The Vector Complementary Problem and Its Equivalences with the Weak Minimal Element in Ordered Spaces

CHEN GUANG-YA

Institute of Systems Science, Academia Sinica, Beijing 100080, People's Republic of China

AND

YANG XIAO-QI

Chongqing Architecture and Engineering Institute, Chongqing, People's Republic of China

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1. INTRODUCTION

Recently, F. Giannessi [11] introduced the vector variational inequality in a finite dimension Euclidean space with further application. In the paper [4] we have proposed the vector variational inequality (VVIP) and have applied it to the vector optimization problem. In the scalar case the variational inequality, the complementary problem, and the extremum problem are equivalent under some conditions. C. W. Cryer [6] discussed the equivalence among the complementary problem, nonlinear program, least element problem, and the unilateral minimization problem in Hilbert space and showed the existence of a common solution of these five problems. R. C. Riddell [16] extended the results to a Banach lattice. J. M. Borwein [3] proposed the following forms of the complementary problem.

Let $X$, $Y$ be paired topological vector spaces with associated bilinear form $(\cdot, \cdot)$, let $C$ be a closed convex cone in $X$, the operator $T: C \rightarrow Y$. One asks for a solution to

$$(T(x) + q, x) = 0, \quad x \in C, \quad T(x) + q \in C^+,$$

for $q$ in $Y$, $C^+ = \{ y \in Y | (y, x) \geq 0, \text{ all } x \in C \}$. This is called the topological complementary problem.
If $X$ is a vector lattice space with positive cone $C$, the operator $T: C \to X$, one asks for a solution to

$$(T(x) + q) \wedge X = 0,$$

for $q$ in $X$. This is called the order complementary problem.

With these two kinds of complementary problems it may be difficult to discuss their relative equivalences with the vector variational inequality and the vector extremum problem.

In this paper, we attempt to give the equivalence and the existence of the vector complementary problem (VCP), the vector variational inequality (VVIP), the vector extremum problem (VEP), the weak minimal element problem (VMEP), and the vector unilateral minimization problem (VUMP) in the Banach space.

Let $X$ be a Banach space over $\mathbb{R}$ and $A$ be a subset in $X$, the coset of $A$ denoted by $A^C$. A nonempty subset $C$ in $X$ is called a convex cone if $C + C = C$, and $\lambda C \subset C$, for any $\lambda > 0$, $\lambda \in \mathbb{R}$. $C$ is called a pointed cone if $C$ is a cone and $C \cap (-C) = \{0\}$, $C$ is called connected if $C \cup (-C) = X$, and $C$ is called reproduced if $C - C = X$. The polar cone is defined as $C^* = \{ x^* \in X^*: (x^*, x) \geq 0, \text{for any } x \in C\}$, where $X^*$ is the conjugate space of $X$. The partial order in $X$ is denoted by $\leq$ and is defined as $x \leq y$ iff $y - x \in C$, for any $x$, $y$ in $X$. A linear order in $X$ is such a partial order which is induced by a convex cone; $C$ is called the positive cone in $X$. A ordered Banach space $X$ means a real Banach space with a linear order, denoted by $(X, C)$. For any $x$, $y$ in $X$, $[x, y] = (x + C) \cap (y - C)$ is called an order interval. The topological interior of a subset $A$ in $X$ is denoted by $\text{int } A$.

Let $(X, C)$ be an ordered Banach space, $\text{int } C \neq \emptyset$, and for any $x$, $y$ in $X$ define $x \triangleleft y$ iff $y - x \in \text{int } C$. We usually call $\triangleleft$ a weak order. The following properties are elementary:

$$x \triangleleft y \iff x + z \triangleleft y + z, \text{for any } x, y, z \in X;$$

$$x \triangleleft y \iff \lambda x \triangleleft \lambda y, \text{for any } \lambda \geq 0.$$

**Corollary.**

1. $0 \triangleleft a > b$ implies $b \succ 0$;
2. $0 \triangleleft a \geq b$ implies $b \succ 0$;
3. $0 \succ a < b$ implies $b \prec 0$;
4. $0 \succ a \leq b$ implies $b \prec 0$.

Given two subsets $A$ and $B$ in $X$, we define

$$A \leq B \iff \text{all } a \in A, b \in B, a \leq b;$$

$$A \triangleleft B \iff \text{all } a \in A, b \in B, a \triangleleft b.$$
Let $Y$ be a real Banach space over $\mathbb{R}$ and $L(X, Y)$ be the set of all linear bounded operators from $X$ to $Y$. We denote the value of $l \in L(X, Y)$ on $x \in X$ by $(l, x)$.

Let $A$ be a subset of an ordered Banach space $(X, C)$. If there is a point $a \in X$ such that $A \subseteq a + C$, $A$ is called bounded below; a point $a \in A$ is said to be a lower (upper) efficient point of the set $A$ if there exists no $a' \in A$ such that $a' \neq a$, and $a' \leq a$ ($a' \geq a$). Suppose that $\text{int} \ C \neq \emptyset$, a point $a \in A$ is said to be a weak lower (upper) efficient point if $a \nless A$ ($a \leq A$). We denote the set of all lower (upper) efficient points by $\text{LEXT} \ A$ ($\text{UEXT} \ A$), and the set of all weak lower (upper) efficient points by $\text{WLEXT} \ A$ ($\text{WUEXT} \ A$).

Let $X$ be a real Banach space and $(Y, P)$ be an ordered Banach space, where $P$ is a positive cone in $Y$, the mapping $T: X \to L(X, Y)$,

(1) $T$ is said to be monotone, if $(T(x) - T(y), x - y) \geq 0$ (all $x, y$ in $X$);

(2) $T$ is said to be strictly monotone, if $(T(x) - T(y), x - y) > 0$ (all $x, y$ in $X, x \neq y$.)

Let $f: X \to Y$, $f$ is said to be convex, if

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y),$$

for all $x, y$ in $X$, $0 \leq \lambda \leq 1$. $f$ is said to be strictly convex, if

$$f(\lambda x + (1 - \lambda) y) < \lambda f(x) + (1 - \lambda) f(y),$$

for all $x, y$ in $X$, $x \neq y$, $0 < \lambda < 1$.

Let $X$, $Y$ be Banach spaces, the map $f: X \to Y$, $x_0 \in X$. $f$ is said to be Fréchet differentiable at $x_0 \in X$, if there exists a linear bounded operator $Df(x_0)$, such that

$$\lim_{x \to 0} \frac{\|f(x_0 + x) - f(x_0) - (Df(x_0), x)\|}{\|x\|} = 0,$$

$Df(x_0)$ is said to be the Fréchet derivative of $f$ at $x_0$, $f$ is said to be Fréchet differentiable on $X$ if $f$ is Fréchet differentiable at each point of $X$.

If for any $y \in X$, the limit

$$\lim_{x \to 0} \frac{(f(x + ty) - f(x))/t}{y}$$

exists, $f$ is said to be Gateaux differentiable. This limit is said to be the Gateaux derivative of $f$ at $x$, denoted by $(f'(x), y)$.

If $f$ is Fréchet differentiable on $X$, then the following propositions are equivalent:
(1) $f$ is convex;
(2) $f(y) - f(x) \geq (Df(x), y - x)$, all $x, y$ in $X$.

**Knaster, Kuratowski, and Mazurkiewicz Theorem (KKM Theorem [15])**. Let $E$ be a subset of the topological vector space $X$. To each $x \in E$, let a closed set $F(x)$ in $X$ be given such that $F(x)$ is compact for at least one $x \in X$. If the convex hull of every finite subset $\{x_1, ..., x_n\}$ of $E$ is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$, then $\bigcap_{x \in E} F(x) \neq \emptyset$.

In this paper, we introduce the vector complementary problem. Let $(X, C)$ and $(Y, P)$ be ordered Banach spaces, int $P \neq \emptyset$. We define the weak dual cone of $C$ with respect to $P$ as

$$C_{p}^{w+} = \{ l \in L(X, Y) \mid (l, x) \triangleq 0, \text{ all } x \in C \}.$$ $C_{p}^{w+}$ is a cone but not convex. Let $T$ be a map from $X$ to $L(X, Y)$, we define a general vector complementary problem (VCP) as

Find a vector $x \in C$, such that $(T(x), x) \triangleright 0$, $T(x) \in C_{p}^{w+}$.

**Remark.** If $Y = \mathbb{R}$, the vector complementary problem coincides with the generalized complementary problem [14]:

Find a vector $x \in C$, such that $(T(x), x) = 0$, $T(x) \in C^{*}$,

where $C^{*} = \{ l \in X^{*} \mid (l, x) \triangleright 0, \text{ all } x \in C \}$ is the polar cone of $C$.

If $X = \mathbb{R}^{n}$, $Y = \mathbb{R}$, $C = R_{n}^{+}$, then $L(X, Y) = \mathbb{R}^{n}$, $(T(x), y) \triangleq 0 \ \forall y \in C$, implies $T(x) \triangleright 0$; $0 \triangleq (T(x), x) \triangleq 0$ implies $(T(x), x) = 0$. Thus the vector complementary problem becomes finding a vector $x \in C$, such that

$$y = T(x) \triangleright 0, \quad (y, x) = 0.$$ This is the usual complementary problem (CP).

In this paper, we consider the following three kinds of vector complementary problems. Let int $P \neq \emptyset$, map $T: X \rightarrow L(X, Y)$.

**Weak Vector Complementary Problem (VCP)**. Find a vector $x \in C$, such that $(T(x), x) \triangleright 0$, $T(x) \in C_{p}^{w+}$, where $C_{p}^{w+} = \{ l \in L(X, Y) \mid (l, x) \triangleq 0, \text{ all } x \in C \}$.

**Positive Vector Complementary Problem (PVCP)**. Find a vector $x \in C$, such that $(T(x), x) \triangleright 0$, $T(x) \in C_{p}^{*+}$, where $C_{p}^{*+} = \{ l \in L(X, Y) \mid (l, x) \triangleright 0, \text{ all } x \in C \}$. 

**Positive Vector Complementary Problem (PVCP)**. Find a vector $x \in C$, such that $(T(x), x) \triangleright 0$, $T(x) \in C_{p}^{*+}$, where $C_{p}^{*+} = \{ l \in L(X, Y) \mid (l, x) \triangleright 0, \text{ all } x \in C \}$. 

Strong Vector Complementary Problem (SVCP). Find a vector \( x \in C \), such that \( (T(x), x) = 0 \), \( T(x) \in C_p^{++} \).

The sets of solutions of (VCP), (PVCP), (SVCP) are denoted with \( N \), \( N_p \), \( N_s \), respectively. Obviously, we have that \( C_p^{++} \subseteq C_p^{++} \), and \( C_p^{++} \) is a convex cone.

2. Existence of a Solution to the Vector Variational Inequality

In this section, we extend the linearization lemma [2], in the sense of the weak order, then we prove the existence theorem of the solution of the vector variational inequality by means of the extended linearization lemma and the KKM theorem.

Let \( (X, C) \) and \( (Y, P) \) be ordered Banach spaces, \( \text{int} P = \emptyset \), \( T : X \rightarrow L(X, Y) \), \( K \) be a nonempty subset in \( X \). We consider the vector variational inequality problem (VVIP)_{K}:

Find a vector \( x \in K \), such that \( (T(x), y - x) \preceq 0 \), all \( y \) in \( K \).

**Definition 2.1.** Let \( X \), \( Y \) be normed spaces and \( T \) be a map from \( X \) to \( L(X, Y) \); \( T \) is called a \( v \)-hemicontinuity if for every \( x, y \) in \( X \) the map \( t \rightarrow (T(x + ty), y) \) is continuous at \( 0^+ \).

**Lemma 2.1 (Generalized Linearization Lemma).** Let \( (X, C) \) and \( (Y, P) \) be ordered Banach spaces and \( T \) be monotone and a \( v \)-hemicontinuous map. Then the following two problems are equivalent for each convex subset \( K \) in \( X \):

(1) \( x \in K \), \( (T(x), y - x) \preceq 0 \), all \( y \) in \( K \);

(II) \( x \in K \), \( (T(y), y - x) \preceq 0 \), all \( y \) in \( K \).

**Proof.** Let \( x \) be the solution given by (I). Since \( T \) is monotone,

\[
(T(y) - T(x), y - x) \geq 0, \quad \text{all } y \text{ in } K,
\]

\[
(T(y), y - x) \geq (T(x), y - x) \preceq 0.
\]

By the Corollary in the preliminaries, we produce

\[
(T(y), y - x) \preceq 0, \quad \text{all } y \text{ in } K,
\]

which is (II).

Suppose (II) holds. For each \( y \) in \( K \), \( 0 < \lambda < 1 \),

\[
(T(\lambda y + (1 - \lambda) x), \lambda y + (1 - \lambda) x - x) \preceq 0.
\]
We divide by \( \lambda \) such that

\[
(T(x + \lambda(y-x)), y-x) \leq 0,
\]

then let \( \lambda \) tend to 0 from the right. We derive (I) by the \( \nu \)-hemicontinuity of \( T \).

**Remark.** (1) If \( Y = \mathbb{R} \), the above result is the lemma in [2].

(2) If \( x \) is an interior point of \( K \), i.e., \( x \) in \( \text{int} \ K \) (for example, \( K = X \)), (I) gives

\[
0 \in (T(x), y) \leq 0, \quad \text{all } y \text{ in } K,
\]

i.e.,

\[
T(x) \in K_p^{w+} \cap (-K_p^{w+}),
\]

where \( K_p^{w+} = \{ l \in L(X; Y) | (l, x) \leq 0, \text{ all } x \text{ in } K \} \).

(3) If \( K = \mathcal{C} \), (I) in Lemma 2.1 is the vector variational inequality discussed in [4]. The generalized linearization lemma establishes the equivalence between the vector variational inequality and problem (II) under certain conditions.

**Definition.** Let \( X \) be a Banach space, \((Y, P)\) an ordered Banach space with \( \text{int} \ P^* \neq \emptyset \), and let \( K \) be a convex unbounded set in \( X \). We shall say that the \( T: K \to L(X, Y) \) is weak coercive on \( K \) if there exists \( x_0 \in K \) and an \( s \in \text{int} P^* \) such that

\[
(s \circ T(x) - s \circ T(x_0), x - x_0)/\|x - x_0\| \to +\infty
\]

whenever \( x \in K, \|x\| \to +\infty \).

It is easy to see that if \( Y = \mathbb{R} \), \( L(X, Y) = X^* \), \( \text{int} P^* = \mathbb{R}_+ \), the weak coercive condition coincides with the coercive condition in the "scalar" variational inequality.

Let

\[
F_1(y) = \{ x \in K | (T(x), y-x) \leq 0 \}, \quad \text{all } y \text{ in } K;
\]

\[
F_1: K \to 2^K \quad \text{(the power set of } K).\]

**Theorem 2.1.** Let \( X \) be a reflexive Banach space, \((Y, P)\) a ordered Banach space with \( \text{int} \ P \neq \emptyset \) and \( \text{int} P^* \neq \emptyset \). Let \( K \) be a nonempty closed convex subset in \( X \), and let \( T: K \to L(X, Y) \) be a monotone and \( \nu \)-hemicontinuous map on \( X \). Then if
(i) \( K \) is bounded, or
(ii) \( T \) is weak coercive on \( K \)

the vector variational inequality \((\text{VVIP})_k\) is solvable.

Proof. For (i) we prove that \( F_1 \) is a KKM map on \( K \). Let 
\( \{x_1, \ldots, x_n\} \subset K, \; \sum_{i=1}^n \alpha_i = 1, \; \alpha_i \geq 0. \) Suppose that \( x = \sum_{i=1}^n \alpha_i x_i \notin \bigcup_{i=1}^n F_i(x_i) \), then

\[
(T(x), x_i - x) < 0 \quad \text{all } i,
\]

\[
(T(x), x) = \sum_{i=1}^n \alpha_i (T(x), x_i) < \sum_{i=1}^n \alpha_i (T(x), x) = (T(x), x).
\]

It is impossible, so we derive

\[
\text{Co}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n F_i(x_i),
\]

i.e., \( F_1 \) is the KKM map on \( K \).

Let

\[
F_2(y) = \{x \in K | (T(y), y - x) < 0\}, \; \text{all } y \in K.
\]

We have \( F_1(y) \subset F_2(y) \) for all \( y \) in \( K \). Indeed, let \( x \in F_1(y) \), so that \( (T(x), y - x) < 0 \). By the monotonicity of \( T \) we have

\[
(T(y), y - x) \geq (T(x), y - x) < 0,
\]

that is, \( x \in F_2(y) \). Thus \( F_2 \) is also the KKM map on \( K \). By Lemma 2.1 we have

\[
\bigcap_{y \in K} F_1(y) = \bigcap_{y \in K} F_2(y).
\]

Obviously, for each \( y \in K \), \( F_2(y) \) is the closed subset since \( T(y) \in L(X, Y) \) and \( Y \setminus \text{int} \; P \) is closed.

We now equip \( X \) with the weak topology. Then \( K \), as a closed bounded convex subset in \( X \), is weakly compact. Thus \( F_2(y) \) is the weakly compact subset in \( K \) since \( F_2(y) \subset K \) for each \( y \in K \). By the KKM theorem

\[
\bigcap_{y \in K} F_1(y) = \bigcap_{y \in K} F_2(y) \neq \emptyset.
\]

Herewith there exists an \( x' \in K \) such that

\[
(T(x'), x - x') \leq 0 \quad \text{all } x \in K.
\]
Consider case (ii). First, we prove the following conclusion: if \( s \in \text{int } P^* \) and \( x_0 \) is a solution of the variational inequality
\[
x \in K \quad (s \circ T(x), y - x) \geq 0 \quad \text{all } y \text{ in } K, \quad (VI),
\]
then \( x_0 \) is the solution of \((\text{VVIP})_K\).

Indeed, suppose that \( x_0 \) is not the solution of \((\text{VVIP})_K\), then \((T(x_0), y - x_0) < 0\) for some \( y \) in \( K \). Thus, by \( s \in \text{int } P^* \)
\[
(s \circ T(x_0), y - x_0) < 0 \quad \text{for some } y \text{ in } K,
\]
i.e., \( x_0 \) is not the solution of \((VI)_s\).

So it is sufficient to prove that there exists the solution of \((VI)_s\), where \( s \) is as in the weak coercive condition. Let \( B \), denote the closed ball of center \( 0 \) and radius \( r \) in \( X \) (for the norm in \( X \)). By the Hartman–Stampacchia theorem there exists the solution \( x_r \) of the variational inequality
\[
x \in K \cap B_r \quad (s \circ T(x), y - x) \geq 0 \quad \text{all } y \text{ in } K \cap B_r.
\]
Choose \( r \geq \|x_0\| \) with \( x_0 \) as in the weak coercive condition. Then we have \((s \circ T(x_k), x_0 - x_r) \geq 0\). Moreover,
\[
(s \circ T(x_r), x_0 - x_r) = -(s \circ T(x_r) - s \circ T(x_0), x_r - x_0)
\]
\[
\geq - (s \circ T(x_r) - s \circ T(x_0), x_r - x_0)
\]
\[
\geq \|s \circ T(x_0)\| \|x_r - x_0\|
\]
\[
\geq \|x_r - x_0\| / \|x_r - x_0\| + \|s \circ T(x_0)\| \|x_r - x_0\| + \|s \circ T(x_0)\|.
\]
Now if \( \|x_r\| = r \) for all \( r \) we may choose \( r \) big enough so that the above inequality and the weak coerciveness of \( T \) imply \((s \circ T(x_r), x_0 - x_r) < 0\), which contradicts \((s \circ T(x_r), x_0 - x_r) \geq 0\). So there exists an \( r \) such that \( \|x_r\| < r \). Now for every \( x \in K \) we can choose \( \varepsilon > 0 \) small enough such that \( x_r + \varepsilon (x - x_r) \in K \cap B_r \), and thus \((s \circ T(x_r), \varepsilon (x - x_r)) \geq 0\), for all \( x \in K \), i.e., \((s \circ T(x_r), x - x_r) \geq 0\), all \( x \) in \( K \), which proves that \( x_r \) is the solution of \((VI)_s\). So \( x_r \) is the solution of \((\text{VVIP})_K\).

Remark. If \( Y = \mathbb{R} \), so \( \text{int } P^* = \{ r \in \mathbb{R} \mid r > 0 \} \), the weak coercive condition coincides with the usual coercive condition in the variational inequality, then Theorem 2.1 coincides with the Hartman–Stampacchia theorem.

**Theorem 2.2.** Let \( X \) be a reflexive Banach space, \((Y, P)\) a ordered
Banach space with $\text{int } P \neq \emptyset$. Let $K$ be a nonempty bounded closed convex subset in $X$, and let $T : K \to L(X, Y)$ be a continuous map on $K$. Then the vector variational inequality $(\text{VVIP})_K$ is solvable.

Proof. We can establish that $F_1$ is a KKM map on $K$ as in Theorem 2.1. We next prove that for each $y \in K$, $F_1(y)$ is closed. Indeed, let $\{x_n\} \subset F_1(y)$, $x_n \to x$ (strongly) for any $y$ in $K$. We have $T(x_n) \to T(x)$ (uniformly) since $T$ is continuous on $K$.

$$
\|(T(x_n), y - x_n) - (T(x), y - x)\| \\
\leq \|(T(x_n), y - x_n) - (T(x), y - x_n)\| \\
\quad + \|(T(x), y - x_n) - (T(x), y - x)\| \\
\leq \|T(x_n) - T(x)\| \|y - x_n\| + \|T(x)\| \|x_n - x\| \to 0,
$$

so that $(T(x), y - x) \not\leq 0$. Thus we derive $F_1(y)$ is closed for all $y$ in $K$, which is bounded.

We now equip $X$ with the weak topology. Then, $K$ is the weakly compact set in $X$, so $F_1(y)$ is the weakly compact subset since $F_1(y) \subset K$ for all $y$ in $K$ and $F_1(y)$ is closed. Using the KKM theorem we have $\bigcap_{y \in K} F_1(y) \neq \emptyset$. This shows the existence of $x$ in $K$ such that

$$(T(x), y - x) \not\leq 0 \quad \text{all } y \text{ in } K.$$

The proof is complete.

3. The Existence of the Solution of the Vector Complementary Problem

J. M. Borwein [7] and R. W. Cottle [5] established the equivalence between the (generalized) linear complementary problem and the mathematical program and discussed the properties of the solution of the (generalized) linear complementary problem by doing those of the mathematical program. Similarly, we establish some equivalences between the (positive) vector complementary problem and the vector extremum problem and treat the sufficient conditions of the existence of the solution of the vector extremum problem: first, the vector extremum problem is changed into the single objective nonconvex minimization problem [1]; second, the sufficient conditions of the existence of the nonconvex minimization are given by means of the results of the nonconvex problem in Elster [10]. So the sufficient conditions of the existence of the solution of the (positive) vector complementary problem are clarified. For convenience, all problems in this section will be in finite dimension Euclidean spaces.
Let $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, $(X, C)$ and $(Y, P)$ be ordered Euclidean spaces, and $\text{int } P \neq \emptyset$. The space $L(X, Y)$ of all linear bounded operators is the set of the matrix of $n \times m$ order. Let

$$T(x) = A_{n \times m}.$$ 

Then $(T(x), y) = A'^t y \in \mathbb{R}^m$ ($A'^t$ transpose of $A$).

Let's consider the following vector complementary problem, the map $T: X \to L(X, Y)$: Find a vector $x \in C$, such that $(T(x), x) \geq 0$, $T(x) \in C^w_+$, where $p$ is the weak dual cone of $C$,

$$C^w_+ = \{ l \in L(X, Y) | (l, x) \leq 0, \text{all } x \text{ in } C \}.$$ 

A feasible set is

$$\mathcal{F} = \{ x \in X | x \in C, T(x) \in C^w_+ \}.$$ 

Let $f(x) = (T(x), x)$, all $x$ in $C$.

We consider the vector extremum problem

$$\text{WLEXT}\{ f(x) | x \in C, T(x) \in C^w_+ \}. \quad (\text{VP})$$

$x$ is called a weak lower efficient solution of $(\text{VP})$, if $f(x)$ is a weak lower efficient point of $(\text{VP})$; we will write the set of all weak lower efficient solutions (point) $E_w(H_w): f(E_w) = H_w$.

**Theorem 3.1.** If $H_w \neq 0$ and there exists $z \in H_w$ such that $z \geq 0$. Then the vector complementary problem $(\text{VCP})$ is solvable.

**Proof.** Let $z \in H_w$ and $z \geq 0$. There exists a point $x \in C$ such that $T(x) \in C^w_+$, $z = f(x) = (T(x), x) \geq 0$. So $x$ is the solution of $(\text{VCP})$.

**Theorem 3.2.** If there exist at most finite solutions of the vector complementary problem, then $(\text{VCP})$ is solvable iff $H_w \neq 0$, and there exists $z \in H_w$ such that $z \geq 0$.

**Proof.** Let $x_1$ be the solution of $(\text{VCP})$. If $x_1 \in E_w$, we are done. If $x_1 \notin E_w$, by the definition of the weak efficient solution, there exists $x_2 \in C$ such that $T(x_2) \in C^w_+$ and $(T(x_2), x_2) < (T(x_1), x_1) \geq 0$. Thus we obtain

$$(T(x_2), x_2) \geq 0,$$

hence $x_2$ is the solution of $(\text{VCP})$ and $x_1 \neq x_2$. Continuing this process, by the finiteness of the solution of $(\text{VCP})$, there exists $x_n \in C$ such that $x_n$ is the solution of $(\text{VCP})$, $x_n \in E_w$, $z = (T(x_n), x_n) \geq 0$, $z \in H_w$.

Conversely, we finish the proof from Theorem 3.1.
Remark. The equivalent relation in Theorem 3.2 is a generalization of the equivalence in J. M. Borwein [3]. If $Y = \mathbb{R}$, we obtain the results in [3].

We next consider when $H_w \neq 0$. For convenience, we consider the following vector extremum problem stronger than the vector extremum problem (VP),

$$\text{LEXT}\{f(x) | x \in \mathcal{F}\},$$

that is, the set of the efficient point of (VP) rather than the set of the weak efficient point. We denote the set of the efficient point and the efficient solution of (VP)$_0$ with $H^0$ and $E^0$, respectively.

Let $g(x) = -f(x)$, all $x$ in $\mathcal{F}$.

Then (VP)$_0$ is equivalent to

$$(VP)' \cup \text{LEXT}\{g(x) | x \in \mathcal{F}\}.$$  

Denoted the set of the efficient point and the efficient solution of (VP)' with $H'$ and $E'$. We have

$$E_w \supset E^0 = E';$$

$$H_w \supset H^0 = -H'.$$

The $p$ dual cone of $P$ is $P^* = \{l \in \mathbb{R}^m | (l, x) \leq 0, \text{ all } x \in P\}$.

Let $\text{int} P^* \neq \emptyset$. In fact by [1],

$$\text{int} P^* = \{l \in \mathbb{R}^m | (l, x) < 0, \text{ all } x \in P \setminus \{0\}\}.$$  

Let $\bar{x}$ be any element fixed in $\mathcal{F}$ and $l_0$ be in $\text{int} P^*$.

Let

$$G = \{x \in \mathcal{F} | g(x) - g(\bar{x}) \in -P\}$$

$$= \{x \in \mathcal{F} | (T(\bar{x}), x) - (T(x), x) \in -P\}.$$  

Since $\bar{x} \in G$, $G \neq 0$. Generally, $G$ is not a convex subset.

Let's consider the single objective maximization problem

$$\varphi_{l_0, \bar{x}}(0) = \sup_{x \in G} (l_0, g(x))$$

$$= \sup_{x \in G} (l_0, -(T(x), x))$$

$$= -\inf_{x \in G} (l_0, (T(x), x)).$$

Theorem 3.3 [1]. If $x'$ is the solution of (P), then $x'$ is the solution of (VP)', hence the solution of (VP).
Consider the nonconvex minimization problem

\[(P)_0 \inf_{x \in G} (l_0, (T(x), x)) = \inf_{x \in G} k(x),\]

where the objective function \(k(x) = (l_0, (T(x), x))\) is nonconvex and \(G\) is also nonconvex because \(T\) is generally not a linear operator.

Let

\[h(x) = -(T(x), x) + (T(\bar{x}), \bar{x}), \quad \text{all } x \in \mathcal{F}.\]

Therefore

\[G = \{ x \in \mathcal{F} \mid -h(x) \in P \};\]

\[D((l_0, (T(\cdot), \cdot))) = X; \quad \text{(Domain)}\]

\[D(h(\cdot)) = \mathcal{F}.\]

Thus

\[D((l_0, (T(\cdot), \cdot))) \cap D(h(\cdot)) = \mathcal{F}.\]

Let \(x \in \mathcal{F}\). Setting

\[A = \{(h(x) + p, k(x) - k(x^0) + \lambda) \mid x \in \mathcal{F}, p \in P, \lambda \in R_+\}\]

\[= \{((T(\bar{x}), \bar{x}) - (T(x), x) + p, (l_0, (T(x), x)))\]

\[-(l_0, (T(x^0), x^0)) + \lambda) \mid x \in C, T(x) \in C_p^{\infty}, p \in P, \lambda \in R_+\}\]

\[= \{((T(\bar{x}), \bar{x}) - (T(x), x) + p, (l_0, (T(x), x)) - (T(x^0), x^0)\]

\[+ \lambda) \mid x \in C, T(x) \in C_p^{\infty}, p \in P, \lambda \in R_+\}, A \subset R^n \times R^1,\]

thus the cone produced by \(A\) is

\[P(A) = \bigcup_{\lambda \in R_+} \lambda A = \{ x = \lambda x_1 \mid x_1 \in A, \lambda \in R_+ \}.\]

The convex hull of \(P(A)\) is

\[\text{Co}(P(A)) = \left\{ x = \sum_{i=1}^n \lambda_i x_i \mid x_i \in (P(A)), \lambda_i \geq 0, \right.\]

\[\left. \sum_{i=1}^n \lambda_i = 1, n \text{ positive integer} \right\}\]

\[= \left\{ x = \sum_{i=1}^n \lambda_i x_i \mid x_i \in A, \lambda_i \geq 0, \lambda_i \geq 0, \right.\]

\[\left. \sum_{i=1}^n \lambda_i = 1, n \text{ positive integer} \right\}.\]
Let \((0, \mu)\) be in \(\text{Co}(P(A))\). Thus
\[
(0, \mu) = \sum_{i=1}^{n} \lambda_i \alpha_i x_i
\]
\[
= \sum_{i=1}^{n} \lambda_i \alpha_i ((T(\bar{x}), \bar{x}) - (T(x_i), x_i) + P_i,
\]
\[
(l_0, (T(x_i), x_i) - (T(x^0), x^0)) + \lambda_i,
\]
where \(\lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, \alpha_i \geq 0, x_i \in C, T(x_i) \in C_{p^+}^u, p_i \in P, \lambda_i \geq 0, \)
\[
0 = \sum_{i=1}^{n} \lambda_i \alpha_i ((T(\bar{x}), \bar{x}) - (T(x_i), x_i) + P_i)
\]
\[
\geq \sum_{i=1}^{n} \lambda_i \alpha_i ((T(\bar{x}), \bar{x}) - (T(x_i), x_i)),
\]
\[
\mu = \sum_{i=1}^{n} \lambda_i \alpha_i ((l_0, (T(x_i), x_i) - (T(x^0), x^0)) + \lambda_i)
\]
\[
\geq \sum_{i=1}^{n} \lambda_i \alpha_i (l_0, (T(x_i), x_i) - (T(x^0), x^0)),
\]
setting \(\mu_1 = \sum_{i=1}^{n} \lambda_i \alpha_i (l_0, (T(x_i), x_i) - (T(x^0), x^0))\). Since \(l_0 \in \text{int} P^* = \{l \in R^m | (l, x) < 0, x \in P \setminus \{0\}\}\), setting
\[
\mu_2 = \left( l_0, \sum_{i=1}^{n} \lambda_i \alpha_i ((T(\bar{x}), \bar{x}) - (T(x_i), x_i)) \right),
\]
thus \(\mu_2 \geq 0\).

We next make the hypothesis \((A)\):
\[
(T(\bar{x}), \bar{x}) + (T(x^0), x^0) \geq 2(T(x_i), x_i), \quad \text{all } x_i \in C, T(x_i) \in C_{p^+}^u. \quad (A)
\]
By the hypothesis \((A)\), we produce
\[
(T(\bar{x}), \bar{x}) - (T(x_i), x_i) \geq (T(x_i), x_i) - (T(x^0), x^0),
\]
\[
\sum_{i=1}^{n} \alpha_i \lambda_i ((T(\bar{x}), \bar{x}) - (T(x_i), x_i))
\]
\[
\geq \sum_{i=1}^{n} \alpha_i \lambda_i ((T(x_i), x_i) - (T(x^0), x^0)).
\]
So we have
\[
\mu_1 \geq \mu_2,
\]
therefore
\[ \mu \geq \mu_1 \geq \mu_2 \geq 0. \]

Under the hypothesis (A), we have proved
\[ (0, \mu) \in \text{Co}(P(A)) \quad \text{implies} \quad \mu \geq 0. \quad (K) \]

**Theorem 3.4 [15].** If \( x^0 \in G \) and \( (K) \) holds, then \( x^0 \) is an optimal of \( (P)_0 \).

\[ x^0 \in G \quad \text{iff} \quad x^0 \in \mathcal{F}, \quad (T(x^0), x^0) - (T(\bar{x}), \bar{x}) \in \mathcal{P}, \]
\[ \text{iff} \quad x^0 \in \mathcal{F}, \quad (T(x^0), x^0) \geq (T(\bar{x}), \bar{x}). \quad (A_2) \]

We make the hypothesis also that
\[ (T(x^0), x^0) \geq (T(x_i), x_i), \quad \text{all} \ x_i \in C, \ T(x_i) \in C^+_p. \quad (A_1) \]

It is obvious that
\[ (A_1) \text{ and } (A_2) \text{ imply } (A). \]

Thus, we prove the following important results.

**Theorem 3.5.** Let \( X = \mathbb{R}^n, \ Y = \mathbb{R}^m, \) and \( (X, C), \ (Y, P) \) be ordered Euclidean spaces, \( \text{Int} \ P^* \neq \emptyset, \) \( \bar{x} \) fixed in \( \mathcal{F}. \) If there exists \( x^0 \) in \( \mathcal{F} \) such that
\[ (T(x^0), x^0) + (T(\bar{x}), \bar{x}) \geq 2(T(x'), x'), \quad \text{for all} \ x' \in C, \ T(x') \in C^+_p \]
then \( x^0 \in E' = E^0 \subset E_w \) and \( H_w \neq \emptyset. \)

**Corollary 3.1.** If \( (T(\bar{x}), \bar{x}) \geq (T(x), x), \) all \( x \) in \( C, \ T(x) \in C^+_p, \) then \( \bar{x} \in E_w. \)

**Proof.** In fact, \( x^0 = \bar{x} \) is satisfied with the conditions in Theorem 3.5.

**Example 3.1.** Let \( X = \mathbb{R}^2, \ Y = \mathbb{R}^3, \ C = \mathbb{R}^2_+, \ P = \mathbb{R}^3_+ . \) Then \( L(X, Y) = \{ \text{the matrix of } 2 \times 3 \text{ order} \}. \) Let
\[ x = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right), \quad y = \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right), \]
\[ T(x) = \left( \begin{array}{ccc} -x_2 & x_1 - x_2 \\ -x_1 & 0 \\ -x_2 & x_1 \end{array} \right). \]
If we want \((T(x), y) \leq 0\), all \(y\) in \(R^2_+\), then
\[
x = \begin{pmatrix} r \\ 0 \end{pmatrix}, \quad r \geq 0 \text{ real number.}
\]

So \(\mathcal{F} = \{(0) | r \geq 0\}\). Let \(\bar{x} = x^0 = (0) \in \mathcal{F}\). \(\bar{x}\) is satisfied with the conditions in Theorem 3.5. So \(\bar{x} = (0) \in E' = E^0 \subset E_w\); i.e., \(H_w \neq \emptyset\), \(z = (T(0), 0) = 0 > 0\) by Theorem 3.1, and the vector complementary problem has a solution, i.e., \(x = 0\).

We next consider the sufficient conditions of the positive vector complementary problem (PVCP). The (PVCP) considers the ordered Banach spaces \((X, C), (Y, P)\), \(\text{int} P \neq \emptyset\), and the map \(T: X \to L(X, Y)\).

Find a vector \(x \in C\) such that \((T(x), x) \geq 0\), \(T(x) \in C^+_p\), where \(P\) is the dual cone of \(C\): \(C^+_p = \{l \in L(X, Y) | (l, x) \geq 0, \text{ all } x \in C\}\). The feasible set is
\[
\mathcal{F}_0 = \{x \in X | x \in C, T(x) \in C^+_p\}.
\]

The vector extremum problem is
\[
\text{LEXT}\{f(x) | x \in C, T(x) \in C^+_p\}, \quad (VP)
\]
We will write the set of all efficient points \(H_p\) and the set of all efficient solutions \(E_p\).

Similarly, we can prove the following results.

**Theorem 3.5.** If \(H_p \neq \emptyset\) and there exists \(z \in H_p\), \(z \geq 0\), then the positive vector complementary problem is solvable.

**Theorem 3.7.** If there exists at most finite solutions of the (PVCP), then the (PVCP) is solvable iff \(H_p \neq \emptyset\), and there exists \(z \in H_p\), such that \(z \geq 0\).

Notice that we don't use properties of \(C^+_p\) when we discuss the sufficient conditions of the (VCP). But the distinction between the (VCP) and the (PVCP) is that the definition of \(C^+_p\) is different from \(C^+_s\). Hence we may give the sufficient conditions of the (PVCP) without the proof.

**Theorem 3.8.** Let \(X = R^n, Y = R^m\), and \((X, C), (Y, P)\) be ordered Euclidean spaces, \(\text{int} P^* \neq \emptyset\), \(\bar{x}\) in \(\mathcal{F}\) fixed. If there exists \(x^0 \in \mathcal{F}\) such that one of the following conditions holds,

1. \((T(\bar{x}), \bar{x}) + (T(x^0), x^0) \geq 2(T(x), x)\) all \(x \in C\), \(T(x) \in C^+_p\),
2. \((T(\bar{x}), \bar{x}) \geq (T(x), x)\), all \(x \in C\), \(T(x) \in C^+_s\),

then \(E_p \neq \emptyset\), i.e., \(H_p \neq \emptyset\).
4. The Equivalences Between the Vector Complementary Problem and the Weak Minimal Element Problem

Let \((X, C)\) and \((Y, P)\) be ordered Banach spaces and \(T: X \to L(X, Y)\) be a given map and \(f: X \to Y\) be a given operator.

Define the feasible set associated to \(T\):

\[ \mathcal{F} = \{ x \in X \mid x \in C, T(x) \in C_p^{w+} \} \]

We shall consider five problems and the first three involve the feasible set.

For a given \(l \in L(X, Y)\), the nonlinear vector extremum problem (VEP) is to find a vector \(x \in \mathcal{F}\) such that \((l, x) \in \text{WEXT}(l, \mathcal{F})\). The weak minimal element problem (WMEP) is to find a vector \(x \in B\) such that \(x \triangleright \mathcal{F}\). The vector complementary problem (VCP) is to find a vector \(x \in \mathcal{F}\) such that \((T(x), x) > 0\). The vector variational inequality problem (VVIP) is to find a vector \(x \in C\) such that 

\[ (T(x), y - x) \leq 0, \text{ all } y \in C. \]

For a given map \(f: X \to Y\), the vector unilateral minimization problem (VUMP) is to find a vector \(x \in C\) such that 

\[ (f, x) \in \text{WEXT}(f, C). \]

**Definition 4.1.** Let \((X, C)\) and \((Y, P)\) be ordered Banach spaces, and \(l: X \to Y\) be a linear operator, \(\text{int } P \neq \emptyset\), \(\text{int } C \neq 0\). \(l\) is called a weak positive linear operator if

\[ x \succ 0 \quad \text{implies} \quad (l, x) \succ 0. \]

Especially when \(Y = R\), \(x \succ 0\) implies \((l, x) \leq 0\), \(l\) is called a weak positive sublinear operator.

Jameson [13] defines \(l\) as the positive operator if \((l, C) \subset P\) (such an operator \(l\) is called monotone in [13], distinguished from the definition of monotonicity in this paper, we call it a positive operator). Generally, there is no inclusion relation between the positive linear operator and the weak positive linear operator.

**Example 4.1.** Let \(X, Y,\) and \(C\) be the positive cones in \(X\), \(\text{int } C \neq \emptyset\). Then the unit operator from \(X\) to itself is a weak positive linear operator.

**Example 4.2.** Let \(X = R^2\), \(Y = R^3\), \(C = R^2_+\), \(P = R^3_+\), \(x = (x_1, x_2) \in X\). Define the operator \(l\),

\[ (l, x) = \begin{pmatrix} x_1 - x_2 \\ 2x_2 \\ x_1 \end{pmatrix} \in R^3. \]

Then \(x \triangleright y\) implies \((l, x) \triangleright (l, y)\). So \(l\) is a weak positive linear operator.
Definition 4.2. Let $X$, $Y$ be Banach spaces and $I$ be a linear operator from $X$ to $Y$. If the image of any bounded set in $X$ is a self-sequentially compact set in $Y$, then $I$ is called completely continuous.

Remark. The definition of completely continuous in Gaung-Gui Ding [8] is that the image of any bounded set in $X$ is the sequentially compact set, a little weaker than one in Definition 4.2.

Definition 4.3. Let $(X, C)$ and $(Y, P)$ be ordered Banach spaces. The norm $\| \cdot \|$ in $X$ is called strictly monotonically increasing on $C$ if for each $y \in C$

$$X \in \{ y \} - \text{int } C \cap C \quad \text{implies} \quad \|x\| < \|y\|.$$ 

Theorem 4.1. Let $(X, C)$ and $(Y, P)$ be ordered Banach spaces, $\text{int } C \neq \emptyset$, $\text{int } P \neq \emptyset$. Suppose that

1. $T = Df$ is the Frechet derivative of the convex operator $f$ from $X$ to $Y$;
2. $I$ is a weak positive linear operator;
3. there exists $x \in F$ such that $T(x)$ is one to one and completely continuous;
4. $X$ is a topological dual space of a real normed space and the norm $\| \cdot \|$ in $X$ is strictly monotonically increasing on $C$; if the vector variational inequality (VVIP) is solvable, then (VEP), (WMEP), (VCP), (VUMP) have a solution, respectively.

Since the assertions which make up the theorem hold in various degrees of generality, we shall treat the parts in a sequence of propositions, each with its own hypotheses.

Proposition 4.1. Let $T = Df$ be the Frechet derivative of $f : X \rightarrow Y$. Then $x$ solves (VUMP) implies that $x$ solves (VVIP); if in addition $f$ is a convex map, then conversely, $x$ solves (VVIP) implies that $x$ solves (VUMP).

Proof. Let $x$ be the solution of (VUMP). Since $C$ is a convex cone, we get

$$f(x) \npreceq f(x + t(w - x)), \quad 0 < t < 1, \ w \in C,$$

$$(f(x + t(w - x))) - f(x)/t \npreceq 0.$$ 

Let $t$ tend to 0 from the right, then

$$(Df(x), w - x) \npreceq 0, \quad \text{all } w \in C,$$

which is (VVIP).
Conversely, let $x$ solve (VVIP), then

$$(T(x), w-x) \leq 0, \quad \text{all } w \in C.$$ 

Since $f$ is a convex operator,

$$f(w) - f(x) \geq (Df(x), w-x) = (T(x), w-x) \leq 0,$$

i.e., $f(w) \leq f(x)$, all $w$ in $C$, which is (VUMP).

**PROPOSITION 4.2.** $x$ solves (VVIP) implies that $x$ solves (VCP).

**Proof.** According to the assumption,

$$(T(x), y-x) \leq 0 \quad \text{all } y \in C.$$ 

Let $y = 0$, $(T(x), x) \geq 0$.

Let $y = w + x$, $w \in C$, $(T(x), w) \leq 0$, i.e., $T(x) \in C_p^w$.

Hence $x$ is a solution of (VCP).

**Remark.** Generally, the inverse relation in this proposition does not hold. If $T$ is conegative ($T$ is called conegative if $(T(x), x) < 0$ holds on $C$), then the inverse holds. In fact, $x$ solves (VCP)

$$(T(x), x) \leq 0 \geq (T(x), y), \quad \text{all } y \in C.$$ 

i.e.,

$$(T(x), y-x) \leq 0 \quad \text{all } y \in C.$$ 

which is (VVIP).

**PROPOSITION 4.3.** Let $l$ be the weak positive linear operator, then $x$ solves (WMEP) implies that $x$ solves (VEP).

**Proof.** This assertion is immediate from the definition of the weak positive linear operator.

**DEFINITION 4.4.** Let $(X, C)$ be an ordered Banach space and $A$ be a nonempty subset in $X$.

1. If for some $x \in X$ the set $A_x = (\{x\} - C) \cap A$ is nonempty, then $A_x$ is called a section of the set $A$.

2. $A$ is called weakly closed if $\{x_n\} \subset A, x \in X$, $(x^*, x_n) \to (x^*, x)$ for each $x^* \in X^*$, then $x \in A$. 


**Lemma 4.1** [12]. Let $A$ be a nonempty subset of an ordered space, $\text{int } C \neq \emptyset$ and $X$ be the topological dual space of a real normed space $(Z, \| \cdot \|_Z)$. Suppose that there exists $x \in X$ such that the section $A_x$ is weakly closed and bounded below and the norm $\| \cdot \|$ in $X$ is strictly monotonically increasing. Then the set $A$ has at least one weak lower efficient point.

**Lemma 4.2.** If the (VVIP) is solvable, then the feasible set $\mathcal{F}$ is nonempty.

**Proof.** Let $x$ be a solution of (VVIP), that is,

$$(T(x), y - x) \leq 0, \quad \text{for all } y \in C.$$ 

Let $y = z + x$, $z \in C$, then $y \in C$ and $(T(x), z) \leq 0$, for all $z$ in $C$. So $T(x) \in C_{p^+}$ and $x \in \mathcal{F}$.

**Lemma 4.3.** If the norm $\| \cdot \|$ in a ordered Banach space $X$ is strictly monotonically increasing, then the order intervals in $X$ are bounded.

**Proposition 4.4.** If the (VVIP) is solvable, and

(1) there exists $x$ in $\mathcal{F}$ such that $T(x)$ is one to one mapping and completely continuous,

(2) $X$ is the topological dual space of a real normed space $(Z, \| \cdot \|_Z)$ and the norm $\| \cdot \|$ in $X$ is strictly monotonically increasing,

then the (WMEP) has at least one solution.

**Proof.** By the assumptions and Lemma 4.2, $\mathcal{F} \neq \emptyset$. Let $x \in \mathcal{F}$ such that $T(x)$ is one to one and completely continuous, and $\{ y_n \} \subset \mathcal{F}$, $y_n \rightarrow y$ (weakly)

$$\mathcal{F}_x = (\{ x \} - C) \cap \mathcal{F} \subset (\{ x \} - C) \cap C = [0, x],$$

by Lemma 4.3. $[0, x]$ is bounded and so is $\mathcal{F}_x$. Since $T(x)$ is completely continuous, $(T(x), \mathcal{F}_x)$ is a self-sequentially compact set, since $(T(x), y_n) \subset (T(x), \mathcal{F}_x)$, there exists a subsequence $(T(x), y_{n_k})$ which converges to $z \in (T(x), \mathcal{F}_x)$. We obtain a point $y_0 \in \mathcal{F}_x$ such that

$$(T(x), y_{n_k}) \rightarrow (T(x), y_0) \quad \text{(strongly)}.$$ 

On the other hand, if $y_n \rightarrow y$ (weakly) and $T(x)$ is completely continuous, we derive

$$(T(x), y_n) \rightarrow (T(x), y) \quad \text{(strongly)}.$$
By the uniqueness of the convergence, we produce

$$(T(x), y_0) = (T(x), y).$$

Since $T(x)$ is one to one, $y_0 = y$, i.e., $y \in F_x$. $F_x$ is weakly closed, by Lemma 4.1; $F$ has a weak minimal element.

The implications and the main conditions among (VEP), (WMEP), (VCP), (VVIP), (VUMP) are presented by the diagram

\[ (VCP) \xrightarrow{T = Df} (VUMP) \xrightarrow{\text{convex}} (VVIP) \xrightarrow{\text{T cone}} (WMEP) \xrightarrow{\text{weak positive}} (VEP) \]

**Definition 4.5.** Let $(X, C)$ and $(Y, P)$ be ordered Banach spaces. The operator $T : X \to L(X, Y)$ is called positive if

$$(T(x), y) \geq 0, \quad \text{for any } x, y \in C.$$ 

Equivalently

$$(T(C), C) \subseteq P.$$ 

The operator $K : X \to Y$ is called positive if

$$(K, x) \geq 0, \quad \text{for any } x \in C.$$ 

Equivalently

$$(K, C) \subseteq P.$$ 

The following corollary is elementary from the definition.

**Corollary 4.1.** $T$ is positive if and only if $T(x)$ is positive for any $x \in C$.

Let $(X, C)$ and $(Y, P)$ be ordered Banach spaces. We next consider such an operator $T$ that is stronger than one in (VCP); that is, $T$ is supposed to be a positive operator. We obtain another form of (VCP), the positive...
vector complementary problem (PVCP): Find a vector \( x \in C \) such that \((T(x), x) \geq 0, T(x) \in C^*_p \), where \( P \) is the dual cone of \( C \),

\[ C^*_p = \{ l \in L(X, Y) \mid (l, x) \geq 0, \text{ all } x \text{ in } C \} \]

Define the feasible set related to the (PVCP) as

\[ \mathcal{F}_0 = \{ x \in X \mid x \in C, T(x) \in C^*_p \} \]

Let's consider the following problems. For a given \( l \in L(X, Y) \), the nonlinear vector extremum problem (VEP) is

Find a vector \( x \in \mathcal{F}_0 \) such that \((l, x) \in \text{WEXT}(l, \mathcal{F}_0)\). \((\text{VEP}_0)\)

The weak minimal element problem (WMEP) is

Find a vector \( x \in \mathcal{F}_0 \) such that \( x \in \mathcal{F}_0 \). \((\text{WMEP}_0)\)

The positive vector complementary problem (PVCP) is

Find a vector \( x \in \mathcal{F}_0 \) such that \( (T(x), x) \geq 0 \). \((\text{PVCP})\)

The vector variational inequality (VVJP) is

Find a vector \( x \in C \) such that \((T(x), y - x) \leq 0\), all \( y \) in \( C \). \((\text{VVIP})\)

For a given map \( f : X \rightarrow Y \), the vector unilateral minimum problem (VUMP) is

Find a vector \( x \in C \) such that \((f, x) \in \text{WLEXT}(f, C)\). \((\text{VUMP})\)

**Proposition 4.5.** Let \( T \) be strictly monotone and \( x \) be the solution of the (PVCP). Then \( x \) is the weak minimal element of \( \mathcal{F}_0 \).

**Proof.** It is elementary that \( x \in \mathcal{F}_0 \subset C \). If \( x \in bd(C) \), then \( x \) solves \((\text{WMEP})_0\). Otherwise, there exists \( x' \in \mathcal{F}_0 \) such that \( x' < x \), so

\[ x = x - x' + x' \in \text{int } C + C \subset \text{int } C \]

a contradiction. If \( x \in \text{int } C \), by the strict monotonicity of \( T \),

\[ (T(x), x - u) > (T(u), x - u), \quad \text{for each } u \in \mathcal{F}_0, u \neq x. \]
If \( x > u \), \( (T(u), x - u) \geq 0 \), we obtain

\[
(T(x), x - u) \geq 0,
\]

\[
0 \geq (T(x), x) \geq (T(x), u) + k, \quad \text{for some } k \in \text{int } P.
\]

Therefore

\[
(T(x), u) + k \not\geq 0. \quad (*)
\]

Since \( k \in \text{int } P \) and \((*)\), \( (T(x), u) \geq 0 \) does not hold. But it is contrary to the assumption of \( x \in \mathcal{F}_0 \). So \( x > u \) does not hold, that is, \( x \not> u \). Hence \( x \) solves \((\text{WMEP})_0\).

**Remark.** In the usual case, one asks: Is there any relationship between the variational inequality and the least element problem? When \( T: X \rightarrow X^* \) is strictly monotone and \( Z\)-map, R. C. Riddell [16] proved that \( x \) solves the variational inequality. This implies that \( x \) solves the least element problem. This proposition gives the vector form of this problem.

**Proposition 4.6.** \( x \) solves \((\text{PVCP})\) implies that \( x \) solves \((\text{VVIP})\).

**Proof.** By the definition of \((\text{PVCP})\), \( x \in \mathcal{C}, \ (T(x), x) \not\geq 0, \ (T(x), y) \geq 0 \), all \( y \) in \( \mathcal{C} \); i.e., for any \( y \in \mathcal{C} \),

\[
(T(x), x) \not\geq 0 \leq (T(x), y) \quad \text{and} \quad (T(x), y - x) \not\leq 0,
\]

which is \((\text{VVIP})\).

**Remark.** In Proposition 4.2 we prove that \( x \) solves \((\text{VVIP})\) implies that \( x \) solves \((\text{VCP})\). In this proposition we prove an inverse relation under the condition that \( T \) is a positive operator. On the other hand, if \( T \) is a positive operator, it is elementary that \( x \) solves \((\text{VVIP})\) implies that \( x \) solves \((\text{PVCP})\). We have shown that \((\text{VVIP})\) and \((\text{PVCP})\) are equivalent if \( T \) is a positive operator.

Similarly, we can obtain other equivalent relations. We have shown the following theorem.

**Theorem 4.2.** Let \((X, C)\) and \((Y, P)\) be ordered Banach spaces, and \( \text{int } C \neq \emptyset, \ \text{int } P \neq \emptyset \). Suppose that

1. \( T = Df \) is the Frechet derivative of the convex operator \( f: X \rightarrow Y \);
2. \( l \) is a weak positive linear operator;
3. \( T \) is strictly monotone.

Then if the \((\text{PVCP})\) is solvable, then \((\text{VEP})_0, (\text{WMEP})_0, (\text{PVCP}), (\text{VVIP}), \) and \((\text{VUMP})\) have at least a common solution.
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REFERENCES