\mathscr{A}$, and $C^k(\mathscr{A})$ denotes $L^0(L^1(L^2(\dots(L^k(\mathscr{A}))\dots)))$.

(14.4) *If $\mathscr{A}$ is well-rooted then so is $L^k(\mathscr{A})$.*

This follows immediately from (14.2), (13.1), (13.3), (14.3), and (4.2).

We say that a graft (G, T) is a *concatenation* over $\mathscr{A}$ if $(G, T) \in C^k(\mathscr{A})$ for some $k \geqslant 0$.

(14.5) *If $\mathscr{A}$ is well-rooted, then the class of all concatenations over $\mathscr{A}$ is well-rooted.*

Proof. By (4.1), $\mathscr{A}$ has some index k_0. Now for $k > k_0$, $L^k(\mathscr{A}) = \mathscr{A}$, and so every concatenation over $\mathscr{A}$ is an element of $C^{k_0}(\mathscr{A})$. But this is well-rooted, from (14.4). The result follows. ∎

15. Tubes

A *cylinder* is a surface homeomorphic to $\Sigma(0, 0, 2)$. Our next objective is to prove (6.1) when Σ is a cylinder.

There are two kinds of O-arc in a cylinder; those that are null-homotopic and bound a disc, and those that wind once around Σ. A *hoop* is an O-arc of the second kind. If F is a hoop then $\Sigma - F$ is naturally divided into two parts, which we call the *sides* of F. If F_1, F_2 are hoops and $F_2 - F_1$ lies completely in one side of F_1, it follows that $F_1 - F_2$ lies completely in one side of F_2. In these circumstances we say that F_1, F_2 are *laminar*. If F_1, F_2 are distinct laminar hoops, then Σ is divided naturally into three parts—the side of F_1 not containing $F_2 - F_1$, the side of F_2 not containing $F_1 - F_2$, and the remainder, which we denote by $\Sigma(F_1, F_2)$.

A *tube* in a cylinder Σ is a triple (Γ, F_1, F_2), where Γ is a drawing in Σ and F_1, F_2 are Γ-normal laminar hoops such that $U(\Gamma) \subseteq \Sigma(F_1, F_2)$. The *width* of a tube (Γ, F_1, F_2) is the largest value of r such that for every Γ-normal hoop $F \subseteq \Sigma(F_1, F_2)$, either $F \cap V(\Gamma) = F_1 \cap V(\Gamma)$, or $F \cap V(\Gamma) = F_2 \cap V(\Gamma)$, or $|F \cap V(\Gamma)| \geqslant r$. If there is no such number r, we say its width is infinite. The *length* of a tube (Γ, F_1, F_2) is the largest value of s such that $|I \cap V(\Gamma)| \geqslant s$ for every Γ-normal $[0, 1]$-arc I with one end in F_1 and the other in F_2.

If (Γ, F_1, F_2) is a tube, and (G, T) is a graft, we say that (Γ, F_1, F_2) is a *framing* of (G, T) if there is a bijection α from $V(G)$ to $V(\Gamma)$ and a bijection β from $E(G)$ to the set of edges of Γ, such that

(i) for $v \in V(G)$ and $e \in E(G)$, v is incident with e if and only if $\alpha(v) \in \overline{\beta(e)}$

(ii) for $v \in V(G)$, $\alpha(v) \in F_1 \cup F_2$ if and only if $v \in T$.

If (G, T) has a framing, we say (G, T) is *tubular*. If (G, T) is a tubular graft, its *width* is the maximum width of all its framings and its *length* is the minimum length of all its framings.

(15.1) *Every tubular graft of width $\geqslant r$ and index $\leqslant k$ is an r-train of tubular grafts of width $\geqslant r+1$ and index $\leqslant k+2r$.*

Proof. Let (G, T) be a tubular graft of width $\geqslant r$ and index $\leqslant k$. We wish to show it is an r-train of tubular grafts of width $\geqslant r+1$ and index $\leqslant k+2r$. If its width is at least $r+1$ we are done. We assume then that it has width r. Let (Γ, F_1, F_2) be a framing of (G, T) with width r, on a cylinder Σ. Choose hoops $H_1, \ldots, H_n$ with n maximum such that

(i) $H_1, \ldots, H_n \subseteq \Sigma(F_1, F_2)$ are Γ-normal, and mutually laminar

(ii) $|H_i \cap V(\Gamma)| \leqslant r$ $(1 \leqslant i \leqslant n)$

(iii) the sets $F_1 \cap V(\Gamma)$, $F_2 \cap V(\Gamma)$ and $H_i \cap V(\Gamma)$ $(1 \leqslant i \leqslant n)$ are all different.

Define $H_0 = F_1$, $H_{n+1} = F_2$. We assume $H_1, \ldots, H_n$ to be numbered in order along Σ, so that for $1 \leqslant i \leqslant n$, $H_i \subseteq \Sigma(H_{i-1}, H_{i+1})$. Let Γ_i be the intersection of Γ with $\Sigma(H_{i-1}, H_i)$ $(1 \leqslant i \leqslant n+1)$. From the maximality of n, (Γ_i, H_{i-1}, H_i) is a tube with width $\geqslant r+1$. Let G_i be the subgraph of

G corresponding to Γ_i, and let Y_i be the set of vertices of G represented by the points in $V(\Gamma)\cap H_i$. Then (G_i, Y_{i-1}, Y_i) $(1\leqslant i\leqslant n+1)$ is a path-decomposition of (G, T) over the class of all tubular grafts with width $\geqslant r+1$ and index $\leqslant k+2r$. It remains to check that

(i) for $1\leqslant i\leqslant n$, $|Y_i|=r$; this is true by (ii) above and the fact that (Γ, F_1, F_2) has width r

(ii) if $n\geqslant 1$, there are r mutually vertex-disjoint paths of Γ between Y_1 and Y_n; this follows from Menger's theorem and the fact that (Γ, F_1, F_2) has width r.

The result follows. ∎

(15.2) *Let (G, T) be a tubular graft of index $\leqslant k$ and let $r\geqslant 0$ be an integer. Then (G, T) is a concatenation of tubular grafts of width $\geqslant r$ and index $\leqslant k+r(r-1)$.*

Proof. Let $\mathscr{G}(t, u)$ be the class of all tubular grafts of width $\geqslant t$ and index $\leqslant u$. Now by (15.1),

$$\mathscr{G}(0, k)\subseteq L^0(\mathscr{G}(1, k))=C^0(\mathscr{G}(1, k)).$$

We prove by induction on t that for $t\geqslant 0$,

$$\mathscr{G}(0, k)\subseteq C^t(\mathscr{G}(t+1, k+(t+1)t)).$$

The result is true if $t=0$, and we assume $t\geqslant 1$. By our inductive hypothesis,

$$\mathscr{G}(0, k)\subseteq C^{t-1}(\mathscr{G}(t, k+t(t-1)))$$

but by (15.1),

$$\mathscr{G}(t, k+t(t-1))\subseteq L^t(\mathscr{G}(t+1, k+(t+1)t)),$$

and so

$$\mathscr{G}(0, k)\subseteq C^{t-1}(L^t(\mathscr{G}(t+1, k+(t+1)t)))=C^t(\mathscr{G}(t+1, k+(t+1)t)).$$

This completes the inductive proof, and the result follows. ∎

Let $\mathscr{A}(k, r, s)$ denote the class of all tubular grafts with index $\leqslant k$ which have a framing (Γ, F_1, F_2) in a cylinder Σ, such that there do not exist Γ-normal laminar hoops $F_1', F_2'\subseteq\Sigma(F_1, F_2)$ such that (Γ', F_1', F_2') has width $\geqslant r$ and length $\geqslant s$, where Γ' is the intersection of Γ and $\Sigma(F_1', F_2')$. Our next object is to prove that $\mathscr{A}(k, r, s)$ is well-rooted.

(15.3) *If $G, T)\in\mathscr{A}(k, r, s)$, then (G, T) is a concatenation of tubular grafts of length $<s$ and index $\leqslant k+r(r-1)$.*

Proof. Take a framing (Γ, F_1, F_2) of (G, T) in a cylinder Σ, as in the definition of $\mathscr{A}(k, r, s)$. If we express (G, T) as a concatenation by beginning with this framing and decomposing it (by the method of (15.1) and (15.2)) we express (G, T) as a concatenation of tubular grafts of width $\geqslant r$ and index $\leqslant k + r(r-1)$, with the additional property that each of these grafts has a framing (Γ', F_1', F_2'), where $F_1', F_2' \subseteq \Sigma(F_1, F_2)$ are Γ-normal and $\Gamma' = \Gamma \cap \Sigma(F_1', F_2')$. But any such graft has length $<s$, since $(G, T) \in \mathscr{A}(k, r, s)$. The result follows. ∎

(15.4) *For any integers $k, r, s \geqslant 0$, $\mathscr{A}(k, r, s)$ is well-rooted.*

Proof. Let $\mathscr{B}$ denote the class of grafts with index $\leqslant k + r(r-1) + 2s$ which embed in a disc. Then $\mathscr{B}$ is well-rooted by the result of Section 12. Let $\mathscr{B}'$ be the class of all tubular grafts with length $<s$ and index $\leqslant k + r(r-1)$. Then $\mathscr{B}' \subseteq W(\mathscr{B})$, and so $\mathscr{B}'$ is well-rooted by (13.3). But any member of $\mathscr{A}(k, r, s)$ is a concatenation over $\mathscr{B}'$, by (15.3), and hence $\mathscr{A}(k, r, s)$ is well-rooted by (14.5). ∎

16. Boundary-Linking in Tubes

Let Σ be a cylinder, and let $r, s \geqslant 0$ be integers. A framing (Γ, F_1, F_2) in Σ is said to be *(r, s)-boundary-linked* if there exist Γ-normal laminar hoops $F_1', F_2' \subseteq \Sigma(F_1, F_2)$ such that

(i) $F_2' \subseteq \Sigma(F_1, F_1')$, $F_1' \subseteq \Sigma(F_2, F_2')$

(ii) $(\Gamma \cap \Sigma(F_1', F_2'), F_1', F_2')$ has width $\geqslant r$ and length $\geqslant s$

(iii) for $i = 1, 2$ there are $|V(\Gamma) \cap F_i|$ mutually vertex-disjoint paths of Γ between F_i and F_i'.

Let $k, r, s \geqslant 0$ be integers, and let $\mathscr{B}(k, r, s)$ denote the class of all grafts of index $\leqslant k$ which have framings in Σ which are not (r, s)-boundary-linked. Let $\mathscr{B}(k)$ denote the class of all tubular grafts with index $\leqslant k$. The main result of this section is the following.

(16.1) *If $\mathscr{B}(k-1)$ is well-rooted then so is $\mathscr{B}(k, r, s)$, for any integers $k, r, s \geqslant 0$.*

[We interpret $\mathscr{B}(-1) = \varnothing$.]
The following lemma is used to prove (16.1).

(16.2) $\mathscr{B}(k, r, s) \subseteq \mathscr{A}(k, r, s) \cup W(\mathscr{B}(k-1) \oplus \mathscr{A}(2k, r, s+2)) \cup W(\mathscr{B}(k-1) \oplus \mathscr{B}(k-1))$.

Proof. Let $(G, T) \in \mathscr{B}(k, r, s)$, and let (Γ, F_1, F_2) be a framing of (G, T) in Σ which is not (r, s)-boundary-linked. Let $|V(\Gamma) \cap F_i| = k_i$ $(i = 1, 2)$. We assume $k_1 \geqslant k_2$ without loss of generality; and certainly $k_1 + k_2 \leqslant 2k$. Let k_3 be the minimum value of $|V(\Gamma) \cap F|$ taken over all Γ-normal hoops $F \subseteq \Sigma(F_1, F_2)$. Then $k_2 \geqslant k_3$ since F_2 is a possible choice of F. Since $k_1 \geqslant k_2 \geqslant k_3$, one of the following three cases must occur.

Case 1. $k_1 = k_2 = k_3$.

In this case we claim that $(G, T) \in \mathscr{A}(k, r, s)$. For suppose that F_1', F_2' are Γ-normal laminar hoops with $F_1', F_2' \subseteq \Sigma(F_1, F_2)$, and $(\Gamma \cap \Sigma(F_1', F_2'), F_1', F_2')$ has width $\geqslant r$ and length $\geqslant s$. We assume without loss of generality that $F_2' \subseteq \Sigma(F_1, F_1')$ and $F_1' \subseteq \Sigma(F_2, F_2')$. But by Menger's theorem there are k_3 mutually vertex-disjoint paths of Γ between F_i and F_i' $(i = 1, 2)$, and so (Γ, F_1, F_2) is (r, s)-boundary-linked, contrary to our hypothesis. Thus such F_1', F_2' do not exist, and so $(G, T) \in \mathscr{A}(k, r, s)$ as claimed.

Case 2. $k_1 > k_2 = k_3$.

In this case we claim that $(G, T) \in W(\mathscr{B}(k-1) \oplus \mathscr{A}(2k, r, s+2))$. For let F be a Γ-normal hoop with $F \subseteq \Sigma(F_1, F_2)$ and with $|V(\Gamma) \cap F| < k_1$, chosen so that the side of F including $F_1 - F$ contains as few vertices of Γ as possible. Let $\Gamma_i = \Gamma \cap \Sigma(F, F_i)$ and let (Γ_i, F, F_i) be a framing of some graft (G_i, T_i) $(i = 1, 2)$.

Now $(G_2, T_2) \in \mathscr{B}(k-1)$ since $|V(\Gamma) \cap F| < k_1$. We claim that $(G_1, T_1) \in \mathscr{A}(2k, r, s+2)$. Certainly $|T_1| \leqslant k_1 + k_3 \leqslant 2k$. Suppose then that $F_1', F_2' \subseteq \Sigma(F_1, F)$ are Γ-normal laminar hoops, and (Γ', F_1', F_2') has width $\geqslant r$ and length $\geqslant s + 2 \geqslant \max(s+1, 2)$, where $\Gamma' = \Gamma \cap \Sigma(F_1', F_2')$. We assume without loss of generality that $F_1' \subseteq \Sigma(F_2, F_2')$, $F_2' \subseteq \Sigma(F_1, F_1')$. It is easy to see that there is a Γ-normal hoop $F_1'' \subseteq \Sigma(F_1', F_2')$, such that the tube (Γ'', F_1'', F_2') has length $\geqslant s$, where $\Gamma'' = \Gamma \cap \Sigma(F_1'', F_2')$, and such that $F_1'' \cap \Gamma \nsubseteq F_1' \cap \Gamma$. Then (Γ'', F_1'', F_2') has width $\geqslant r$ and length $\geqslant s$. Moreover, there are k_2 mutually disjoint paths of Γ between F_1 and F_2, since $k_3 \geqslant k_2$, and hence there are k_2 mutually disjoint paths of Γ between F_2'' and F_2. Also, from the choice of F, there are k_1 mutually disjoint paths of Γ between F_1'' and F_1. Thus (Γ, F_1, F_2) is (r, s)-boundary-linked, a contradiction. Hence there do not exist such F_1', F_2' and so $(G_1, T_1) \in \mathscr{A}(2k, r, s+2)$ as claimed. It follows that

$$(G, T) \in W(\mathscr{B}(k-1) \oplus \mathscr{A}(2k, r, s+2)).$$

Case 3. $k_2 > k_3$.

Let F be a Γ-normal hoop with $|V(\Gamma) \cap F| < k_2$; and let

$\Gamma_i = \Gamma \cap \Sigma(F, F_i)$ $(i = 1, 2)$. Then (Γ_i, F, F_i) is a framing of some member of $\mathscr{B}(k-1)$ since $k_3 < k_1, k_2$ $(i = 1, 2)$; and so

$$(G, T) \in W(\mathscr{B}(k-1) \oplus \mathscr{B}(k-1)).$$

In each case then the theorem is satisfied. This completes the proof. ∎

Proof of (16.1). If $\mathscr{B}(k-1)$ is well-rooted then the right hand side of (16.2) is well-rooted, by (4.2), (13.1), (13.3), and (15.4), and hence so is $\mathscr{B}(k, r, s)$. ∎

17. Conclusion of the Cylinder Case

Now we prove that $\mathscr{B}(k)$ (as defined in Section 16) is well-rooted. Let Σ be a cylinder with cuffs C_1, C_2, and let $\mathscr{B}(k_1, k_2)$ be the set of all grafts with drawings Γ in Σ such that $|\Gamma \cap C_i| = k_i$ $(i = 1, 2)$. Clearly every tubular graft embeds in Σ, and in fact we have the following, as is easily seen.

(17.1) *For any integer $k \geqslant 0$, $\mathscr{B}(k) = \bigcup(\mathscr{B}(k_1, k_2))$, the union being taken over all integers $k_1, k_2 \geqslant 0$ with $k_1 + k_2 \leqslant k$.*

We shall need the following lemma, which is a consequence of [11, Theorem (9.3)].

(17.2) *Let $(H, U) \in \mathscr{B}(k_1, k_2)$. Then there are integers $r, s \geqslant 0$ such that for every $(G, T) \in \mathscr{B}(k_1, k_2)$, either (H, U) is simulated in (G, T) or $(G, T) \in \mathscr{B}(k_1 + k_2, r, s)$.*

From this we deduce the following.

(17.3) *For any integer $k \geqslant 0$, $\mathscr{B}(k)$ is well-rooted.*

Proof. We proceed by induction on k, and assume that $\mathscr{B}(k-1)$ is well-rooted. (We recall that $\mathscr{B}(-1)$ is interpreted as $\emptyset$, which is certainly well-rooted.) By (17.1) and (4.2), it suffices to prove that $\mathscr{B}(k_1, k_2)$ is well-rooted for all choices of k_1, k_2 with $k_1 + k_2 \leqslant k$. Let us then choose integers $k_1, k_2 \geqslant 0$ with $k_1 + k_2 \leqslant k$. Let $(G_1, T_1), (G_2, T_2), \ldots$ be a countable sequence of members of $\mathscr{B}(k_1, k_2)$. We wish to show that there exist $j > i \geqslant 1$ such that (G_i, T_i) is simulated in (G_j, T_j). If (G_1, T_1) is simulated in (G_i, T_i) for some $i > 1$ we are done, and so we assume that this is false. By (17.2) there are integers $r, s \geqslant 0$ such that $(G_i, T_i) \in \mathscr{B}(k_1 + k_2, r, s)$ for all $i \geqslant 2$. But $\mathscr{B}(k_1 + k_2, r, s) \subseteq \mathscr{B}(k, r, s)$, and $\mathscr{B}(k, r, s)$ is well-rooted by (16.1) and our inductive hypothesis that $\mathscr{B}(k-1)$ is well-rooted. Hence there exist $j > i \geqslant 2$ such that (G_i, T_i) is simulated in (G_j, T_j), as required. ∎

18. The General Surface

So far we have proved (6.1) when Σ is a disc or cylinder. In this section we prove it in all other cases.

If C is a cuff of a surface Σ, $\Sigma + \hat{C}$ denotes the surface obtained from Σ by "pasting" a disc onto C, thereby reducing the number of cuffs by one. An O-arc F in Σ is said to be *planar* in Σ if it bounds a disc in Σ; it *surrounds* a cuff C if it is planar in $\Sigma + \hat{C}$ but not in Σ. It is *near-planar* if either it is planar in Σ or it is planar in $\Sigma + \hat{C}$ for some cuff C.

Let G_1 be the graph with two vertices u, v say and three edges, one a loop on u, one a loop on v, and one joining u, v. Let G_2 be the graph with one vertex and two edges. Let G_3 be the graph with two vertices and three edges, mutually parallel. Let G_4 be the graph with two vertices and two edges, exactly one of which is a loop. Let X_1, X_2, X_3, X_4 be the topological space associated with G_1, G_2, G_3, G_4, respectively. We recall that an X_i-arc in Σ is a subset of Σ homeomorphic to X_i. We define the *end* of an X_4-arc to be the point representing the monovalent vertex of G_4.

Let Σ be a surface. By a *schism* in Σ we mean a subset of Σ which is one of the following:

(i) an O-arc F which is not near-planar, with $|F \cap bd\,\Sigma| \leqslant 1$

(ii) a $[0, 1]$-arc with its ends in distinct cuffs, containing no other point of $bd\,\Sigma$

(iii) a $[0, 1]$-arc F with both ends in the same cuff C, such that $F \cup C$ is not near-planar, and such that $|F \cap bd\,\Sigma| = 2$

(iv) an X_1 or X_2-arc in Σ such that both its O-arcs surround distinct cuffs of Σ, and which contains no point of $bd\,\Sigma$

(v) an X_3-arc in Σ such that all three of its O-arcs surround distinct cuffs, and which contains no point of $bd\,\Sigma$

(vi) an X_4-arc with its end in one cuff and its O-arc surrounding another cuff, containing only one point of $bd\,\Sigma$.

If X is a schism in Σ, we may "cut" along X in the natural way, to obtain a new surface Σ'. There is a natural surjection $\theta: \Sigma' \to \Sigma$. If Γ is a drawing in Σ and X is Γ-normal, we define

$$\theta^{-1}(\Gamma) = (\theta^{-1}(U(\Gamma)), \theta^{-1}(V(\Gamma))).$$

Then $\theta^{-1}(\Gamma)$ is a drawing in Σ'.

We require the following lemma, which is a consequence of [11, Theorem (9.1)], and the discussion in Section 8 of that paper.

(18.1) *Let Σ be a connected surface, not a sphere, disc, or cylinder. Let Δ be a drawing of a graft (H, U) in Σ. Then there is a number N such that for* *every graft* (G, T) *and every drawing* Γ *of* (G, T) *in* Σ, *one of the following holds*:

(i) (H, U) *is simulated in* (G, T)

(ii) *for some cuff* C *of* Σ, $|V(\Gamma) \cap C| \neq |V(\Delta) \cap C|$

(iii) *for some* Γ*-normal schism* F, $|V(\Gamma) \cap F| \leqslant N$

(iv) *for some cuff* C *of* Σ *there is a* Γ*-normal O-arc* F *surrounding* C *with* $F \cap bd\,\Sigma \subseteq C$ *and with* $|V(\Gamma) \cap F| < |V(\Gamma) \cap C|$.

Let Σ be a connected surface. Then $\Sigma \cong \Sigma(a, b, c)$ for some choice of a, b, c, and we define $g(\Sigma) = 4a + 2b + c$. (This definition does not depend on the choice of a, b, c because c is fixed, being the number of components of $bd\,\Sigma$, and $4a + 2b$ is also fixed, being equal to $2(2 - e - c)$ where e is the Euler characteristic of Σ.) For a general surface Σ, we define $g(\Sigma)$ to be the maximum of $g(\Sigma')$ taken over all components Σ' of Σ.

(18.2) *Let* Σ *be a connected surface, not a sphere, disc or cylinder, and let* X *be a schism in* Σ. *Let* Σ' *be obtained from* Σ *by cutting along* X. *Then* $g(\Sigma') < g(\Sigma)$.

Proof. Let Σ_1 be a component of Σ' and Σ_2 be the union of the other components. Let $\Sigma, \Sigma', \Sigma_1, \Sigma_2$ have Euler characteristics e, e', e_1, e_2 and have c, c', c_1, c_2 cuffs, respectively. Let $a, b, c, k_2, a_2, b_2, c_2$ be such that $\Sigma \cong \Sigma(a, b, c)$, and Σ_2 has k_2 components, and Σ_2 may be obtained from the union of k_2 disjoint spheres by adding a_2 handles and b_2 cross-caps and removing the interiors of c_2 disjoint closed discs (where of course $k_2 = a_2 = b_2 = c_2 = 0$ if Σ' is connected). The schism X is of one of the types listed in the definition of "schism," which gives us several cases to consider. In each case we can find $e' - e$ by considering a triangulation of Σ in which X is a union of 0- and 1-simplices, and it can be verified that

$$2(e' - e) + e' - c + 4a_2 + 2b_2 + c_2 > 4k_2.$$

But $e' = e_1 + 2k_2 - 2a_2 - b_2 - c_2$ and $c' = c_1 + c_2$, and it follows by substitution that $2e_1 + c_1 > 2e + c$. Now $g(\Sigma_1) = 4 - 2e_1 - c_1$, and $g(\Sigma) = 4 - 2e - c$, and so $g(\Sigma_1) < g(\Sigma)$. Since this holds for every component Σ_1 of Σ' we deduce that $g(\Sigma') < g(\Sigma)$, as required. ∎

We come now to the proof of (6.1). If Σ is a surface and $k \geqslant 0$ is an integer, we define $\mathscr{C}(\Sigma, k)$ to be the class of all grafts of index $\leqslant k$ which embed in Σ. If $h, k \geqslant 0$ are integers, we define $\mathscr{C}(h, k)$ to be the union of $\mathscr{C}(\Sigma, k)$, taken over all surfaces Σ with $g(\Sigma) \leqslant h$. Then (6.1) is a consequence of the following.

(18.3) *For any integers* $h, k \geqslant 0$, $\mathscr{C}(h, k)$ *is well-rooted.*

Proof. We make two inductive hypotheses—that $\mathscr{C}(h', k')$ is well-rooted for all h', k' with $h' < h$, and that $\mathscr{C}(h, k')$ is well-rooted for all $k' < k$.

Let Σ be a connected surface with $g(\Sigma) \leqslant h$. We shall show first that $\mathscr{C}(\Sigma, k)$ is well-rooted. For let the cuffs of Σ be $C_1, ..., C_c$ in some order. Let $\mathscr{C}(\Sigma; k_1, ..., k_c)$ denote the class of all grafts which have a drawing Γ in Σ with

$$|V(\Gamma) \cap C_i| = k_i \qquad (1 \leqslant i \leqslant c).$$

We claim that for any integers $k_1, ..., k_c \geqslant 0$ with $k_1 + \cdots + k_c \leqslant k$, $\mathscr{C}(\Sigma; k_1, ..., k_c)$ is well-rooted. For if Σ is a sphere, disc, or cylinder then $\mathscr{C}(\Sigma; k_1, ..., k_c) \subseteq \mathscr{B}(k)$ and the result follows from (17.3). We assume then that Σ is not a sphere, disc, or cylinder. Let (G_1, T_1), (G_2, T_2), ... be a countable sequence of grafts in $\mathscr{C}(\Sigma; k_1, ..., k_c)$. We may assume that (G_1, T_1) is not simulated in any of (G_2, T_2), (G_3, T_3), ... By (18.1) and (18.2) there is a number N such that for $i = 2, 3, ...$, either

(i) $(G_i, T_i) \in W(\mathscr{C}(h-1, k+2N+2))$, or

(ii) $(G_i, T_i) \in W(\mathscr{C}(h, k-1) \oplus \mathscr{B}(2k-1))$

depending on whether (18.1)(iii) or (18.1)(v) applies. Thus in either case $(G_i, T_i) \in \mathscr{D}$, where

$$\mathscr{D} = W(\mathscr{C}(h-1, k+2N+2)) \cup W(\mathscr{C}(h, k-1) \oplus \mathscr{B}(2k-1)).$$

But $\mathscr{C}(h-1, k+2N+2)$ is well-rooted, by our first inductive hypothesis; and $\mathscr{C}(h, k-1)$ is well-rooted, by our second inductive hypothesis; and $\mathscr{B}(2k-1)$ is well-rooted, by (17.3). Hence $\mathscr{D}$ is well-rooted by (13.1) and (13.3), and so there exist $j > i \geqslant 1$ such that (G_i, T_i) is simulated in (G_j, T_j). Thus $\mathscr{C}(\Sigma; k_1, ..., k_c)$ is well-rooted, as claimed.

It follows that $\mathscr{C}(\Sigma, k)$ is well-rooted. For $\mathscr{C}(\Sigma, k) = \bigcup \mathscr{C}(\Sigma; k_1, ..., k_c)$, the union being taken over all choices of $k_1, ..., k_c \geqslant 0$ with $k_1 + \cdots + k_c \leqslant k$. Since there are only finitely many such choices, $\mathscr{C}(\Sigma, k)$ is well-rooted by (4.2).

Let $\mathscr{D}(h, k)$ be the union of $\mathscr{C}(\Sigma, k)$ taken over all connected surfaces Σ with $g(\Sigma) \leqslant h$. If $\Sigma_1 \cong \Sigma_2$ then $\mathscr{C}(\Sigma_1, k) = \mathscr{C}(\Sigma_2, k)$, and up to homeomorphism there are only finitely many connected surfaces Σ with $g(\Sigma) \leqslant h$. Thus there are finitely many connected surfaces $\Sigma_1, ..., \Sigma_n$ with $g(\Sigma_i) \leqslant h$ $(1 \leqslant i \leqslant n)$ such that

$$\mathscr{D}(h, k) = \bigcup (\mathscr{C}(\Sigma_i, k) : 1 \leqslant i \leqslant n).$$

Since each $\mathscr{C}(\Sigma_i, k)$ is well-rooted, it follows from (4.2) that $\mathscr{D}(h, k)$ is well-rooted.

In particular, $\mathscr{D}(h, 0)$ is well-rooted. It follows from (3.3) that $\mathscr{C}(h, 0)$ is well-rooted; for if $(G, \varnothing) \in \mathscr{C}(h, 0)$ there are grafts $(G_1, \varnothing), ..., (G_r, \varnothing) \in \mathscr{D}(h, 0)$ such that

$$(G, \varnothing) = (G_1, \varnothing) \oplus \cdots \oplus (G_r, \varnothing).$$

(In contrast, for $k > 0$ the class of all grafts expressible as the "disjoint union" of members of $\mathscr{D}(h, k)$ is not well-rooted, by (4.1).) Let $\mathscr{E}(h, k)$ be $\mathscr{D}(h, k) \oplus \cdots \oplus \mathscr{D}(h, k)$, with k summands. By (13.1), $\mathscr{E}(h, k)$ is well-rooted, and hence by (13.1) again, so is $\mathscr{E}(h, k) \oplus \mathscr{C}(h, 0)$.

Let $(G, T) \in \mathscr{C}(h, k)$, and let Γ be a drawing of (G, T) in some surface Σ with $g(\Sigma) \leqslant h$. Then $V(\Gamma) \cap bd(\Sigma') \neq \varnothing$ for at most k components Σ' of Σ, since $|T| \leqslant k$, and so we may express (G, T) as $(G_1, T_1) \oplus (G_2, \varnothing)$ where $(G_1, T_1) \in \mathscr{E}(h, k)$ and $(G_2, T_2) \in \mathscr{C}(h, 0)$. It follows that

$$\mathscr{C}(h, k) \subseteq \mathscr{E}(h, k) \oplus \mathscr{C}(h, 0)$$

and so $\mathscr{C}(h, k)$ is well-rooted, as required. ∎

References

1. D. Archdeacon, "A Kuratowski Theorem for the Projective Plane," Thesis, The Ohio State University, 1980.
2. D. Archdeacon and P. Huneke, A Kuratowski theorem for non-orientable surfaces, *J. Combin. Theory Ser. B* **46** (1989), 173–231.
3. H. Glover, P. Huneke, and C. S. Wang, 103 graphs that are irreducible for the projective plane, *J. Combin. Theory Ser. B* **27** (1979), 332–370.
4. G. Higman, Ordering by divisibility in abstract algebras, *Proc. London Math. Soc.* **2** (1952), 326–336.
5. J. Kruskal, Well-quasi-ordering, the tree theorem, and Vázsonyi's conjecture, *Trans. Amer. Math. Soc.* **95** (1960), 210–225.
6. K. Kuratowski, Sur le problème des courbes gauches en topologie, *Fund. Math.* **15** (1930), 271–283.
7. N. Robertson and P. D. Seymour, Graph minors. III. Planar tree-width, *J. Combin. Theory Ser. B* **36** (1984), 49–64.
8. N. Robertson and P. D. Seymour, Graph minors. IV. Tree-width and well-quasi-ordering, *J. Combin. Theory Ser. B* **48** (1990), 227–254.
9. N. Robertson and P. D. Seymour, Graph minors. V. Excluding a planar graph, *J. Combin. Theory Ser. B* **41** (1986), 92–114.
10. N. Robertson and P. D. Seymour, Graph minors. VI. Disjoint paths across a disc, *J. Combin. Theory Ser. B* **41** (1986), 115–138.
11. N. Robertson and P. D. Seymour, Graph minors. VII. Disjoint paths on a surface, *J. Combin. Theory Ser. B* **45** (1988), 212–254.