Topology and its Applications 26 (1987) 287-291
North-Holland

SHORT PATHS IN HOMOGENEOUS CONTINUA

David P. BELLAMY*
Department of Mathematics, University of Delaware, Newark, DE 19716, USA

Received 10 January 1986

Homogeneous arcwise connected metric continua are shown to, in effect, be arcwise connected by arcs of bounded length. Specifically, for any positive ε, there is a natural number n such that every two points can be joined by an arc which is the union of n subarcs of diameter less than ε.

AMS (MOS)Subj. Class.: 54C05, 54F15, 54F20, 54H15

homogeneous continuum | arcwise connected | uniform path connectedness

K. Kuperberg has asked whether every homogeneous arcwise connected continuum is locally connected. This problem remains unsolved, but it will be shown here that homogeneous arcwise connected continua have a weak form of uniform path connectedness, indicating that one need not use arcs which are too wildly oscillatory to connect pairs of points in such a continuum. The Warsaw circle, for example, does not have this property, while every locally connected continuum and every contractible continuum does.

A *continuum* here means a compact, connected metric space. A topological space is *homogeneous* provided that for every $x, y \in X$, there is a homeomorphism $h: X \to X$ such that $h(x) = y$. I is the interval $[0, 1]$. From now on, the letter X denotes a continuum. If $\omega: I \to X$ is a path, n is a positive integer, and $\varepsilon > 0$, we say that ω has *ε-length* less than or equal to n if and only if there is an increasing sequence $\langle x_i \rangle_{i=0}^{n}$ of $n+1$ points in I such that $x_0 = 0$, $x_n = 1$, and for every $i \geq 1$, the diameter of $\omega[x_{i-1}, x_i]$ is less than ε. The sequence $\langle x_i \rangle_{i=0}^{n}$ is said to *realize* an ε-length of n for ω. (Thus, the ε-length of ω is the smallest n such that there exists a sequence realizing an ε-length of n for ω.)

A path $\omega: I \to X$ is a path *from* x if $\omega(0) = x$ and is a path to x if $\omega(1) = x$. A path *joining* x and y is a path from either to the other. X^I is the family of all paths in X with the compact open topology, and if $p \in X$, $X^I(p)$ denotes the subspace of paths from p.

Kuperberg [4] has defined a continuum X to be *uniformly path connected*, or u.p.c., provided there exists a family P of paths in X such that the following hold:

(i) For every $x, y \in X$ there is a path $\omega \in P$ joining x and y.

* The author acknowledges the support of National Science Foundation grant number MCS8302176.

(ii) For every $\varepsilon > 0$ there exists a natural number n such that every $\omega \in P$ has ε-length $\leq n$.

In the same paper, Kuperberg showed that a continuum X is a continuous image of the cone over the Cantor set if and only if X is u.p.c.

The property used there is related to, but perhaps weaker than, uniform path connectedness. A continuum X is *connected by uniformly short paths* (c.u.s.p.) if and only if for every $\varepsilon > 0$ there is a natural number n such that for every $x, y \in X$, there is a path of ε-length less than or equal to n joining x to y.

It is easy to see that every u.p.c. continuum is c.u.s.p. The converse of this statement is open.

A set $A \subseteq X$ is an *analytic set* provided A is a continuous image of the space of irrationals, or equivalently a continuous image of a Borel subset of some complete separable metric space. A has the *Baire property* [5, p. 87] provided that there exists an open set O such that $(A - O) \cup (O - A)$ is first category. The largest such O is the *quasi-interior* of A. The fact which we shall need in this paper is that an analytic set has the Baire property, and thus is second category if and only if it has nonempty quasi-interior.

Lemma 1. *If $\varepsilon > 0$, $\omega \in X^I$, and ω has ε-length $\leq n$, then for some $\delta < \varepsilon$, ω has δ-length $\leq n$ as well.*

Proof. Let $\langle x_i \rangle_{i=0}^n$ be a sequence realizing an ε-length of n for ω. Then if r is the maximum of the diameters of $\omega[x_{i-1}, x_i]$, it follows that $r < \varepsilon$. Consequently, any δ with $r < \delta < \varepsilon$ satisfies the requirement. □

Lemma 2. *If $\delta > \varepsilon > 0$ and $w \in X^I$ has ε-length less than or equal to n, then ω also has δ-length $\leq n$.*

Proof. This is clear. □

If ω is a path in X from x to y and σ a path from y to z, the *juxtaposition path* $\omega^* \sigma$ is the path from x to z defined by

$$\begin{aligned} \omega^*\sigma(t) &= \omega(2t), & t &\in [0, \tfrac{1}{2}] \\ &= \sigma(2t-1), & t &\in (\tfrac{1}{2}, 1]. \end{aligned}$$

Lemma 3. *If ω is a path from x to y and σ a path from y to z; and ω has ε-length $\leq n$ while σ has ε-length $\leq m$, then $\omega^*\sigma$ has ε-length $\leq n + m$.*

Proof. This is also clear. □

Let X be a continuum and $p \in X$. $W(\varepsilon, n)$ denotes the set $\{\omega \in X^I \mid \omega$ *has* ε-length $\leq n\}$ and $W(\varepsilon, n; p) = W(\varepsilon, n) \cap X^I(p)$. The map $E : X^I(p) \to X$ is defined by $E(\omega) = \omega(1)$.

Lemma 4. *For any $\varepsilon > 0$ and any natural number n, $W(\varepsilon, n; p) \subseteq W(\varepsilon, n+1; p)$.*

Proof. This is clear. $\square$

Lemma 5. *For any $\varepsilon > 0$ and any natural number n, $W(\varepsilon, n)$ is open in X^I.*

Proof. Let $\omega \in W(\varepsilon, n)$. Let $\langle x_i \rangle_{i=0}^n$ be a sequence realizing this ε-length for ω. For each i, let U_i be an open set in X of diameter less that ε such that $\omega([x_{i-1}, x_i]) \subseteq U_i$. Then $\bigcap_{i=1}^n \{\sigma \in X^I \mid \sigma([x_{i-1}, x_i]) \subseteq U_i\}$ is a basic open set in X^I containing ω and contained in $W(\varepsilon, n)$, completing the proof. $\square$

The proofs of the next four lemmas are easy.

Lemma 6. $\bigcup_{n=1}^{\infty} W(\varepsilon, n) = X^I$ *and* $\bigcup_{n=1}^{\infty} W(\varepsilon, n; p) = X^I(p)$.

Lemma 7. *If X is arcwise connected, then $\bigcup_{n=1}^{\infty} E(W(\varepsilon, n; p)) = X$.*

Lemma 8. *Each $E(W(\varepsilon, n; p))$ is an analytic set.*

Lemma 9. *If ω is a path in X of ε-length $\leqslant n$ and h is a homeomorphism of X within δ of the identity, then $(h \circ \omega)$ is a path in X of $(\varepsilon + 2\delta)$-length $\leqslant n$.*

The Effros Theorem

By now, the Effros Theorem is a standard (and profoundly powerful) tool for the study of homogeneous continua. It is stated in the form in which it is needed here, not in its full generality.

Effros' Theorem. *Let X be a homogeneous continuum, H the group of homeomorphisms of X with the compact-open topology, and $x \in X$. Then the evaluation map $e_x : H \to X$ defined by $e_x(h) = h(x)$ is continuous and open.*

Proof. This is a special case of Theorem 2.1(1) of [2, p. 39] when there is only one orbit.

Let X be a homogeneous continuum and H its homeomorphism group. If $A \subseteq X$ and $U \subseteq H$, then define $UA = \{h(a) \mid h \in U \text{ and } a \in A\}$.

What we shall actually use is the:

Corollary to Effros' Theorem. *If X is a homogeneous continuum, H its homeomorphism group, and U is open in H, then for any $A \subseteq X$, UA is open in X.*

Proof.

$$UA = \bigcup_{a \in A} U\{a\} = \bigcup_{a \in A} e_a(U).$$

The main result

We can now state and prove the following.

Theorem. *Every homogeneous, arcwise connected continuum is connected by uniformly short paths.*

Proof. This proof has several parts, which are going to be listed as additional Lemmas. Throughout, X is a homogeneous, arcwise connected continuum; p a fixed point in X, $\varepsilon > 0$, and $\langle \varepsilon_k \rangle_{k=1}^{\infty}$ is a strictly increasing sequence of positive reals converging to ε. (e.g., $\varepsilon_k = \varepsilon(1 - 1/(k+1))$). H is the homeomorphism group of X with supremum metric.

Lemma 10. *There exists a natural number n such that $E(W(\varepsilon, n; p))$ is second category.*

Proof. X is not a countable union of first category sets. Apply Lemma 7. □

Lemma 11. *If $E(W(\varepsilon, n; p))$ is second category then there exists m such that, for every $k \geq m$, $E(W(\varepsilon_k, n; p))$ is second category.*

Proof. $E(W(\varepsilon, n; p) = \bigcup_{j=1}^{\infty} E(W(\varepsilon_k, n; p))$. The Baire Category Theorem again yields that for some m, $E(W(\varepsilon_m, n; p))$ is second category. The rest follows since $E(W(\varepsilon_k, n; p))$, $k = 1, 2, \ldots$ is an increasing sequence of sets. □

Lemma 12. *Let n be a natural number such that $E(W(\varepsilon, n; p))$ is second category in X. Then $E(W(\varepsilon, n; p))$ is contained in the interior of $E(W(\varepsilon, 3n; p))$.*

Proof. Let $x \in E(W(\varepsilon, n; p))$. We shall prove that x is an interior point of $E(W(\varepsilon, 3n; p))$. Let ω be a path from p to x of ε-length $\leq n$. Then, for some k, ω has ε_k-length $\leq n$ and $E(W(\varepsilon_k, n; p))$ has nonempty quasi-interior. Let y be a point in the quasi-interior of $E(W(\varepsilon_k, n; p))$, and choose $\delta > 0$ small enough to satisfy the following two conditions:

(i) The δ-neighborhood of y is contained, except for a first category subset, in $E(W(\varepsilon_k, n; p))$.

(ii) $\delta < \frac{1}{2}(\varepsilon - \varepsilon_k)$.

Let U be the δ-neighborhood of the identity in H. Then $UE(W(\varepsilon_k, n; p))$ is an open set, by the corollary to the Effros Theorem. It remains to be shown that

$$UE(W(\varepsilon_k, n; p)) \subseteq E(W(\varepsilon, 3n; p)).$$

Let $q \in UE(W(\varepsilon_k, n; p))$. Then there is an $h \in U$ such that $q \in h(E(W(\varepsilon_k, n; p)))$, and so $h^{-1}(q) \in E(W(\varepsilon_k, n; p))$. Further, since h is within δ of the identity, by condition (i) on δ it follows that $E(W(\varepsilon_k, n; p)) \cap h(E(W(\varepsilon_k, n; p))) \neq \emptyset$ (see [1, Lemma 2, p. 392]). Let z belong to this intersection. Then $z \in E(W(\varepsilon_k, n; p))$ and $h^{-1}(z) \in E(W)(\varepsilon_k, n; p))$. Let ω, σ, and τ be paths of ε_k-length $\leq n$ from p to

z, from $h^{-1}(z)$ to p, and from p to $h^{-1}(q)$, respectively. Then ω, $h \circ \sigma$, and $h \circ \tau$ are paths from p to z, from z to $h(p)$, and from $n(p)$ to q, respectively. Since $\delta < \frac{1}{2}(\varepsilon - \varepsilon_k)$, by Lemma 9 it follows that each of these has ε-length $\leqslant n$. By Lemma 3, $(\omega^* h \circ \sigma)^* h \circ \tau$ is a path from p to q of ε-length$\leqslant 3n$, so that $q \in E(W(\varepsilon, 3n; p))$, and the proof of Lemma 12 is done.

Returning to the proof of the Theorem, choose m such that $E(W(\varepsilon, m; p))$ has nonempty quasi-interior. Then

$$X = \bigcup_{n=1}^{\infty} E(W(\varepsilon, n; p)) = \bigcup_{k=1}^{\infty} E(W(\varepsilon, 3^{k-1} \cdot m; p))$$

$$\subseteq \bigcup_{k=1}^{\infty} \operatorname{Int}(E(W(\varepsilon, 3^k \cdot m; p))),$$

this latter being an ascending union of open sets. By compactness, there exists k such that $X \subseteq \operatorname{Int}(E(W(\varepsilon, 3^k \cdot m; p)))$; hence $E(W(\varepsilon, 3^k \cdot m; p)) = X$. Consequently, for any y in X, there is a path joining p and y of ε-length at most $3^k \cdot m$. Juxtaposing two of these, any two points of X can be joined by a path of ε-length at most $2 \cdot 3^k \cdot m$. Thus, X is c.u.s.p., as required.

Questions. (1) Is every homogeneous arcwise connected continuum uniformly path connected?

(2) More generally, are connectedness by uniformly short paths and uniform path connectedness equivalent for all continua?

References

[1] D.P. Bellamy and L. Lum, The cyclic connectivity of homogeneous arcwise connected continua, Trans. Amer. Math. Soc. 266 (1981) 389-396.

[2] E.G. Effros, Transformation groups and C^* algebras, Ann. of Math 81 (2) (1965) 38-55.

[3] K. Kuperberg, A locally connected, micro-homogeneous, non-homogeneous continuum, Bull. Acad. Polon. Sci. Ser. Sci. Math. 28 (1980) 627-630.

[4] W. Kuperberg, Uniformly pathwise connected continua, Studies in Topology, Proc. of the Charlotte Topology Conference, 1974 (Academic Press, New York, 1975) 315-324.

[5] K. Kuratowski, Topology I (PWN, Warsaw; Academic Press, New York, 1966).