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Controllability cost of conservative systems: resolvent condition and transmutation

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Abstract

This article concerns the exact controllability of unitary groups on Hilbert spaces with unbounded control operator. It provides a necessary and sufficient condition not involving time which blends a resolvent estimate and an observability inequality. By the transmutation of controls in some time L for the corresponding second-order conservative system, it is proved that the cost of controls in time T for the unitary group grows at most like $\exp(\alpha L^2/T)$ as T tends to 0. In the application to the cost of fast controls for the Schrödinger equation, L is the length of the longest ray of geometric optics which does not intersect the control region. This article also provides observability resolvent estimates implying fast smoothing effect controllability at low cost, and underscores that the controllability cost of a system is not changed by taking its tensor product with a conservative system.

Keywords: Controllability; Observability; Control cost; Resolvent estimates; Smoothing; Schrödinger equation; Transmutation

1. Introduction

Let H_0 and Y be Hilbert spaces with respective norms $\|\cdot\|_0$ and $\|\cdot\|$. Let A be a self-adjoint, positive and boundedly invertible unbounded operator on H_0 with domain $D(A)$. We introduce the Sobolev scale of spaces based on A. For any positive

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integer p, let H_p denote the Hilbert space $D(A^{p/2})$ with the norm $||x||_p = ||A^{p/2}x||_0$ (which is equivalent to the graph norm $||x||_0 + ||A^{p/2}x||_0$). We identify H_0 and Y with their duals. Let H_{-p} denote the dual of H_p. Since H_p is densely continuously embedded in H_0, the pivot space H_0 is densely continuously embedded in H_{-p}, and H_{-p} is the completion of H_0 with respect to the norm $||x||_{-p} = ||A^{-p/2}x||_0$. We still denote by A the restriction of A to H_p with domain H_{p+2}. It is self-adjoint with respect to the H_p scalar product. We denote by A' its dual with respect to the duality between H_p and H_{-p}, which is an extension of A to H_{-p} with domain H_{2-p}.

Let $C \in \mathscr{L}(H_2; Y)$ and let $B \in \mathscr{L}(Y, H_{-2})$ denote the dual of C (where $\mathscr{L}(X, Y)$ denotes the Banach space of continuous operators from X to Y).

The "generator" A' and the control operator B define the first- and second-order differential equations:

$$\dot{\phi}(t) - iA'\phi(t) = Bu(t), \quad \phi(0) = \phi_0 \in H_{-1}, \quad u \in L^2_{\text{loc}}(\mathbb{R}; Y), \tag{1}$$

$$\ddot{\zeta}(t) + A'\zeta(t) = Bv(t), \quad \zeta(0) = \zeta_0 \in H_0, \quad \dot{\zeta}(0) = \zeta_1 \in H_{-1}, \quad v \in L^2_{\text{loc}}(\mathbb{R}; Y), \tag{2}$$

where each dot denotes a derivative with respect to the time variable t, u and v are the input functions.

Eqs. (1) and (2) with $u = 0 = v$ describe reversible conservative systems. For example, if A is the positive Laplacian and B is a boundary control operator, then (2) is a boundary controlled scalar wave equation, (1) is a boundary controlled Schrödinger equation (Sections 2 and 10 elaborate on this example).

We assume that B is *an admissible control operator for* (2), i.e.

$$\forall T > 0, \ \forall v \in L^2(0, T; Y), \quad \int_0^T e^{it\sqrt{A'}} Bv(t)\, dt \in H_{-1}, \tag{3}$$

so that the solution $\zeta \in C^0(\mathbb{R}; H_0) \cap C^1(\mathbb{R}; H_{-1})$ of (2) is defined by the following integral formula where $S(t) = (\sqrt{A'})^{-1} \sin(t\sqrt{A'})$ and $\dot{S}(t) = \cos(t\sqrt{A'})$:

$$\zeta(t) = \dot{S}(t)\zeta_0 + S(t)\zeta_1 + \int_0^t S(t-s)Bv(s)\, ds.$$

In Section 4 the control system (2) and its dual observation system are reduced to the standard first-order setting for the theory of observation and control. We also assume that B is *an admissible control operator for* (1), i.e. $\forall T > 0$, $\exists K_{1,T} > 0$,

$$\forall u \in L^2(0, T; Y), \quad \left|\left| \int_0^T e^{itA'} Bu(t)\, dt \right|\right|_{-1}^2 \leqslant K_{1,T} \int_0^T ||u(t)||^2\, dt, \tag{4}$$

so that the solution $\phi \in C^0(\mathbb{R}; H_{-1})$ of (1) is defined by the integral formula:

$$\phi(t) = e^{itA'}\phi_0 + \int_0^t e^{i(t-s)A'} Bu(s)\, ds. \tag{5}$$

Definition 1.1. System (1) is *exactly controllable in time T* if for all ϕ_0 in H_{-1}, there is a u in $L^2(\mathbb{R}; Y)$ such that $u(t) = 0$ for $t \notin [0, T]$ and $\phi(T) = 0$. The *controllability cost* for (1) in time T is the smallest positive constant $\kappa_{1,T}$ in the following inequality for all such ϕ_0 and u:

$$\int_0^T ||u(t)||^2 \, dt \leqslant \kappa_{1,T} ||\phi_0||_{-1}^2. \tag{6}$$

System (2) is *exactly controllable in time T* if for all ζ_0 in H_0 and ζ_1 in H_{-1}, there is a v in $L^2(\mathbb{R}; Y)$ such that $v(t) = 0$ for $t \notin [0, T]$ and $\zeta(T) = \dot{\zeta}(T) = 0$. The *controllability cost* for (2) in time T is the smallest positive constant $\kappa_{2,T}$ in the following inequality for all such ζ_0, ζ_1 and v:

$$\int_0^T ||v(t)||^2 \, dt \leqslant \kappa_{2,T} (||\zeta_0||_0^2 + ||\zeta_1||_{-1}^2). \tag{7}$$

Remark 1.2. Strictly speaking, the properties above define the null-controllability of the systems but, for such reversible systems, they are equivalent to exact controllability. We refer to Section 4 for the dual notions of observability.

The main results of the paper, stated in Section 3, are consequences of the controllability of the wave-like system (2) on the controllability of the Schrödinger-like system (1). In particular, upper bounds on the controllability cost $\kappa_{1,T}$ of (1) as T tends to zero are given. Applications to the boundary controllability of the Schrödinger equation, based on the geodesic condition of Bardos et al. [2] for the controllability of the wave equation, are presented in Section 10: a new proof of the result of Lebeau [9], an extension of this result to product manifolds, and an upper bound on the cost of fast controls in the same context.

The main tool presented in this paper is the *control transmutation method* which can be seen as an adaptation to the theory of control of the kernel estimates method of Cheeger et al. [5]. It consists in explicitly constructing controls v in any time T for the Schrödinger-like system (1) in terms of controls u in time L for the corresponding wave-like system (2), i.e. $u(t, x) = \int_{\mathbb{R}} k(t, s) v(s) \, ds$, where the compactly supported kernel k which depends on T and L is some *fundamental controlled solution* on the segment $[-L, L]$ controlled at both ends. In Section 2, we recall an earlier estimate on the optimal fast control cost rate for a one dimensional system. We use it to construct the fundamental controlled solution k in Section 8 and perform the transmutation in Section 9.

This paper also contains results of independent interest on the controllability of systems defined by unitary groups. In Section 4, we recall admissibility, observability and controllability notions for such systems, their duality, and reduce the second-order system (2) to this first-order setting. In Section 5, we state a necessary and sufficient condition on the resolvent of the generator and the observation operator for exact observability. In Section 6, we state a sufficient condition on the resolvent of the generator and the observation operator for the existence of controls steering any state to a smooth state in any positive time T at a cost bounded from above by a negative power of T. In Section 7, we prove that the controllability cost of a system is not changed by taking its tensor product with a system defined by a unitary group.

In the companion paper [12], we apply the control transmutation method to the simpler case of the first-order equation $\dot{\phi}(t) + e^{i\theta}A'\phi(t) = Bu(t)$ with $|\theta| < \pi/2$ (in particular, the corresponding semigroup is holomorphic), but in the more general setting where A' generates a cosine operator function in a Banach space. The relationship between the controllability of this first order equation and the second-order equation (2) has been investigated earlier in various settings (cf. references in [12]) but only with $\theta = 0$. Eq. (1) corresponds to the case $|\theta| = \pi/2$.

2. Boundary control of the Schrödinger equation on a segment

Our estimate of the cost of fast controls for (1) builds, through the control transmutation method, on the same estimate for a one-dimensional control system of type (1), i.e. the Schrödinger equation on a segment $[0, L]$ with Dirichlet ($N = 0$) or Neumann ($N = 1$) condition at the left end controlled at the right end through a Dirichlet condition:

$$\partial_t\phi + i\partial_s^2\phi = 0 \text{ on }]0, T[\times]0, L[, \ \partial_s^N\phi\rceil_{s=0} = 0, \ \phi\rceil_{s=L} = u, \ \phi\rceil_{t=0} = \phi_0. \quad (8)$$

With the notations of Section 1, $A = -\partial_s^2$ on $H_0 = L^2(0, L)$ with $D(A) = \{f \in H^2(0, L) \mid \partial_s^N f(0) = f(L) = 0\}$, C with values in $Y = \mathbb{C}$ is defined by $Cf = \partial_s f(L)$, and $H_1 = H_N^1$ is one of the following Sobolev spaces on the segment $[0, L]$:

$$H_1^1(0, L) = \{f \in H^1(0, L) \,|\, f(L) = 0\} \quad \text{and} \quad H_0^1(0, L) = \{f \in H_1^1(0, L) \,|\, f(0) = 0\}.$$

Definition 2.1. The rate α_* is the smallest positive constant such that for all $\alpha > \alpha_*$ there exists $\gamma > 0$ such that, for all $N \in \{0, 1\}$, $L > 0$, $T \in]0, \inf(\pi, L)^2]$ the controllability cost $\kappa_{L,T}$ of system (8) satisfies: $\kappa_{L,T} \leqslant \gamma \exp(\alpha L^2/T)$.

It is well-known that the controllability of this system reduces by spectral analysis to classical results on nonharmonic Fourier series. The study of upper bounds of the controllability cost for short times was initiated by Seidman (cf. references in [13]). We recall a theorem of [13] which improves his estimate of the optimal rate α_* (computing α_* is an interesting open problem and its solution does not have to rely on the analysis of series of complex exponentials).

Theorem 2.2. *The optimal fast control cost rate for the one-dimensional system* (8) *in Definition* 2.1 *satisfies*: $1/2 \leqslant \alpha_* \leqslant 8(36/37)^2 < 8$.

3. Main results

The main result of this paper, proved in Section 9, is a generalization of Theorem 2.2 to the first-order system (1) under some condition on the second-order system (2):

Theorem 3.1. *If system* (2) *is exactly controllable for times greater than* L_*, *then system* (1) *is exactly controllable in any time* T. *Moreover, the controllability cost* $\kappa_{1,T}$ *of* (1) *satisfies the following upper bound* (*with* α_* *as in Theorem* 2.2):

$$\limsup_{T\to 0}\ T\ln\kappa_{1,T} \leqslant \alpha_* L_*^2. \tag{9}$$

Remark 3.2. The upper bound (9) means that the smallest norm of an input function u steering system (1) from an initial state ϕ_0 to zero grows at most like $\gamma\|\phi_0\|\exp(\alpha L_*^2/(2T))$ as the control time T tends to zero (with any $\alpha>\alpha_*$ and some $\gamma>0$). The falsity of the converse of Theorem 3.1 is well-known, e.g. in the setting of Section 10.

Remark 3.3. As observed in [4] (9) yields a logarithmic modulus of continuity for the minimal time function $T_{\min}: H_{-1}\to[0,+\infty)$ of (1); i.e. $T_{\min}(\phi_0)$, defined as the infimum of the times $T>0$ for which there is a u in $L^2(\mathbb{R};Y)$ such that $\int_0^T \|u(t)\|^2\,dt\leqslant 1$, $u(t)=0$ for $t\notin[0,T]$ and $\phi(T)=0$, satisfies: for all $\alpha>\alpha_*$, there is a $c>0$ such that, for all ϕ_0 and ϕ_0' in H_{-1} with $\|\phi_0-\phi_0'\|_{-1}$ small enough, $|T_{\min}(\phi_0)-T_{\min}(\phi_0')|\leqslant \alpha L_*^2/\ln(c/\|\phi_0-\phi_0'\|_{-1})$.

Replacing the notion of exact controllability by the controllability to a subspace with finite spectrum, which is enough to steer any initial state to a smooth final state, we obtain a much better upper bound for the cost of fast controls. The spectral projection on $[\lambda_1,\lambda_2]$ is denoted by $\mathbf{1}_{\lambda_1\leqslant A\leqslant\lambda_2}$.

Theorem 3.4. *If system* (2) *is exactly controllable, then* $\exists\kappa>0$, $\exists d>0$, $\forall T\in]0,1]$, $\forall\phi_0\in H_{-1}$, $\exists u\in L^2(\mathbb{R};Y)$ *such that the solution* $\phi\in C^0([0,\infty);H_{-1})$ *of* (1) *satisfies* $\mathbf{1}_{|A|\geqslant d/T^2}\phi(T)=0$ *and* $\int_0^T\|u(t)\|^2\,dt\leqslant\frac{\kappa}{T}\|\mathbf{1}_{|A|\geqslant d/T^2}\phi_0\|_{-1}^2$. *In particular, for all* $p\in\mathbb{N}$: $\phi(T)\in H_{p-1}$ *and* $\|\phi(T)\|_{p-1}\leqslant(1+\sqrt{\kappa K_{1,T}/T})(d/T^2)^{p/2}\|\phi_0\|_{-1}$.

Theorem 3.1 still holds when system (1) is replaced by its tensor product with a conservative system. If we consider A as a self-adjoint operator on H_1 and if $\tilde{A}$ is an other self-adjoint operator on an other Hilbert space $\tilde{H}$, then the operator $A\otimes I+I\otimes\tilde{A}$ defined on the algebraic tensor product $D(A)\otimes D(\tilde{A})$ is closable and its closure, denoted $A+\tilde{A}$, is a self-adjoint operator on the closure of the algebraic tensor products $H_1\otimes\tilde{H}$, denoted $H_1\bar{\otimes}\tilde{H}$ (cf. Theorem VIII.33 in [14]). The self-adjoint operator $A'+\tilde{A}$ is defined similarly. Thanks to Lemma 7.1 proved in

Section 7 (and the duality between observability and controllability), Theorem 3.1 implies:

Theorem 3.5. *Let $\tilde{A}$ be a self-adjoint operator on an other Hilbert space $\tilde{H}$. If system* (2) *is exactly controllable in times greater than L_*, then for all positive time T there is a positive constant $\tilde{\kappa}_T$ satisfying* (*with α_* as in Theorem* 2.2):

$$\forall F \in H_1 \bar{\otimes} \tilde{H}, \quad \int_0^T ||(C \otimes I)e^{it(A+\tilde{A})}F||^2 \, dt \leqslant \tilde{\kappa}_T ||F||^2 \text{ and } \limsup_{T\to 0} T \ln \tilde{\kappa}_T \leqslant \alpha_* L_*^2.$$

This is equivalent to the exact controllability in time T at cost $\tilde{\kappa}_T$ of the equation $\dot{\Phi}(t) - i(A' + \tilde{A})\Phi(t) = (B \otimes I)u(t)$ with $\Phi(0) = \Phi_0 \in H_{-1} \bar{\otimes} \tilde{H}$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}; Y \bar{\otimes} \tilde{H})$.

4. Preliminaries on conservative control systems

In this section, we review the general setting for conservative control systems: admissibility, observability and controllability notions and their duality (cf. [6,16]). We recall the characterization of solutions in the weak sense. We prove that smoother data can be controlled with smoother input functions. We reduce the second-order system (2) to this first-order setting.

Let X and Y be Hilbert spaces. Let $\mathscr{A} : D(\mathscr{A}) \to X$ be a self-adjoint operator. Equivalently, $i\mathscr{A}$ generates a strongly continuous group $(e^{it\mathscr{A}})_{t\in\mathbb{R}}$ of unitary operators on X. Let X_1 denote $D(\mathscr{A})$ with the norm $||x||_1 = ||(\mathscr{A} - \beta)x||$ for some $\beta \notin \sigma(\mathscr{A})$ ($\sigma(\mathscr{A})$ denotes the spectrum of $\mathscr{A}$, this norm is equivalent to the graph norm and X_1 is densely and continuously embedded in X) and let X_{-1} be the completion of X with respect to the norm $||\xi||_{-1} = ||(\mathscr{A} - \beta)^{-1}\xi||$. Let X' denote the dual of X with respect to the pairing $\langle \cdot, \cdot \rangle$ (linear in the first variable and conjugate-linear in the second variable). The dual of $\mathscr{A}$ is a self-adjoint operator $\mathscr{A}'$ on X'. The dual of X_1 is the space X'_{-1} which is the completion of X' with respect to the norm $||\xi||_{-1} = ||(\mathscr{A}' - \bar{\beta})^{-1}\xi||$ and the dual of X_{-1} is the space X'_1 which is $D(\mathscr{A}')$ with the norm $||x||_1 = ||(\mathscr{A}' - \bar{\beta})x||$.

Let $\mathscr{C} \in \mathscr{L}(X_1, Y)$ and let $\mathscr{B} \in \mathscr{L}(Y', X'_{-1})$ denote its dual. Note that the same theory applies to any $\mathscr{A}$-bounded operator $\mathscr{C}$ with a domain invariant by $(e^{it\mathscr{A}})_{t\geqslant 0}$ since it can be represented by an operator in $\mathscr{L}(X_1, Y)$ (cf. [16]).

We consider the dual observation and control systems with output function y and input function u:

$$\dot{x}(t) - i\mathscr{A}x(t) = 0, \quad x(0) = x_0 \in X, \quad y(t) = \mathscr{C}x(t), \tag{10}$$

$$\dot{\xi}(t) - i\mathscr{A}'\xi(t) = \mathscr{B}u(t), \quad \xi(0) = \xi_0 \in X', \quad u \in L^2_{\mathrm{loc}}(\mathbb{R}; Y'). \tag{11}$$

We make the following equivalent admissibility assumptions on the observation operator $\mathscr{C}$ and the control operator $\mathscr{B}$ (cf. [16]): $\forall T>0$, $\exists K_T>0$,

$$\forall x_0 \in D(\mathscr{A}), \quad \int_0^T ||\mathscr{C} e^{it\mathscr{A}} x_0||^2 \, dt \leqslant K_T ||x_0||^2, \tag{12}$$

$$\forall u \in L^2(\mathbb{R}; Y'), \quad \left|\left| \int_0^T e^{it\mathscr{A}'} \mathscr{B} u(t) \, dt \right|\right|^2 \leqslant K_T \int_0^T ||u(t)||^2 \, dt. \tag{13}$$

With this assumption, the output map $x_0 \mapsto y$ from $D(\mathscr{A})$ to $L^2_{\mathrm{loc}}(\mathbb{R}; Y)$ has a continuous extension to X. Eqs. (10) and (11) have unique solutions $x \in C(\mathbb{R}, X)$ and $\xi \in C(\mathbb{R}, X')$ defined by

$$x(t) = e^{it\mathscr{A}} x_0, \quad \xi(t) = e^{it\mathscr{A}'} \xi(0) + \int_0^t e^{i(t-s)\mathscr{A}} \mathscr{B} u(s) \, ds. \tag{14}$$

These so-called mild solutions are also the unique solutions in the weak sense (cf. [1]): $x(0) = x_0$, $\xi(0) = \xi_0$,

$$\forall \varphi \in D(\mathscr{A}'), \ t \mapsto \langle x(t), \varphi \rangle \in H^1(\mathbb{R}), \ \frac{d}{dt} \langle x(t), \varphi \rangle + \langle x(t), i\mathscr{A}' \varphi \rangle = 0, \tag{15}$$

$$\forall \varphi \in D(\mathscr{A}), \ t \mapsto \langle \xi(t), \varphi \rangle \in H^1(\mathbb{R}), \ \frac{d}{dt} \langle \xi(t), \varphi \rangle + \langle \xi(t), i\mathscr{A} \varphi \rangle = \langle u(t), \mathscr{C} \varphi \rangle. \tag{16}$$

The following dual notions of observability and controllability are equivalent (cf. [6]).

Definition 4.1. System (10) is *exactly observable* in time T at cost κ_T if the following observation inequality holds:

$$\forall x_0 \in X, \quad ||x_0||^2 \leqslant \kappa_T \int_0^T ||y(t)||^2 \, dt. \tag{17}$$

System (11) is *exactly controllable* in time T at cost κ_T if for all ξ_0 in X', there is a u in $L^2(\mathbb{R}; Y')$ such that $u(t) = 0$ for $t \notin [0, T]$, $\xi(T) = 0$ and:

$$\int_0^T ||u(t)||^2 \, dt \leqslant \kappa_T ||\xi_0||^2. \tag{18}$$

The *controllability cost* for (1) in time T is the smallest constant in (18), or in (17), still denoted κ_T.

In this setting, smoother data can be controlled by smoother input functions. The Sobolev space $H^1_0(0, T; Y')$ is endowed with the homogeneous norm defined by

$||u||_1^2 = \int_0^T ||\frac{d}{dt}(e^{-i\beta t}u(t))||^2\, dt$, and its dual is $H^{-1}(0,T;Y)$ with dual norm $||\cdot||_{-1}$. Integrating by parts, for all $x_0 \in X_{-1}$, $y(t) = \mathscr{C}e^{it\mathscr{A}}x_0$ satisfies

$$||y||_{-1} = \inf_{\phi \in H_0^1(0,T;Y')} \left| \int_0^T \left\langle \mathscr{C}e^{it(\mathscr{A}-\beta)}(\mathscr{A}-\beta)^{-1}x_0, \frac{d}{dt}(e^{-i\beta t}\phi(t))\, dt \right\rangle \right| \Big/ ||\phi||_1.$$

With this remark (and the usual duality argument) we obtain:

Lemma 4.2. *Let* $\beta_* = \sup_{t\in[0,T]} |e^{-i\beta t}|^{-2}$ *and* $\beta^* = \sup_{t\in[0,T]}|e^{-i\beta t}|^2$. *The admissibility assumptions* (12) *and* (13) *imply*: $\forall x_0 \in X$, $||\mathscr{C}e^{it\mathscr{A}}x_0||_{-1}^2 \leqslant \beta^* K_T ||x_0||_{-1}^2$, *and* $\forall u \in H_0^1(\mathbb{R};Y')$, $||\int_0^T e^{it\mathscr{A}'}\mathscr{B}u(t)\,dt||_1^2 \leqslant \beta^* K_T ||u||_1^2$. *Definition* 4.1 *implies*: $\forall x_0 \in X_{-1}$, $||x_0||_{-1}^2 \leqslant \beta_* \kappa_T ||y||_{-1}^2$, *and equivalently*: *for all* ξ_0 *in* X_1', *there is a* u *in* $H^1(\mathbb{R};Y')$ *such that* $u(t) = 0$ *for* $t \notin (0,T)$, $\xi(T) = 0$ *and* $||u||_1^2 \leqslant \beta_* \kappa_T ||\xi_0||_1^2$.

The first-order control system (1) and its dual observation system:

$$\dot{f}(t) - iAf(t) = 0, \quad f(0) = f_0 \in H_1, \quad y(t) = Cf(t), \tag{19}$$

fit into the present setting: $X = H_1$, $X' = H_{-1}$, $\mathscr{A}$ is A with $D(\mathscr{A}) = H_3$, $\mathscr{A}'$ is A' with $D(\mathscr{A}') = H_1$, $\beta = 0$, $\beta_* = \beta^* = 1$, $\mathscr{C}$ is the $\mathscr{A}$-bounded operator C with $D(\mathscr{C}) = H_2$ invariant by $(e^{it\mathscr{A}})_{t\geqslant 0}$. We shall now explain how the second-order control system (2) and its dual observation system:

$$\ddot{z}(t) + Az(t) = 0, \quad z(0) = z_0 \in H_1, \quad \dot{z}(0) = z_1 \in H_0, \quad y(t) = Cz(t), \tag{20}$$

also fit into the present setting.

The states $x(t)$ and $\xi(t)$ of systems (20) and (2) at time t and their state spaces X and X' are defined by

$$x(t) = (z(t), \dot{z}(t)) \in X = H_1 \times H_0, \quad \xi(t) = (\zeta(t), \dot{\zeta}(t)) \in X' = H_0 \times H_{-1}.$$

X is a Hilbert space with the "energy norm" defined by $||(z_0,z_1)||^2 = ||\sqrt{A}z_0||_0^2 + ||z_1||_0^2$, X' is a Hilbert space with norm defined by $||(\zeta_0,\zeta_1)||_\prime^2 = ||\zeta_0||_0^2 + ||\zeta_1||_{-1}^2$, and X is densely continuously embedded in X'. These spaces are dual with respect to the pairing $\langle (\zeta_0,\zeta_1),(z_0,z_1)\rangle = \langle A^{-1/2}\zeta_1, A^{1/2}z_0\rangle_0 - \langle \zeta_0, z_1\rangle_0$.

The dual second-order systems (20) and (2) rewrite as dual first-order systems (10) and (18), where $u = v$, $\mathscr{A}$ is defined on the domain $D(\mathscr{A}) = D(A) \times D(\sqrt{A})$ by $\mathscr{A}(z_0,z_1) = -i(z_1, -Az_0)$, $\mathscr{A}'$ is an extension of $\mathscr{A}$ to X' with domain X, $\beta = 0$, $\beta_* = \beta^* = 1$, X_1 is $H_2 \times H_1$ with the norm defined by $||(z_0,z_1)||^2 = ||\mathscr{A}(z_0,z_1)||^2 = ||\sqrt{A}z_1||_0^2 + ||Az_0||_0^2$, $\mathscr{C} \in \mathscr{L}(X_1, Y)$ is defined by $\mathscr{C}(z_0,z_1) = Cz_0$ and $\mathscr{B} \in \mathscr{L}(Y, X_{-1}')$ is defined by $\mathscr{B}y = (0, By)$.

The following admissibility assumptions are then equivalent: (3) for B, (13) for $\mathscr{B}$, (12) for $\mathscr{C}$, and the admissibility of C for (20), i.e. $\forall T>0$, $\exists K_{2,T}>0$,

$$\forall x_0 = (z_0, z_1) \in D(\mathscr{A}), \quad \int_0^T ||Cz(t)||^2\, dt \leqslant K_{2,T}(||z_0||_1^2 + ||z_1||_0^2). \tag{21}$$

In particular, ζ is the unique solution of (2) in $C^0(\mathbb{R}; H_0) \cap C^1(\mathbb{R}; H_{-1})$ in the following weak sense: $\zeta(0) = \zeta_0$, $\dot{\zeta}(0) = \zeta_1$, for all φ in $D(A)$,

$$t \mapsto \langle \zeta(t), \varphi \rangle_0 \in H^2(\mathbb{R}) \quad \text{and} \quad \frac{d^2}{dt^2}\langle \zeta(t), \varphi \rangle_0 + \langle \zeta(t), A\varphi \rangle_0 = \langle v(t), C\varphi \rangle_0. \tag{22}$$

The exact controllability for (2) in Definition 1.1 is the usual notion for (11) in Definition 4.1. Similarly, the usual notion of observability for (10) in Definition 4.1 yields the following definition for the exact observability in time T at cost κ_T of system (20):

$$\forall z_0 \in H_1,\ \forall z_1 \in H_0, \quad ||\sqrt{A}z_0||_0^2 + ||z_1||_0^2 \leqslant \kappa_T \int_0^T ||C\dot{z}(t)||^2\, dt. \tag{23}$$

5. Observability resolvent estimate

In the general setting for conservative control systems described in Section 4, we consider the following observability resolvent estimate:

$$\exists M>0,\ \exists m>0,\ \forall x \in D(\mathscr{A}),\ \forall \lambda \in \mathbb{R}, \quad ||x||^2 \leqslant M||(\mathscr{A} - \lambda)x||^2 + m||\mathscr{C}x||^2. \tag{24}$$

Theorem 5.1. *System* (10) *is exactly observable if and only if the observability resolvent estimate* (24) *holds. More precisely, for all* $\varepsilon > 0$ *there is a* $C_\varepsilon > 0$ *such that* (24) *implies* (17) *for all* $T > \sqrt{M(\pi^2 + \varepsilon)}$ *with* $\kappa_T = C_\varepsilon m T/(T^2 - M(\pi^2 + \varepsilon))$.

We begin by proving two lemmas which do not rely on the assumption that A is self-adjoint.

Lemma 5.2. *For all* $T > 0$, $x_0 \in D(\mathscr{A})$, $\lambda \in \mathbb{R}$:

$$\int_0^T ||\mathscr{C}e^{it\mathscr{A}}x_0||^2\, dt \leqslant 2T||\mathscr{C}x_0||^2 + T^2 \int_0^T ||\mathscr{C}e^{it\mathscr{A}}(\mathscr{A} - \lambda)x_0||^2\, dt. \tag{25}$$

In particular, if system (10) *is exactly observable then* (24) *holds.*

Proof. Set $x(t) = e^{it\mathscr{A}}x_0$, $z(t) = x(t) - e^{it\lambda}x_0$ and $f = i(\mathscr{A} - \lambda)x_0$. Since $\dot{x}(t) = i\mathscr{A}x(t) = e^{it\mathscr{A}}(i\lambda x_0 + f) = i\lambda x(t) + e^{it\mathscr{A}}f$, we have $\dot{z}(t) = i\lambda z(t) + e^{it\mathscr{A}}f$ and therefore $z(t) = \int_0^t e^{i(t-s)\lambda} e^{is\mathscr{A}} f \, ds$. We plug it in $x(t) = e^{it\lambda} x_0 + z(t)$ to estimate

$$\int_0^T ||\mathscr{C} x(t)||^2 \, dt \leqslant 2 \int_0^T |e^{it\lambda}|^2 \, dt ||\mathscr{C} x_0||^2 + 2 \int_0^T t \int_0^t |e^{i(t-s)\lambda}|^2 ||\mathscr{C} e^{is\mathscr{A}} f||^2 \, ds \, dt. \tag{26}$$

Since $\lambda \in \mathbb{R}$, we have $|e^{it\lambda}| = |e^{i(t-s)\lambda}|^2 = 1$. Now the inequality:

$$\int_0^T t \int_0^t F(s) \, ds \, dt \leqslant \int_0^T t \int_0^T F(s) \, ds \, dt = (T^2/2) \int_0^T F(s) \, ds$$

with $F(s) = ||\mathscr{C} e^{is\mathscr{A}} f||^2$ completes the proof of (25).

The second statement of Lemma 5.2 results from applying (12) and (17) to (25): it yields (24) with $M = T^2 \kappa_T K_T$ and $m = 2T\kappa_T$. □

Lemma 5.3. *If* (24) *holds then for all* $\chi \in C^1_{\mathrm{comp}}(\mathbb{R})$

$$\forall x_0 \in X, \int ||e^{it\mathscr{A}} x_0||^2 (\chi^2(t) - M\dot{\chi}^2(t)) \, dt \leqslant m \int ||\mathscr{C} e^{it\mathscr{A}} x_0||^2 \chi^2(t) \, dt. \tag{27}$$

Proof. Let $x_0 \in D(\mathscr{A})$. Set $x(t) = e^{it\mathscr{A}} x_0$, $z = \chi x$ and $f = \dot{z} - i\mathscr{A} z$. Since $\dot{x} - i\mathscr{A} x = 0$, we have $f = \dot{\chi} x$. The Fourier transform of f with respect to time is $\hat{f}(\tau) = (-i\tau - iA)\hat{z}(\tau)$. Applying (24) to $\hat{z}(\tau)$, integrating in time, and the unitarity of the Fourier transform yield

$$\int ||z(t)||^2 \, dt \leqslant M \int ||f(t)||^2 \, dt + m \int ||\mathscr{C} z(t)||^2 \, dt. \tag{28}$$

Subtracting the first term of the right-hand side and the density of $D(\mathscr{A}) \in x$ complete the proof of (27). □

Proof of Theorem 5.1. The implication is the second part of Lemma 5.2. The converse results from Lemma 5.3 and the following remark (as in [3]).

Taking $\chi(t) = \phi(t/T)$ with $\phi \in C^\infty_{\mathrm{comp}}(]0,1[)$, we have

$$\int ||\mathscr{C} e^{it\mathscr{A}} x_0||^2 \chi^2(t) \, dt \leqslant ||\phi||^2_{L^\infty} \int_0^T ||\mathscr{C} e^{it\mathscr{A}} x_0||^2 \, dt \tag{29}$$

and, since $(e^{it\mathscr{A}})_{t \geqslant 0}$ is assumed to be a unitary group:

$$\int ||e^{it\mathscr{A}} x_0||^2 (\chi^2(t) - M\dot{\chi}^2(t)) \, dt = ||x_0||^2 I_T \tag{30}$$

with

$$I_T = \int \left(\phi^2\left(\frac{t}{T}\right) - \frac{M}{T^2} \dot{\phi}^2\left(\frac{t}{T}\right) \right) dt = T \int \phi^2(t) \, dt - \frac{M}{T} \int \dot{\phi}^2(t) \, dt. \tag{31}$$

For $\phi \neq 0$ and T large enough, $I_T > 0$ so that (27) implies (23) with $\kappa_T = m||\phi||^2_{L^\infty}/I_T$. In particular, since

$$\kappa_T = mT \frac{||\phi||^2_{L^\infty}}{\int \phi^2(t)\, dt}\left(T^2 - M\frac{\int \dot{\phi}^2(t)\, dt}{\int \phi^2(t)\, dt}\right)^{-1} \quad \text{and} \quad \inf_{\phi \in C^\infty_{\mathrm{comp}}(]0,1[)} \frac{\int \dot{\phi}^2(t)\, dt}{\int \phi^2(t)\, dt} = \pi^2,$$

for all $\varepsilon > 0$, there is a $\phi_\varepsilon \in C^\infty_{\mathrm{comp}}(]0,1[)$ such that $T > M(\pi^2 + \varepsilon)$ implies $\kappa_T = C_\varepsilon mT/(T^2 - M(\pi^2+\varepsilon))$ with $C_\varepsilon = ||\phi_\varepsilon||^2_{L^\infty}/\int \phi^2_\varepsilon(t)\, dt$. $\square$

Remark 5.4. Observability resolvent estimates like (24) are introduced in [3] as sufficient conditions for exact observability. Theorem 5.1 for $\mathscr{C}$ bounded on X is proved in [17], using a more involved strategy of Liu [10] which our proof shortcuts. Liu had proved that, for a conservative first-order systems with bounded control operator, exact controllability is equivalent to exponential stability. From this equivalence and the Huang–Prüss condition for exponential stability, he deduced an observability resolvent condition for conservative second-order systems with bounded observation operator which he called a frequency domain inequality.

6. Fast smoothing controllability

In this section, as a substitute to the smoothing effect of holomorphic semigroup (used in [12]), we introduce the notion of smoothing effect controllability. More precisely, in the general setting for conservative control systems described in Section 4, we prove that controllability to a subspace with finite spectrum and a power-like bound on the cost of fast controls is implied by the following observability resolvent estimates (a stronger form of (24)):

$$\begin{aligned} &\exists m > 0, \exists \varepsilon \in]0,1[, \exists M : \mathbb{R} \to (0, +\infty), \limsup_{|\lambda| \to \infty} |\lambda|^\varepsilon M(\lambda) < \infty \ \text{ such that :} \\ &\forall x \in D(\mathscr{A}),\ \forall \lambda \in \sigma(\mathscr{A}), \quad ||x||^2 \leqslant M(\lambda)||(\mathscr{A} - \lambda)x||^2 + m||\mathscr{C}x||^2. \end{aligned} \tag{32}$$

Theorem 6.1. *Assume that* $\mathscr{A}$ *satisfies* (32). $\exists \kappa > 0$, $\exists d > 0$, $\forall T \in]0,1]$, $\forall \xi_0 \in X'$, $\exists u \in L^2(\mathbb{R}; Y)$ *such that the solution* $\xi \in C^0([0,\infty); X')$ *of*

$$\dot{\xi}(t) - i\mathscr{A}'\xi(t) = Bu(t), \quad \xi(0) = \xi_0,$$

satisfies $\mathbf{1}_{|\mathscr{A}'|^\varepsilon \geqslant d/T^2}\xi(T) = 0$ *and* $\int_0^T ||u(t)||^2\, dt \leqslant \frac{\kappa}{T}||\mathbf{1}_{|\mathscr{A}'|^\varepsilon \geqslant d/T^2}\xi_0||^2$. *In particular, for all positive* s, $\xi(T) \in D(\mathscr{A}'^s)$ *and* $||\mathscr{A}'^s\xi(T)|| \leqslant (1 + \sqrt{\kappa K_T/T})(d/T^2)^{s/\varepsilon}||\xi_0||$.

Proof. Note that (32) still holds for all $\lambda \in \mathbb{R}$. Replacing $\mathscr{C}$ by $\sqrt{m}\mathscr{C}$, we assume that $m = 1$ without loss of generality.

The second statement of the theorem results from applying the first statement and (13) to the integral formula expressing $\xi(T)$ in (14).

The first statement of the theorem is the exact controllability in time T of the projection of the data on the spectral subspace of X with spectrum greater than $(d/T^2)^{1/\varepsilon}$. By duality, it is equivalent to the following exact observability of data in this spectral subspace: $\exists\kappa>0$, $\exists d>0$, $\forall T\in]0,1]$,

$$\forall x_0\in X' \text{ such that } \mathbf{1}_{|\mathscr{A}|^\varepsilon\geqslant d/T^2}x_0=x_0,\quad ||x_0||^2\leqslant\frac{\kappa}{T}\int_0^T||\mathscr{C}e^{it\mathscr{A}}x_0||^2\,dt. \tag{33}$$

Let χ_T denote a smooth truncation defined by $\chi_T(t)=\chi(t/T)$ and $\chi\in C_{\mathrm{comp}}(]0,1[)$. Set $x(t)=e^{it\mathscr{A}}x_0$, $z=\chi x$ and $f=i\dot{z}+\mathscr{A}z$. Since $i\dot{x}+\mathscr{A}x=0$, we have $f=i\dot{\chi}_Tx$. The Fourier transform of f with respect to time is $\hat{f}(\tau)=(\mathscr{A}-\tau)\hat{z}(\tau)$. With $x=\hat{z}(\tau)$ and $\lambda=\tau$, the inequality in (32) writes: $||\hat{z}(\tau)||^2\leqslant M(\tau)||\hat{f}(\tau)||^2+||\mathscr{C}\hat{z}(\tau)||^2$. Applying this inequality for τ greater than a threshold $\mu>0$ and using the unitarity of the Fourier transform yield:

$$\begin{aligned}\int||z(t)||^2\,dt\leqslant{}&\sup_{|\tau|\geqslant\mu}\left|\frac{\tau}{\mu}\right|^\varepsilon M(\tau)\int||f(t)||^2\,dt+\int||\mathscr{C}z(t)||^2\,dt\\&+\int_{|\tau|<\mu}||\hat{z}(\tau)||^2\,dt.\end{aligned} \tag{34}$$

Setting $\mu=2(d/T^2)^{1/\varepsilon}$ we have $\mathbf{1}_{2|\mathscr{A}|\geqslant\mu}x_0=x_0$. For $|\tau|<2\mu$ we have $||(\mathscr{A}-\tau)^{-1}\mathbf{1}_{2|\mathscr{A}|\geqslant\mu}x_0||\leqslant||x_0||(2\mu-|\tau|)^{-1}$, so that using $i^{-1}\partial_te^{it(\mathscr{A}-\tau)}x_0=e^{it(\mathscr{A}-\tau)}(\mathscr{A}-\tau)x_0$, and integrating by parts yield $\hat{z}(\tau)=i^{-1}\int\dot{\chi}_T(t)e^{it(\mathscr{A}-\tau)}(\mathscr{A}-\tau)^{-1}\mathbf{1}_{2|\mathscr{A}|\geqslant\mu}x_0\,dt$ and $||\hat{z}(\tau)||\leqslant||\dot{\chi}||_{L^1}(2\mu-|\tau|)^{-1}||x_0||$. Therefore $\int_{|\tau|\leqslant\mu}||\hat{z}(\tau)||^2\,dt\leqslant\frac{2}{\mu}||\dot{\chi}||_{L^1}^2||x_0||^2$. Moreover $\int||\mathscr{C}z(t)||^2\,dt\leqslant||\chi||_{L^\infty}^2\int||\mathscr{C}x(t)||^2\,dt$, $\int||z(t)||^2\,dt=T||\chi||_{L^2}^2||x_0||^2$ and $\int||f(t)||^2\,dt=T^{-1}||\dot{\chi}||_{L^2}^2||x_0||^2$. Hence (34) implies

$$||x_0||^2\left(||\chi||_{L^2}^2-\frac{||\dot{\chi}||_{L^2}^2}{\mu^\varepsilon T^2}\sup_{|\tau|\geqslant\mu}|\tau|^\varepsilon M(\tau)-\frac{2||\dot{\chi}||_{L^1}^2}{\mu T}\right)\leqslant\frac{||\chi||_{L^\infty}^2}{T}\int||\mathscr{C}x(t)||^2\,dt.$$

Replacing μ and x by their values, there is a κ' depending on χ and ε such that

$$||x_0||^2\left(1-\frac{\kappa'}{d}\sup_{|\tau/2|^\varepsilon\geqslant d/T}|\tau|^\varepsilon M(\tau)-\frac{\kappa' T^{\frac{2}{\varepsilon}-1}}{d^{1/\varepsilon}}\right)\leqslant\frac{\kappa'}{T}\int||\mathscr{C}e^{it\mathscr{A}}x_0||^2\,dt.$$

Since $\limsup_{|\lambda|\to\infty}|\lambda|^\varepsilon M(\lambda)<\infty$, $\frac{2}{\varepsilon}-1>0$ and $T<1$, taking d large enough independently of T yields a $\kappa>0$ such that (33) holds. $\square$

Proof of Theorem 3.4. Since system (2) is exactly controllable, Theorem 5.1 implies the corresponding observability resolvent estimate (24), i.e.

$$\exists M>0,\ \exists m>0,\ \forall z_0\in H_2,\ \forall z_1\in H_1,\ \forall\lambda\in\mathbb{R},$$

$$||\sqrt{A}z_0||_0^2+||z_1||_0^2\leqslant M\Big(||\sqrt{A}(-iz_1-\lambda z_0)||_0^2+||iAz_0-\lambda z_1||_0^2\Big)+m||Cz_0||^2.$$

For $\lambda\neq 0$ and $z_1=i\lambda^{-1}Az_0$, this estimate writes:

$$\forall z_0\in H_3,\ \forall\lambda\in\mathbb{R}^*,\ ||\sqrt{A}z_0||_0^2+\frac{1}{|\lambda|^2}||Az_0||_0^2\leqslant\frac{M}{|\lambda|^2}||\sqrt{A}(A-\lambda^2)z_0||_0^2+m||Cz_0||^2.$$

In particular, since A is positive:

$$\forall z_0\in H_3,\ \forall\tau\in\sigma(A),\quad ||z_0||_1^2\leqslant\frac{M}{|\tau|}||(A-\tau)z_0||_1^2+m||Cz_0||^2.$$

Hence the observability resolvent (32) corresponding to system (1) holds with $M(\lambda)=M/|\lambda|$ and $\varepsilon=1$. Applying Theorem 6.1 with $s=p/2$ completes the proof of Theorem 3.4. □

7. Tensor product with a conservative system

Theorem 3.5 results from Theorem 3.1 and the following lemma. This trivial lemma is of independent interest. It says that the controllability cost of a system is not changed by taking its tensor product with a conservative system. It simplifies greatly and improves on previous results concerning conservative systems distributed in rectangles (or other product spaces like cylinders or parallelepipeds): boundary controllability from one whole side (cf. [8]) and semi-internal controllability (cf. [7]). Some applications are given in Section 10 and [13].

Lemma 7.1. *Let X, $\tilde{X}$ and Y be Hilbert spaces and I denote the identity operator on each of them. Let $\mathscr{A}:D(\mathscr{A})\to X$ and $\tilde{\mathscr{A}}:D(\tilde{\mathscr{A}})\to\tilde{X}$ be generators of strongly continuous semigroups of bounded operators on X and $\tilde{X}$. Let $\mathscr{C}:D(\mathscr{C})\to Y$ be a densely defined operator on X such that $e^{t\mathscr{A}}D(\mathscr{C})\subset D(\mathscr{C})$ for all $t>0$. Let $X\bar{\otimes}\tilde{X}$ and $Y\bar{\otimes}\tilde{X}$ denote the closure of the algebraic tensor products $X\otimes\tilde{X}$ and $Y\otimes\tilde{X}$ for the natural Hilbert norms. The operator $\mathscr{C}\otimes I:D(\mathscr{C})\otimes\tilde{X}\to Y\bar{\otimes}\tilde{X}$ is densely defined on $X\bar{\otimes}\tilde{X}$.*

(i) *The operator $\mathscr{A}\otimes I+I\otimes\tilde{\mathscr{A}}$ defined on the algebraic $D(\mathscr{A})\otimes D(\tilde{\mathscr{A}})$ is closable and its closure, denoted $\mathscr{A}+\tilde{\mathscr{A}}$, generates a strongly continuous semigroup of bounded operators on $X\bar{\otimes}\tilde{X}$ satisfying*

$$\forall t\geqslant 0,\ \forall(x,\tilde{x})\in D(\mathscr{C})\times\tilde{X},\ ||(\mathscr{C}\otimes I)e^{t(\mathscr{A}+\tilde{\mathscr{A}})}(x\otimes\tilde{x})||=||\mathscr{C}e^{t\mathscr{A}}x||\,||e^{t\tilde{\mathscr{A}}}\tilde{x}||.\qquad(35)$$

(ii) *If $i\tilde{\mathscr{A}}$ is self-adjoint, then for all $T \geqslant 0$:*

$$\inf_{z \in X \bar{\otimes} \tilde{X}, ||z||=1} \int_0^T ||(\mathscr{C} \otimes I) e^{t(\mathscr{A}+\tilde{\mathscr{A}})} z||^2 \, dt = \inf_{x \in X, ||x||=1} \int_0^T ||\mathscr{C} e^{t\mathscr{A}} x||^2 \, dt. \tag{36}$$

Remark 7.2. When $\mathscr{C}$ is an admissible observation operator, (36) says that the cost of observing $t \mapsto e^{t(\mathscr{A}+\tilde{\mathscr{A}})}$ through $\mathscr{C} \otimes I$ in time T is exactly the cost of observing $t \mapsto e^{t\mathscr{A}}$ through $\mathscr{C}$ in time T. The proof of part (i) of Lemma 7.1 is still valid if X, $\tilde{X}$ and Y are Banach spaces and $X \bar{\otimes} \tilde{X}$ and $Y \bar{\otimes} \tilde{X}$ are closures with respect to some uniform cross norms (cf. [15]).

Proof. Let G denote the generator of the strongly continuous semigroup $t \mapsto e^{t\mathscr{A}} \otimes e^{t\tilde{\mathscr{A}}}$ (defined since the natural Hilbert norm is a uniform cross norm, cf. [15]). Since $D(\mathscr{A}) \otimes D(\tilde{\mathscr{A}})$ is dense in $X \otimes \tilde{X}$ and invariant by $t \mapsto e^{tG}$, it is a core for G (cf. theorem X.49 in [14]). Since $\mathscr{A} \otimes I + I \otimes \tilde{\mathscr{A}} = G_{\rceil D(\mathscr{A}) \otimes D(\tilde{\mathscr{A}})}$, it is closable and $\mathscr{A} + \tilde{\mathscr{A}} = G$. Therefore $e^{t(\mathscr{A}+\tilde{\mathscr{A}})} = e^{t\mathscr{A}} \otimes e^{t\tilde{\mathscr{A}}}$ and (35) follows (by the cross norm property).

To prove point (ii), we denote the left- and right-hand sides of (36) by $\mathscr{I}_{\mathscr{A}+\tilde{\mathscr{A}}}$ and $\mathscr{I}_{\mathscr{A}}$. Taking $z = x \otimes \tilde{x}$ with $||\tilde{x}|| = 1$, $\mathscr{I}_{\mathscr{A}+\tilde{\mathscr{A}}} \leqslant \mathscr{I}_{\mathscr{A}}$ results from (35). To prove $\mathscr{I}_{\mathscr{A}+\tilde{\mathscr{A}}} \geqslant \mathscr{I}_{\mathscr{A}}$, we only consider the case in which both X and $\tilde{X}$ are infinite dimensional and separable (this simplifies the notation and the other cases are similar). Let $(e_n)_{n \in \mathbb{N}}$ and $(\tilde{e}_n)_{n \in \mathbb{N}}$ be orthonormal bases for X and $\tilde{X}$. Since $(e_n \otimes \tilde{e}_m)_{n,m \in \mathbb{N}}$ is an orthonormal base for $X \bar{\otimes} \tilde{X}$, any $z \in X \bar{\otimes} \tilde{X}$ writes

$$z = \sum_m x_m \otimes \tilde{e}_m \text{ with } x_m = \sum_n c_{n,m} e_n \text{ and } ||z||^2 = \sum_{n,m} |c_{n,m}|^2 = \sum_m ||x_m||^2.$$

Since $i\tilde{\mathscr{A}}$ is self-adjoint, $t \mapsto e^{t\tilde{\mathscr{A}}}$ is unitary for all $t \geqslant 0$ so that $(e^{t\tilde{\mathscr{A}}} \tilde{e}_n)_{n \in \mathbb{N}}$ is orthonormal. Therefore, using (35):

$$||\mathscr{C} e^{t(\mathscr{A}+\tilde{\mathscr{A}})} z||^2 = \left\| \sum_m (\mathscr{C} e^{t\mathscr{A}} x_m) \otimes (e^{t\tilde{\mathscr{A}}} \tilde{e}_m) \right\|^2 = \sum_m ||\mathscr{C} e^{t\mathscr{A}} x_m||^2.$$

By definition, $\int_0^T ||\mathscr{C} e^{t\mathscr{A}} x_m||^2 dt \geqslant \mathscr{I}_{\mathscr{A}} ||x_m||^2$. Summing up over $m \in \mathbb{N}$, we obtain

$$\int_0^T ||(\mathscr{C} \otimes I) e^{t(\mathscr{A}+\tilde{\mathscr{A}})} z||^2 \, dt = \int_0^T \sum_m ||\mathscr{C} e^{t\mathscr{A}} x_m||^2 \geqslant \mathscr{I}_{\mathscr{A}} \sum_m ||x_m||^2 = \mathscr{I}_{\mathscr{A}} ||z||^2.$$

This proves $\mathscr{I}_{\mathscr{A}+\tilde{\mathscr{A}}} \geqslant \mathscr{I}_{\mathscr{A}}$ and completes the proof of Lemma 7.1. $\square$

8. The fundamental controlled solution

In this section we use Theorem 2.2 to construct a "fundamental controlled solution" k of the Schrödinger equation on a segment controlled by Dirichlet conditions at both ends.

The following proposition shows that the upper bound for the controllability cost of the Schrödinger equation on the segment $[0, L]$ controlled at one end is the same as the controllability cost of the Schrödinger equation on the twofold segment $[-L, L]$ controlled at both ends.

Proposition 8.1. *For any* $\alpha > \alpha_*$ (*cf. Definition* 2.1), *there exists* $\gamma > 0$ *such that, for all* $L > 0$, $T \in]0, \inf(\pi/2, L)^2]$ *and* $\phi_0 \in H^{-1}(-L, L)$, *there are* g_- *and* g_+ *in* $L^2(0, T)$ *such that the solution* $\phi \in C^0([0, T]; H^{-1}(-L, L))$ *of the following Schrödinger equation on* $[-L, L]$ *controlled by* g_- *and* g_+:

$$\partial_t \phi + i\partial_s^2 \phi = 0 \quad \text{in }]0, T[\times] - L, L[, \quad \phi \rceil_{s=\pm L} = g_\pm, \quad \phi \rceil_{t=0} = \phi_0 \tag{37}$$

satisfies $\phi = 0$ *at* $t = T$ *and* $\int_0^T |g_\pm(t)|^2 \, dt \leqslant \gamma e^{\alpha L^2/T} ||\phi_0||^2_{H^{-1}(-L,L)}$.

Proof. By duality (cf. [6]), it is enough to prove the observation inequality: $\exists \gamma > 0, \ \forall \phi_0 \in H_0^1(-L, L), ||\phi_0||^2_{H^1} \leqslant \gamma e^{\alpha L^2/T} ||\partial_s e^{it\Delta} \phi_0 \rceil_{s=\pm L}||^2_{L^2(0,T)^2}$, where Δ denotes ∂_s^2 on $[-L, L]$ with Dirichlet boundary conditions. Applying Theorem 2.2 with $N = 0$ to the odd part of ϕ_0 and with $N = 1$ to the even part of ϕ_0 completes the proof of Proposition 8.1. $\square$

Expressing the solution of (37) with $\phi_0 = \delta \in H^{-1}(-L, L)$ (the Dirac distribution at the origin) in terms of $g_\pm$ by the integral formula and applying Proposition 8.1 yields the following family of null-controlled solutions (depending on $L > 0$ and $T > 0$ with a good cost estimate) which we refer to as fundamental controlled solutions.

Corollary 8.2. *For any* $\alpha > \alpha_*$ (*cf. Definition* 2.1), *there exists* $\gamma > 0$ *such that* $\forall L > 0$, $\forall T \in]0, \inf(\pi/2, L)^2]$, $\exists k \in C^0([0, T]; H^{-1}(] - L, L[))$ *satisfying*

$$\partial_t k + i\partial_s^2 k = 0 \quad \text{in } \mathscr{D}'(]0, T[\times] - L, L[), \tag{38}$$

$$k \rceil_{t=0} = \delta \quad \text{and} \quad k \rceil_{t=T} = 0, \tag{39}$$

$$\int_0^T ||k(t, \cdot)||^2_{H^{-1}(]-L,L[)} \, dt \leqslant \gamma e^{\alpha L^2/T}. \tag{40}$$

9. The transmutation of second-order controls into first-order controls

In this section we perform a transmutation of a control for the second-order system (2) into a control for the first-order system (1) (cf. (47)), then combine it with Theorem 3.4 into Theorem 3.1. The control transmutation method outlined in Section 1 proves Theorem 3.1 only for smoother data, i.e. :

Proposition 9.1. *If system* (2) *is exactly controllable in times greater than* L_* (*cf. Definition* 1.1), *then* $\exists\alpha>0$, $\exists\gamma>0$, $\forall L>L_*$, $\forall T\in]0,\inf(1,L)^2]$, $\forall\phi_0\in H_1$, $\exists u\in L^2(0,T;Y)$ *such that the solution* ϕ *of* (1) *satisfies* $\phi(T)=0$ *and* $\int_0^T ||u(t)||^2\,dt\leqslant\kappa_{2,L}\gamma e^{\alpha L^2/T}||\phi_0||_1^2$, *where* $\kappa_{2,L}$ *is defined in* (7).

Proof. Let $L>L_*$. Since (2) is exactly controllable in time L, by Lemma 4.2 (applied to the reduction of (2) to the first-order setting described after the statement of this lemma): for all $\zeta_0\in H_1$ and $\zeta_1\in H_0$, there is a $v\in H^1(\mathbb{R};Y)$ such that $v(s)=0$ for $s\notin(0,L)$, the solution ζ of (2) satisfies $\zeta(L)=\dot\zeta(L)=0$ and

$$\int ||\dot v(t)||^2\,dt\leqslant\kappa_{2,L}(||\zeta_0||_1^2+||\zeta_1||_0^2). \tag{41}$$

Let $\alpha>\alpha_*$ and $T\in]0,\inf(1,L^2)[$. Let $\gamma>0$ and $k\in C^0([0,T];H^{-1}(]-L,L[))$ be the corresponding constant and fundamental controlled solution given by Corollary 8.2. We define $\underline{k}\in C^0([0,\infty);H^{-1}(\mathbb{R}))$ as the extension of k by zero, i.e. $\underline{k}(t,s)=\bar k(t,s)$ on $[0,T]\times]-L,L[$ and $\underline{k}$ is zero everywhere else. It inherits from k the following properties:

$$\partial_t\underline{k}+i\partial_s^2\underline{k}=0\quad\text{in }\mathscr{D}'(]0,T[\times]-L,L[), \tag{42}$$

$$\underline{k}_{\rceil t=0}=\delta\quad\text{and}\quad\underline{k}_{\rceil t=T}=0, \tag{43}$$

$$\int_0^T ||\underline{k}(t,\cdot)||^2_{H^{-1}(\mathbb{R})}\,dt\leqslant\gamma e^{\alpha L^2/T}. \tag{44}$$

Let $\phi_0\in H_1$ be an initial data for (1). Let ζ and v be the corresponding solution and control function for (2) with data $\zeta_0=\phi_0$ and $\zeta_1=0$. We define $\underline{\zeta}\in C^0(\mathbb{R};H_1)\cap C^1(\mathbb{R};H_0)$ and $\underline{v}\in H^1(\mathbb{R};Y)$ as the extensions of ζ and v by reflection with respect to $s=0$, i.e. $\underline{\zeta}(s)=\zeta(s)=\underline{\zeta}(-s)$ and $\underline{v}(s)=v(s)=\underline{v}(-s)$ for $s\geqslant0$. Since $\zeta_1=\zeta(L)=\dot\zeta(L)=0$, $\underline{\zeta}$ is the unique solution in $C^0(\mathbb{R};H_0)\cap C^1(\mathbb{R};H_{-1})$ of

$$\ddot{\underline{\zeta}}(t)+A'\underline{\zeta}(t)=B\underline{v}(t),\quad\underline{\zeta}(0)=\phi_0,\ \dot{\underline{\zeta}}(0)=0,$$

in particular in the following weak sense (as in (22)): for all φ in H_2,

$$s \mapsto \langle \underline{\zeta}(s), \varphi \rangle_0 \in H^2(\mathbb{R}) \text{ and } \frac{d^2}{ds^2}\langle \underline{\zeta}(s), \varphi \rangle_0 + \langle \underline{\zeta}(s), A\varphi \rangle_0 = \langle \underline{v}(s), C\varphi \rangle_0. \quad (45)$$

Eq. (41) implies the following cost estimate for $\underline{v}$:

$$\int \|\dot{\underline{v}}(s)\|^2 \, ds \leqslant 2 \int \|v(s)\|^2 \, ds \leqslant 2\kappa_{2,L} \|\phi_0\|_1^2. \quad (46)$$

The main idea of our proof is to use $\underline{k}$ as a kernel to transmute $\underline{\zeta}$ and $\underline{v}$ into a solution ϕ and a control u for (1). The transmutation formulas

$$\phi(t) = \int \underline{k}(t,s)\underline{\zeta}(s) \, ds \quad \text{and} \quad \forall t > 0, \; u(t) = -i \int \underline{k}(t,s)\underline{v}(s) \, ds, \quad (47)$$

define $\phi \in C^0([0,\infty); H_0)$ and $u \in L^2([0,\infty); Y)$ since $\underline{k} \in C^0([0,\infty); H^{-1}(\mathbb{R}))$, $\underline{\zeta} \in H^1(\mathbb{R}; H_0)$ and $\underline{v} \in H^1(\mathbb{R}; Y)$. Property (43) of $\underline{k}$ implies $\phi(0) = \phi_0$ and $\phi(T) = 0$. Since $\underline{\zeta}(s) = \dot{\underline{\zeta}}(s) = 0$ for $|s| = L$, Eqs. (45) and (42) imply, by integrating by parts, for all φ in H_3:

$$t \mapsto \langle \phi(t), \varphi \rangle_0 \in H^1(0,\infty), \; \frac{d}{dt}\langle \phi(t), \varphi \rangle_0 + \langle \phi(t), iA\varphi \rangle_0 = \langle u(t), C\varphi \rangle_0. \quad (48)$$

This is Eq. (16) corresponding to (1), i.e. with the settings described after (19). Therefore ϕ and u satisfy (5).

Since $\int_0^T \|u(t)\|^2 \, dt \leqslant \int_0^T \|\underline{k}(t,\cdot)\|_{H^{-1}(\mathbb{R})}^2 \, dt \int \|\dot{\underline{v}}(s)\|^2 \, ds$, Eqs. (44) and (46) imply the cost estimate which completes the proof of Proposition 9.1. $\square$

Proof of Theorem 3.1. Let $\alpha > \alpha_*$, $L > L_*$ and $\varepsilon \in]0,1[$.

According to Theorem 3.4 with $p = 2$: $\exists \kappa > 0$, $\exists d > 0$, $\forall T \in]0,1]$, $\forall \phi_0 \in H_{-1}$, $\exists u_1 \in L^2([0,\varepsilon T]; Y)$ such that the solution $\phi \in C^0([0,\varepsilon T]; H_{-1})$ of (1) with $u = u_1$ on $[0,\varepsilon T]$ satisfies $\phi(T) \in H_1$, $\|\phi(T)\|_1 \leqslant \|\phi_0\|_{-1}(1 + \sqrt{\kappa K_{1,\varepsilon T}/\varepsilon T})d/(\varepsilon T)^2$, and $\int_0^T \|u_1(t)\|^2 \, dt \leqslant \frac{\kappa}{\varepsilon T}\|\phi_0\|_{-1}^2$. Therefore, according to Proposition 9.1, $\exists \alpha > 0$, $\exists \gamma > 0$, $\forall T \in]0, \inf(1,L)^2]$, $\forall \phi_0 \in H_{-1}$, $\exists u_2 \in L^2([\varepsilon T, T]; Y)$ such that the solution $\phi \in C^0([0,T]; H_{-1})$ of (1) with $u = u_1$ on $[0,\varepsilon T]$ and $u = u_2$ on $[\varepsilon T, T]$ satisfies $\phi(T) = 0$ and $\int_{\varepsilon T}^T \|u_2(t)\|^2 \, dt \leqslant \kappa_{2,L}\gamma e^{\alpha L^2/(T-\varepsilon T)}\|\phi_0\|_2^2$. Since $\int_0^T \|u(t)\|^2 \, dt = \int_0^{\varepsilon T} \|u_1(t)\|^2 \, dt + \int_{\varepsilon T}^T \|u_2(t)\|^2 \, dt$, the controllability cost $\kappa_{1,T}$ in Definition 1.1 satisfies for all $T \in]0, \inf(1,L)^2]$:

$$\kappa_{1,T} \leqslant \frac{\kappa}{\varepsilon T} + \left(1 + \sqrt{\kappa K_{1,1}/\varepsilon T}\right)^2 \frac{d^2}{(\varepsilon T)^4} \kappa_{2,L}\gamma \exp \frac{\alpha L^2}{(1-\varepsilon)T}.$$

Therefore $\limsup_{T \to 0} T \ln \kappa_{1,T} \leqslant \alpha L^2/(1-\varepsilon)$. Letting α, L and ε tend, respectively, to α_*, L_* and 0 completes the proof of (9). $\square$

10. Geometric bounds on the cost of fast boundary controls for Schrödinger equations

When the second-order equation (2) has a finite propagation speed and is controllable, the control transmutation method yields geometric upper bounds on the cost of fast controls for the first-order equation (1). This was illustrated in [13] on the internal controllability of Schrödinger equations on Riemannian manifolds which have the wave equation as corresponding second-order equation. Similar lower bounds proved in [13] (without assuming the controllability of the wave equation) imply that the upper bounds are optimal with respect to time dependence. In this section, we illustrate the control transmutation method on the analogous boundary control problem for Schrödinger equations.

Let (M, g) be a smooth connected compact n-dimensional Riemannian manifold with metric g and smooth boundary ∂M. When $\partial M \neq \emptyset$, M denotes the interior and $\bar{M} = M \cup \partial M$. Let Δ denote the (negative) Laplacian on (M, g) and ∂_ν denote the exterior Neumann vector field on ∂M. The characteristic function of a set S is denoted by χ_S.

Let $H_0 = L^2(M)$. Let A be defined by $Af = -\Delta f$ on $D(A) = H^2(M) \cap H_0^1(M)$. Let C be defined from $D(A)$ to $Y = L^2(\partial M)$ by $Cf = \partial_\nu f_{\rceil \Gamma}$ where Γ is an open subset of ∂M. With this setting, (1) is a Schrödinger equation, (2) is a scalar wave equation, and these equations are controlled by the Dirichlet boundary condition on Γ. In particular (2) writes:

$$\partial_t^2 \zeta - \Delta \zeta = 0 \text{ on } \mathbb{R}_t \times M, \quad \zeta = \chi_\Gamma v \text{ on } \mathbb{R}_t \times \partial M,$$
$$\zeta(0) = \zeta_0 \in L^2(M), \ \dot{\zeta}(0) = \zeta_1 \in H^{-1}(M), \ v \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\partial M)). \tag{49}$$

It is well known that C is an admissible observation operator for the wave equation (20) and the Schrödinger equation (19) (cf. e.g. [2,9], Corollary 3.9). To ensure the exact controllability of the wave equation we use the geometric optics condition of Bardos–Lebeau–Rauch (specifically Example 1 after Corollary 4.10 in [2]):

There is a positive constant L_Γ such that every generalized geodesic
of length greater than L_Γ passes through Γ at a non-diffractive point. (50)

Generalized geodesics are the rays of geometrical optics (we refer to [11] for a presentation of this condition with a discussion of its significance). We make the additional assumption that they can be uniquely continued at the boundary ∂M. As in [2], to ensure this, we may assume either that ∂M has no contacts of infinite order with its tangents (e.g. $\partial M = \emptyset$), or that g and ∂M are real analytic. For instance, we recall that (50) holds when Γ contains a closed hemisphere of a Euclidean ball M of diameter $L_\Gamma/2$, or when $\Gamma = \partial M$ and M is a strictly convex bounded Euclidean set which does not contain any segment of length L_Γ.

Theorem 10.1 (Bardos et al. [2]). *If* (50) *holds then the wave equation* (49) *is exactly controllable in any time greater than* L_Γ.

Thanks to this theorem, Theorem 3.5 implies:

Theorem 10.2. *Let* $\tilde{M}$ *be a smooth complete* $\tilde{n}$*-dimensional Riemannian manifold and* $\tilde{\Delta}$ *denote the Laplacian on* $\tilde{M}$ *with the Dirichlet boundary condition. Let* γ *denote the subset* $\Gamma \times \tilde{M}$ *of* $\partial(M \times \tilde{M})$. *If* (50) *holds then the Schrödinger equation*:

$$i\partial_t \phi - (\Delta + \tilde{\Delta})\phi = 0 \text{ on } \mathbb{R}_t \times M \times \tilde{M}, \quad \phi = \chi_\gamma u \text{ on } \mathbb{R}_t \times \partial(M \times \tilde{M}),$$

$$\phi(0) = \phi_0 \in L^2(\tilde{M}; H^{-1}(M)), \; u \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\partial(M \times \tilde{M}))),$$

is exactly controllable in any time T *at a cost* $\tilde{\kappa}_T$ *which satisfies the following upper bound* (*with* α_* *as in Theorem* 2.2): $\limsup_{T\to 0} T \ln \tilde{\kappa}_T \leqslant \alpha_* L_\Gamma^2$.

Remark 10.3. For $\tilde{M} = \emptyset$, the controllability was proved in [9]. As in [9], this results extends to the plate equation. The boundary controllability of a rectangular plate from one side was proved in [8] (Theorem 2.2). When M is a segment and $\tilde{M}$ is a line, Theorem 10.2 extends this result to an infinite strip.

Remark 10.4. In particular, Theorem 10.2 shows that the geometric optics condition is not necessary for the controllability cost of the Schrödinger equation to grow at most like $\exp(C/T)$ as T tends to 0. Indeed, any geodesic of $\tilde{M}$ yields a geodesic of $M \times \tilde{M}$ in a slab $\{x\} \times \tilde{M}$ with $x \in M$, and this geodesic does not pass through the control region γ since the slab does not intersect the boundary set $\partial M \times \tilde{M}$.

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