Consistency degrees of theories and methods of graded reasoning in $n$-valued $R_0$-logic (NM-logic)

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Abstract

In the present paper the idea of Wang [G.J. Wang, Theory of truth degrees of formulas in Łukasiewicz $n$-valued propositional logic and a limit theorem, Sci. China Inform. Sci. E 35(6) (2005) 561–569 (in Chinese)] is firstly extended to the $n$-valued $R_0$-logic $L_n$ and the concept of truth degrees of formulas in $L_n$ is proposed. A limit theorem saying that the truth function $\tau_n$ induced by truth degrees converges to the integrated truth function $\tau$ when $n$ converges to infinity is obtained. This theorem builds a bridge between discrete valued $R_0$-logic and continuous valued $R_0$-logic. Secondly, based on deduction theorem, completeness theorem and the concept of truth degrees of formulas in $L_n$, the concept of consistency degrees of theories is given. It is proved that a theory $\Gamma$ over $L_n$ is a useless theory (i.e., the deductions of $\Gamma$ are all tautologies) iff the consistency degree $\text{consist}_n(\Gamma)$ of $\Gamma$ is equal to 1, $\Gamma$ is consistent iff $\frac{1}{2} \leq \text{consist}_n(\Gamma) \leq 1$, and $\Gamma$ is inconsistent iff $\text{consist}_n(\Gamma) = 0$. Lastly, the concept of consistency degrees of theories is generalized and a method of graded reasoning in $L_n$ is obtained.

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1. Introduction

Whether a theory (i.e., a set of formulas) is consistent or not is one of the crucial questions in any logic system. The reason is that in classical logic, a contradictory theory (i.e., a theory which is not consistent) proves anything. The same result also holds in fuzzy and many-valued logic systems in which the interpretation of the logical implication satisfies $0 \rightarrow x = 1$, where $0$, $x$, and $1$ are truth values [4]. Moreover, how to measure the extent to which a theory is consistent is also one of the crucial questions in logic systems. For trying to grade the extent of consistency of different theories, many authors have proposed different methods in fuzzy (continuous valued) logic systems and have obtained many good results [3,7,13,21,19,20,17]. Especially in [20], the authors, from logical point of view and based on deduction theorems, completeness theorems and the concept of truth degrees of formulas, introduced, in classical and fuzzy propositional logic systems, a more natural and reasonable definition of consistency degrees of theories. In other words, we have studied successfully the consistency of theories where the set of truth values jumped from $\{0,1\}$ to $[0,1]$. A natural question then arises: How do we harmoniously fill in the gap of consistency of theories between classical and fuzzy logic systems? That is to say, how do we establish the concept of consistency degrees of theories in $n$-valued logic systems such that it approximates the consistency of theories in fuzzy logic system when $n$ turns to infinity, and it takes the classical case as a special case when $n = 2$?

In order to put fuzzy reasoning on a solid foundation, the second author proposed in 1997 a new formal deductive system $L/C^{3}$ (also called $R_{0}$-logic) and a kind of new algebraic structures, called $R_{0}$-algebras (see [12,14]). To formalize the logic of nilpotent minimums, Esteva and Godo introduced in 2001 a nilpotent minimum logic, NM for short (NM is a schematic extension of the Monoidal t-norm based logic, MTL for short, as the logic of left-continuous t-norms, see [1]). So far the authors of [8,6] have proved that NM and $L^{+}$ are equivalent and NM-algebras and $R_{0}$-algebras are the same algebraic structures. It is proved also that $L^{+}$ is standard complete [10] (or see [1] for the standard completeness of NM) and the $n$-valued $R_{0}$-logic $L_{n}^{+}$ is also complete with respect to the standard $n$-valued $R_{0}$-algebra $R_{n}$ (see [11]). Moreover, the $R_{0}$-logic $L^{+}$ has another good property that the structure of its generalized deduction theorem ($\Gamma \cup \{A\} \vdash B$ iff $\Gamma \vdash A^{2} \rightarrow B$, see Theorem 2.4) is decided. Therefore we try to choose the $n$-valued $R_{0}$-logic $L_{n}^{+}$ to answer the question above. As we will see from the paper, the presented results can be easily adapted also to each finite-valued Łukasiewicz logic, $\mathcal{L}_{n}$ for short, which is standard complete with respect to the standard $MV_{n}$-chain [2].

The present paper is arranged like this: In Section 2, we recall the presentation of the $n$-valued $R_{0}$-logic $L_{n}^{+}$ including syntax and semantics, and deduction theorem and completeness theorem. In Section 3, we, based on the measure theory on the set $\Omega_{n}$ of all valuations from the set $\mathcal{F}(S)$ of all formulas into the set of truth values, introduce the concept of truth degrees of formulas in $L_{n}^{+}$ as done in Łukasiewicz $n$-valued logic system [15]. With the concept of truth degrees of formulas, we propose successfully in Section 4, the concept of consistency degrees of theories after taking on a deep analysis on what it means for a theory to be inconsistent. The idea of the present paper has a good intuitive meaning, and it can be
easily generalized by replacing the contradiction 0 with a general formula \( A \) to establish a method of graded reasoning in \( \mathcal{L}^*_n \). We will discuss this issue in Section 5. The last part, Section 6, is some conclusion summarizing the results and outlining problems and further research topics.

2. Preliminaries

First let us recall the formal deduction system \( \mathcal{L}^* \) (also called \( R_0 \)-logic) proposed by the second author in 1997 [12], which is proved now equivalent to the nilpotent minimum logic, NM for short, introduced by the authors of [1].

**Definition 2.1** [14]. Suppose that \( S = \{ p_1, p_2, \ldots \} \) is a countable set of propositional variables. The set \( F(S) \) of well-formed formulas over \( \mathcal{L}^* \) is defined inductively as follows: each propositional variable \( p \in S \) is a formula; if \( A \) and \( B \) are formulas, then \( \neg A \), \( A \lor B \) and \( A \to B \) are formulas, where \( \neg \) is a unary operator, and \( \lor \) and \( \to \) are binary ones respectively.

The simplified axiom schemes of \( \mathcal{L}^* \) given in [14] are as follows:

\[
\begin{align*}
(L^1) & \quad A \to (B \to A \land B) \\
(L^2) & \quad (\neg A \to \neg B) \to (B \to A) \\
(L^3) & \quad (A \to (B \to C)) \to (B \to (A \to C)) \\
(L^4) & \quad (B \to C) \to ((A \to B) \to (A \to C)) \\
(L^5) & \quad A \to \neg \neg A \\
(L^6) & \quad A \to A \lor B \\
(L^7) & \quad A \lor B \to B \lor A \\
(L^8) & \quad (A \to C) \land (B \to C) \to (A \lor B \to C) \\
(L^9) & \quad (A \land B \to C) \to (A \to C) \lor (B \to C) \\
(L^{10}) & \quad (A \to B) \lor ((A \to B) \to \neg A \lor B)
\end{align*}
\]

where \( A \land B = (\neg \neg A \lor \neg \neg B) \).

The deduction rule of \( \mathcal{L}^* \) is Modus Ponens (briefly, MP): from \( A \) and \( A \to B \) infer \( B \).

**Remark 2.2.** (i) To formalize the logic of nilpotent minimum \( t \)-norms, Esteva and Godo introduced, in 2001, a nilpotent minimum logic, NM for short, which is a schematic extension of the Monoidal \( t \)-norm based logic, MTL for short, as the logic of left-continuous \( t \)-norms [1]. But now Pei, and Liu and Li have proved in [8] and [6] respectively, the equivalence of NM and \( \mathcal{L}^* \).

(ii) The connectives \( \neg, \lor \) and \( \to \) in \( \mathcal{L}^* \) are not independent, Wang et al. proved in [18] that \( A \lor B \) is an abbreviation of \( \neg (\neg ((A \to (A \to B)) \to A) \to A) \to \neg ((A \to B) \to B) \) under provable equivalence, \( A, B \in F(S) \).

**Definition 2.3.** In \( \mathcal{L}^* \).

(i) A subset \( \Gamma \) of \( F(S) \) is called a theory.

(ii) Let \( \Gamma \) be a theory, \( A \in F(S) \). A deduction of \( A \) from \( \Gamma \), in symbols, \( \Gamma \vdash A \), is a finite sequence of formulas \( A_1, \ldots, A_n = A \) such that for each \( 1 \leq i \leq n \), either \( A_i \) is an axiom, or \( A_i \in \Gamma \), or there are \( j, k \in \{ 1, \ldots, i - 1 \} \) such that \( A_i \) follows from \( A_j \) and \( A_k \) by MP. Equivalently, we say that \( A \) is a consequence of \( \Gamma \). The set of all consequences of \( \Gamma \) is denoted by \( D(\Gamma) \). By a proof of \( A \) we shall henceforth mean a deduction of \( A \) from the empty set. We shall also write \( \vdash A \) in place of \( \emptyset \vdash A \) and \( A \) will be called to be a theorem.
(iii) Let \( A, B \in F(S) \). If \( \vdash A \to B \) and \( \vdash B \to A \) hold, then \( A \) and \( B \) are called provably equivalent.

(iv) A theory \( \Gamma \subset F(S) \) is called inconsistent if \( \Gamma \vdash \overline{0} \), otherwise, consistent, where \( \overline{0} \) is a refutable formula (or contradiction), i.e., \( \vdash \neg \overline{0} \) holds.

The classical deduction theorem is one of the most important theorem in classical two-valued logic system, and it says that

\[
\Gamma \cup \{ A \} \vdash B \iff \Gamma \vdash A \to B.
\]

Because the left-to-right direction of the classical deduction theorem above depends on the following axiom of classical logic:

\[
(A \to (B \to C)) \to ((A \to B) \to (A \to C)),
\]

which is not assumed in \( L' \), it is no longer valid in \( L' \). Fortunately, it has been proved that there exists in \( L' \) a weak form of the deduction theorem (here called generalized deduction theorem) [9].

**Theorem 2.4** [9]. Suppose that \( \Gamma \) is a theory, \( A, B \in F(S) \), then in \( L' \) holds the following generalized deduction theorem:

\[
\Gamma \cup \{ A \} \vdash B \iff \Gamma \vdash A^2 \to B
\]

where \( A^2 = A \& A = \neg (A \to \neg A) \).

It is easy to check that \((A \& B) \to C \) and \( A \to (B \to C) \) are provably equivalent, hence by the definition of deduction, the above generalized deduction theorem in \( L' \) can be described equivalently as:

\[
\Gamma \vdash A \iff \exists A_1, \ldots, A_m \in \Gamma, \text{ s.t. } \vdash A^2_1 \& \cdots \& A^2_m \to A.
\]

The following is about the semantics of \( L' \).

Let \( N : [0,1] \to [0,1] \) be a strong negation (i.e., an involutive function such that \( N(0) = 1 \)). Define a binary operator \( \otimes_N : [0,1]^2 \to [0,1] \) as follows:

\[
x \otimes_N y = \begin{cases} 
  x \land y, & y > N(x), \\
  0, & \text{otherwise},
\end{cases}
\]

where \( x \land y = \min(x, y) \), \( x, y \in [0,1] \). The operator \( \otimes_N \) is called a nilpotent minimum.

Clearly, the operator \( \otimes_N \) is a left-continuous t-norm, thus we can obtain the corresponding residuum of \( \otimes_N \) \( R_N : [0,1] \to [0,1] \) as follows:

\[
R_N(x,y) = \begin{cases} 
  1, & x \leq y, \\
  N(x) \lor y, & \text{otherwise},
\end{cases}
\]

where \( x \lor y = \max(x, y) \), \( x, y \in [0,1] \). Particularly, if \( N \) is a standard strong negation \( \neg : [0,1] \to [0,1], \neg x = 1 - x \), then the operator \( \otimes_N \) is called the standard nilpotent minimum, and we obtain the so-called \( R_0 \) implication operator and the corresponding t-norm (see [12,14]):

\[
R_0(x,y) = \begin{cases} 
  1, & x \leq y, \\
  (1 - x) \lor y, & \text{otherwise},
\end{cases}
\]

\[
x \otimes y = \begin{cases} 
  x \land y, & x + y > 1, \\
  0, & \text{otherwise}.
\end{cases}
\]
Based on the $R_0$ implication operator and on the background of $L^*$-Lindenbaum algebra, the second author proposed the $R_0$-algebra [14], which is an algebraic structure equivalent to NM-algebra [1].

**Definition 2.5** [14]. Let $M$ be an algebra of type $(\neg, \lor, \to)$, where $\neg$ is a unary operator, $\lor$ and $\to$ are binary operations. If there is a partial order $\leq$ such that $(M, \leq)$ is a bounded lattice, $\lor$ is the supremum operation with respect to $\leq$, $\neg$ is an order-reversing involution, and the following conditions hold for any $x, y, z \in M$.

\begin{align*}
(R1) \quad & -x \to -y = y \to x \\
(R2) \quad & 1 \to x = x, \quad x \to x = 1 \\
(R3) \quad & y \to z \leq (x \to y) \to (x \to z) \\
(R4) \quad & x \to (y \to z) = y \to (x \to z) \\
(R5) \quad & x \to (y \lor z) = (x \to y) \lor (x \to z), \quad x \to (y \land z) = (x \to y) \land (x \to z) \\
(R6) \quad & (x \to y) \lor ((x \to y) \to (\neg x \lor y)) = 1
\end{align*}

For example, define on $[0,1]$ a unary operator and two binary operator as follows:

\begin{align*}
\neg x &= 1-x, \quad x \lor y = \max\{x, y\}, \quad x \to y = R_0(x, y). \quad \text{Then } ([0,1], \neg, \lor, \to) \text{ is a } R_0\text{-algebra, called the standard } R_0\text{-algebra.}
\end{align*}

The finite set $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ with the operations $\neg, \lor, \to$ as defined in the standard $R_0$-algebra is also a $R_0$-algebra, denoted by $R_n$.

**Definition 2.6.** Let $([0,1], \neg, \lor, \to)$ be the standard $R_0$-algebra, $A \in F(S)$.

(i) A homomorphism $\nu : F(S) \to [0,1]$ of type $(\neg, \lor, \to)$ from $F(S)$ into the standard $R_0$-algebra, i.e., $\nu(\neg A) = \neg \nu(A), \nu(A \lor B) = \nu(A) \lor \nu(B), \nu(A \to B) = \nu(A) \to \nu(B) = R_0(\nu(A), \nu(B))$, is called a valuation of $F(S)$. The set of all valuations will be denoted by $\Omega$.

(ii) A formula $A$ is called a tautology if $\forall \nu \in \Omega, \nu(A) = 1$. $A$ is called a contradiction if $\forall \nu \in \Omega, \nu(A) = 0$.

It has been proved that the algebraic semantic and the syntax of $L^*$ are in perfect harmony, i.e. the standard completeness theorem holds in $L^*$.

**Theorem 2.7** [10]. $\forall A \in F(S)$ in $L^*$, $A$ is a theorem iff $A$ is a tautology.

To grade the truth degrees of formulas in $L^*$, the second author proposed the concept of integrated truth degrees of formulas [16].

Let $A = A(p_1, \ldots, p_m)$ be a formula of $R_0$-logic whose all propositional variables are among $p_1, \ldots, p_m$. Then the value of $\overline{A}(x_1, \ldots, x_m)$ is obtained from the truth valuations $v(p_i) = x_i, \quad i = 1, \ldots, m$, in $[0,1]$ using the homomorphism $\nu$. For example, if $A = p_1 \lor p_2 \to \neg p_3$, then $\overline{A}(x_1, x_2, x_3) = (x_1 \lor x_2) \to (1 - x_3) = R_0((x_1 \lor x_2), 1 - x_3)$. It must be stressed that this has sense only in standard semantics of $R_0$-logic (luckily, this logic, more precisely the NM-logic, is standard complete).

**Definition 2.8** [16]. Suppose that $A(p_1, \ldots, p_m)$ is a formula in $L^*$, define

$$
\tau(A) = \int_0^1 \cdots \int_0^1 \overline{A}(x_1, \ldots, x_m) dx_1 \cdots dx_m,
$$

then $\tau(A)$ is called the integrated truth degree of $A$. 

Now we are ready to investigate the $n$-valued $R_0$ logic $L_n^*$. \\

**Definition 2.9** [11]. (i) The axioms of the $(2n + 1)$-valued $R_0$ logic $L_{2n+1}^*$ are those of $L^*$ plus

$(L^{*11}) A_n$

(ii) The axioms of the $(2n + 2)$-valued $R_0$ logic $L_{2n+2}^*$ are those of $L^*$ plus

$(L^{*12}) B_{n+1}$

$(L^{*15}) (A \rightarrow \neg A) \land (\neg A \rightarrow A) \rightarrow (A \lor \neg A)$

where

\[
A_1 = \neg C_1 \lor ((C_1 \rightarrow \neg C_1) \rightarrow (\neg C_1 \rightarrow C_1)),
\]

\[
A_{k+1} = (C_{k+1} \rightarrow A_k) \lor C_{k+1}, k = 1, 2, \ldots,
\]

\[
B_1 = C_1 \lor \neg C_1,
\]

\[
B_{k+1} = (C_{k+1} \rightarrow B_k) \lor C_{k+1}, k = 1, 2, \ldots
\]

and \{C_n\} is an arbitrary sequence of formulas.

If it is not necessary to state whether $n$ is odd or even, we denote the above extension by $L_n^*$. The inference rule of $L_n^*$ is MP.

The definitions of theories, theorems and deductions of $L_n^*$ are almost the same as those of $L^*$. For example, a theory $\Gamma$ over $L_n^*$ is a set of formulas. $\Gamma \vdash A$ (or, more precisely, $\Gamma \vdash_n A$) means that $A$ is provable in $\Gamma$, i.e., there is a $L_n^*$-proof in $\Gamma$ (a finite sequence each of whose members either is an axiom of $L_n^*$, or an element of $\Gamma$ or follows from some preceding members by MP.) Moreover, since $L_n^*$ is an extension of $L^*$, the generalized deduction theorem of $L_n^*$ also holds in $L_n^*$, i.e., $\Gamma \cup \{A\} \vdash_n B$ iff $\Gamma \vdash_n A^2 \rightarrow B$.

The set of truth values over $L_n^*$ is the $n$-valued $R_0$-algebra $R_n$. A homomorphism \(v : F(S) \rightarrow R_n\) is called a valuation of $F(S)$ in $L_n^*$. The set of all valuations is denoted by $\Omega_n$. A formula $A$ is called a tautology if $\forall v \in \Omega_n, v(A) = 1$. $A$ and $B$ are called logically equivalent if $\forall v \in \Omega_n, v(A) = v(B)$. Similarly for a formula $A = A(p_1, \ldots, p_m) \in F(S)$, $A$ induces a function $A : R^m_n \rightarrow R_n$ in the natural way.

D.W. Pei and S.M. Wang has now proved that $L_n^*$ is complete w.r.t. $R_n$ [11].

**Theorem 2.10** [15]. $L_n^*(n \geq 2)$ is complete, i.e., $\forall A \in F(S)$,

\[A \text{ is a theory iff } A \text{ is a tautology.}\]

3. Truth degrees of formulas in $L_n^*$

In this section we extend the idea of [15] to $L_n^*$ and introduce the concept of truth degrees of formulas in $L_n^*$.

Suppose that $X_k = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ and $\mu_k$ is the evenly distributed probability measure on $X_k$, i.e., $\mu_k(\emptyset) = 0, \mu_k(X_k) = 1, \mu_k(\{\frac{i}{n-1}\}) = \frac{1}{n}, i = 0, \ldots, n - 1, k = 1, 2, \ldots$. Assume that $X = \prod_{k=1}^{\infty} X_k$ and $\mu$ is the infinite product of $\mu_1, \mu_2, \ldots$ [5].

Let $v \in \Omega_n$, then $v$ is determined by its restriction \(v|S\) because $F(S)/\sim$ (where $\sim$ is the relation of “logical equivalence in the logic $L_n^*$”) is the free algebra generated by $S$. Assume that $v(p_k) = v_k (k = 1, 2, \ldots)$, then an infinite dimensional vector $\overrightarrow{v} = (v_1, v_2, \ldots)$ in $X = \prod_{k=1}^{\infty} X_k$ is obtained. Conversely, let $\overrightarrow{v} = (v_1, v_2, \ldots)$ be any element of
$X = \prod_{k=1}^{\infty} X_k$, then there exists a unique $v \in \Omega_n$ such that $v(p_k) = v_k (k = 1, 2, \ldots)$. Hence there exists a bijection $\varphi : \Omega_n \rightarrow X = \prod_{k=1}^{\infty} X_k$ defined by $\varphi(v) = \overline{v}$.

**Definition 3.1.** The above-mentioned mapping is called the measurized mapping of $\Omega_n$.

**Definition 3.2.** Suppose that $A \in F(S), n \geq 2$, define

$$\tau_n(A) = \sum_{i=0}^{n-1} \frac{i}{n-1} \mu \left( \frac{[A]}{n-1} \right) = \sum_{i=1}^{n-1} i \mu \left( \frac{[A]}{n-1} \right),$$

where $[A]_{n-1} = \{ \overline{v} \in R_n^n \mid \overline{v} = \varphi(v), v(A) = \frac{i}{n-1}, v \in \Omega_n \}$ is the class of $\frac{i}{n-1}$-models of $A$, $i = 0, 1, \ldots, n-1$. Then $\tau_n(A)$ is called the $n$-valued truth degree of $A$.

**Example 3.3.** (i) $\tau_n(p) = \sum_{i=1}^{n-1} \frac{i}{n-1} \mu \left( \frac{[p]}{n-1} \right) = \sum_{i=1}^{n-1} \frac{i}{n-1} \cdot \frac{1}{n} = \frac{1}{n(n-1)} \sum_{i=1}^{n-1} i = \frac{1}{2}.$

(ii) Let $p$ and $q$ be two different propositional variables, and $v \in \Omega_n$, then $v(p \land q) = \frac{i}{n-1}$ iff $v(p) = \frac{i}{n-1}$ and $v(q) \geq \frac{i}{n-1}$, or $v(q) = \frac{i}{n-1}$ and $v(p) > \frac{i}{n-1} (i = 1, 2, \ldots, n-1)$. Since there are a total of $n^2$ pairs of $(v(p), v(q))$ and among them only $2(n - i - 1)$ 's imply that $v(p \land q) = \frac{i}{n-1}$, hence

$$\mu \left( \frac{[p \land q]}{n-1} \right) = \frac{2(n - i) - 1}{n^2}, \quad i = 1, 2, \ldots, n - 1.$$}

Therefore $\tau_n(p \land q) = \sum_{i=1}^{n-1} \frac{i}{n-1} \mu \left( \frac{[p \land q]}{n-1} \right) = \frac{1}{n(n-1)} \sum_{i=1}^{n-1} i [2(n - i) - 1] = \frac{2n - 1}{6n}$.

By a similar analysis one can obtain

$$\mu \left( \frac{[p \lor q]}{n-1} \right) = \frac{2i + 1}{n^2}, \quad i = 1, 2, \ldots, n - 1.$$}

Therefore $\tau_n(p \lor q) = \sum_{i=1}^{n-1} \frac{i}{n-1} \cdot \frac{2i + 1}{n^2} = \frac{4n + 1}{6n}$.

(iii) Consider the truth degree $\tau_n(p \rightarrow q)$ of $p \rightarrow q$. Note that

$$v(p \rightarrow q) = v(p) \rightarrow v(q) = R_0(v(p), v(q)) = \begin{cases} 1, & v(p) \leq v(q), \\ (1 - v(p)) \lor v(q), & v(p) > v(q). \end{cases}$$

Let $v(p \rightarrow q) = 1$, then $v(p) \leq v(q)$, there a total of $[n + (n - 1) + \cdots + 1] = \frac{n(n + 1)}{2}$ pairs $(v(p), v(q))$ such that $v(p \rightarrow q) = 1$. Next let $v(p \rightarrow q) = \frac{i}{n-1} (1 \leq i \leq n - 2)$, equivalently, $(1 - v(p)) \lor v(q) = \frac{i}{n-1}$ and $v(p) > v(q)$. It is easy to check that there are a total of $(n - 1)$ solutions if $i = \frac{n+1}{2}$ and $n$ is odd, $(2i + 1)$ ones if $i < \frac{n+1}{2}$ and $n$ is odd, $2(n - 1 - i)$ ones if $i > \frac{n+1}{2}$ and $n$ is odd, and $(2i + 1)$ ones if $i < \frac{n+1}{2}$ and $n$ is even, $2(n - 1 - i)$ ones if $i > \frac{n+1}{2}$ and $n$ is even. Thus, if $n$ is odd, then

$$\tau_n(p \rightarrow q) = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} + \frac{2}{n^2} \cdot \frac{n-1}{n} + \sum_{i=1}^{n-1} \frac{i}{n-1} \cdot \frac{2i + 1}{n^2}$$

$$+ \sum_{i=\frac{n+1}{2}+1}^{n-2} \frac{i}{n-1} \cdot \frac{2(n-1-i)}{n^2} = \frac{6n^2 + n - 1}{8n^2},$$
and if $n$ is even, then
\[
\tau_n(p \rightarrow q) = \frac{1}{n^2} \times \frac{n(n+1)}{2} + \sum_{i=1}^{\frac{n-1}{2}} i \cdot \frac{2i+1}{n^2} + \sum_{i=\frac{n+1}{2}}^{\frac{n-1}{2}} i \cdot \frac{2(n-1-i)}{n^2} = \frac{6n^2 - 5n - 2}{8n(n-1)}.
\]

(iv) Now, to calculate the truth degree $\tau_n(p^2)$ of $p^2 = \neg(p \rightarrow \neg p)$ in $\mathcal{L}^*_n$. Let $v(\neg p) = \frac{i}{n-1}$, i.e., $v(p) \otimes v(p) = \frac{i}{n-1}$, $i = 1, 2, \ldots, n-1$, where $\otimes$ is the nilpotent minimum $t$-norm adjoint to the $R_0$ implication operator defined above. Hence if $i \geq \frac{n-1}{2} + 1$, then $v(p^2) = \frac{i}{n-1}$, otherwise $v(p^2) = 0$. Thus we get
\[
\mu\left([p^2]_{\frac{n-1}{2}}\right) = \begin{cases} 
\frac{i}{n}, & i \geq \frac{n-1}{2} + 1, \\
0, & 1 \leq i < \frac{n-1}{2} + 1.
\end{cases}
\]

Therefore
\[
\tau_n(p^2) = \sum_{i=1}^{n-1} \frac{i}{n-1} \mu\left([p^2]_{\frac{n-1}{2}}\right) = \sum_{i=\frac{n-1}{2}+1}^{n-1} \frac{i}{n-1} \cdot \frac{1}{n} = \begin{cases} 
\frac{3n-1}{8n(n-1)}, & n \text{ is odd}, \\
\frac{3n-2}{8n(n-1)}, & n \text{ is even}.
\end{cases}
\]

(v) It is left to the reader to check that $\tau_n(p_1 \land p_2 \land \cdots \land p_m) = \frac{1}{n^{m(n-1)}} \sum_{k=1}^{n-1} k^m$.

In the sequel, we give the following lemma summarizing the basic properties of the truth degree function $\tau_n$, and the proof is obvious and omitted.

**Lemma 3.4.** Suppose $A$ and $B$ are formulas in $\mathcal{L}^*_n$, then

(i) $A$ is a tautology iff $\tau_n(A) = 1$.
(ii) $A$ is a contradiction iff $\tau_n(A) = 0$.
(iii) $\tau_n(\neg A) = 1 - \tau_n(A)$.
(iv) If $\neg A \rightarrow B$ holds, then $\tau_n(A) \leq \tau_n(B)$.
(v) If $A$ and $B$ are logically equivalent, then $\tau_n(A) = \tau_n(B)$.
(vi) If define
\[
\rho_n(A, B) = 1 - \tau_n((A \rightarrow B) \land (B \rightarrow A)), \quad A, B \in F(S),
\]
then $\rho_n$ is a pseudo-metric on $F(S)$.

From Lemma 3.4 one may see that the truth degree function $\tau_n$ looks like a valuation of $F(S)$, but it is not the case. Indeed, as shown in Example 3.3(i) and (ii), $\tau_n$ does not commute with the min-conjunction. $\tau_n$ is actually an intrinsic means to measure the size of the model class of a formula.

What is the relationship between the two truth degree functions $\tau_n$ and $\tau$ when $n$ converges to infinity? The following limit theorem answers the above question.

**Theorem 3.5 (Limit theorem).** $\forall A \in F(S), \lim_{n \to \infty} \tau_n(A) = \tau(A)$. 
Proof. Although the implication operator $R_0$ is not continuous on $[0,1]^2$, the set of its discontinuity points is of zero-measure. So the proof is similar to that of Theorem 2 of [15] and is omitted. □

Example 3.6. Use Definition 2.8 to calculate $\tau(p)$, $\tau(p \land q)$, $\tau(p \lor q)$, $\tau(p \rightarrow q)$ and $\tau(p^2)$, and compare with the corresponding results obtained in Example 3.3.

Solution. (i) $\tau(p) = \int_0^1 x \, dx = \frac{1}{2}$.
(ii) $\tau(p \land q) = \int_0^1 \int_0^1 x \land y \, dx \, dy = \int_0^1 \int_0^1 y \, dx \, dy + \int_0^1 \int_y^1 x \, dx \, dy = \frac{1}{2}$. Similarly, $\tau(p \lor q) = \frac{3}{2}$.
(iii) $\tau(p \rightarrow q) = \int_0^1 \int_0^1 (x \rightarrow y) \, dx \, dy = \int_0^1 \int_y^1 \, dx \, dy + \int_0^1 \int_0^1 (1-y) \land y \, dx \, dy = \frac{3}{4}$.
(iv) $\tau(p^2) = \int_0^1 x^2 \, dx = \frac{3}{8}$.

These results together with the corresponding ones of Example 3.3 further verify Theorem 3.5.

4. Consistency degrees of theories based on deduction theorems in $L_n^*$

First let us take an analysis on the inconsistency of a theory in $L_n^*$. Suppose that $\Gamma$ is a theory and $\Gamma$ is inconsistent, then the contradiction $\bot$ is a consequence of $\Gamma$, that is to say, $\Gamma \vdash \bot$ holds. It follows from the generalized deduction theorem that there exists a finite string of formulas $A_1, \ldots, A_m \in \Gamma$ such that $\vdash_{n} A_1 \land \ldots \land A_m \rightarrow \bot$ holds, i.e., the formula $A_1 \land \ldots \land A_m \rightarrow \bot$ is a theorem of $L_n^*$. By the completeness theorem of $L_n^*$, $A_1 \land \ldots \land A_m \rightarrow \bot$ is a tautology, then the truth degree $\tau_{n}(A_1 \land \ldots \land A_m \rightarrow \bot) = 1$ following from Lemma 3.4.

Conversely, if there is a finite sequence of formulas $A_1, \ldots, A_m \in \Gamma$ such that the truth degree $\tau_{n}(A_1 \land \ldots \land A_m \rightarrow \bot) = 1$, then we obtain more truth degrees of such formulas from $\Gamma$ and it is necessary to decide, which of them should be taken as the result. By the completeness theorem of $L_n^*$, the larger the truth degrees of such formulas are, the closer $\Gamma$ is to be inconsistent. Therefore, it is natural and reasonable for us using the supremum of truth degrees of all formulas with the form $A_1 \land \ldots \land A_m \rightarrow \bot$, where $A_1, \ldots, A_m \in \Gamma$, to measure the inconsistency of $\Gamma$.

Definition 4.1. Suppose that $\Gamma$ is a theory, $2(\Gamma)$ is the set of all finite subsets of $\Gamma$, $\Sigma = \{A_1, \ldots, A_m\} \in 2(\Gamma)$. Let

$\Sigma^2 \rightarrow \bot = \begin{cases} A_1 \land \ldots \land A_m \rightarrow \bot, & m > 0, \\ \bot, & m = 0, \end{cases}$

and define

$\mu_\nu(\Gamma) = \sup\{\tau_{n}(\Sigma^2 \rightarrow \bot) | \Sigma \in 2(\Gamma)\}$. 

Then $\mu_n(\Gamma)$ is called the degree of entailment of $\bar{0}$ from $\Gamma$, or say, $\bar{0}$ is a consequence of $\Gamma$ in the degree $\mu_n(\Gamma)$, we may also call it the degree of inconsistency of $\Gamma$.

**Remark 4.2.** (i) $\forall A, B \in F(S)$, $A\& B \rightarrow C$ is equivalent to $B\& A \rightarrow C$ because $\&$ is commutative, hence the definition of $R$ does not depend on the order of $A_i$’s.

(ii) Since $\tau_n(A\& B \rightarrow C) \geq \tau_n(A \rightarrow C) \land \tau_n(B \rightarrow C)$, one easily gets that $\tau_n(\Sigma^2 \rightarrow \bar{0}) \leq \tau_n(\Sigma^2 \rightarrow \bar{0})$, where $\Sigma_1 \subset \Sigma_2$, $\Sigma_1$, $\Sigma_2 \in 2^{2(G)}$.

(iii) Suppose that $\Gamma$ is finite, then it follows from (ii) that $\mu_n(\Gamma) = \tau_n(\Gamma^2 \rightarrow \bar{0})$.

(iv) **Definition 4.1** indeed offers a method to evaluate the extent to which the contradiction $\bar{0}$ is a consequence of the theory $\Gamma$. For example, suppose that $\Gamma$ is finite and $\mu_n(\Gamma) = 1$, then it is to say $\bar{0}$ is (100%) a consequence of $\Gamma$. Of course, if $\Gamma$ is infinite then “$\mu_n(\Gamma) = 1$” may not imply that $\bar{0}$ is (100%) a consequence of $\Gamma$ (please see Example 4.4(iii)), but in a very high degree converging to 1.

(v) If we introduce in $L_n^*$ the concept of divergence degrees of theories by the truth degrees of formulas as done in [16]:

$$\text{div}_n(\Gamma) = \sup\{\rho_n(A, B) \mid A, B \in D(\Gamma)\},$$

where $\rho_n$ is defined in **Lemma 3.4**. Then we can prove that in $L_n^*$, $\text{div}_n(\Gamma) = \mu_n(\Gamma)$ (please see the proof of *Theorem 3.8* of [20]). But in the present paper we prefer $\mu_n(\Gamma)$ because it not only has a very strong intuitive meaning-the extent to which $\Gamma$ is inconsistent, but also it can be easily generalized to give a method of graded reasoning in $L_n^*$.

The calculation of $\mu_n(\Gamma)$ can be simplified as follows:

**Lemma 4.3.** Suppose that $\Gamma$ is a theory in $L_n^*$, then

$$\mu_n(\Gamma) = 1 - \inf\{\tau_n(A^2_1 \& \cdots \& A^2_m) \mid A_1, \ldots, A_m \in \Gamma, m \in \mathbb{N}\}.$$ 

**Proof.** It follows from the fact $\neg A$ and $A \rightarrow \bar{0}$ are logically equivalent that

$$\mu_n(\Gamma) = \sup\{\tau_n(A^2_1 \& \cdots \& A^2_m \rightarrow \bar{0}) \mid A_1, \ldots, A_m \in \Gamma, m \in \mathbb{N}\}$$

$$= \sup\{\tau_n(\neg(A^2_1 \& \cdots \& A^2_m)) \mid A_1, \ldots, A_m \in \Gamma, m \in \mathbb{N}\}$$

$$= \sup\{1 - \tau_n(A^2_1 \& \cdots \& A^2_m) \mid A_1, \ldots, A_m \in \Gamma, m \in \mathbb{N}\}$$

$$= 1 - \inf\{\tau_n(A^2_1 \& \cdots \& A^2_m) \mid A_1, \ldots, A_m \in \Gamma, m \in \mathbb{N}\}.$$ 

The proof is completed. \(\square\)

**Example 4.4.** Calculate $\mu_n(\Gamma)$ for (i) $\Gamma = \emptyset$, (ii) $\Gamma = \{p\}$, (iii) $\Gamma = S = \{p_1, p_2, \ldots\}$.

**Solution.** (i) If $\Gamma = \emptyset$, then $\forall \Sigma \in 2^{2(G)}$, $\Sigma = \emptyset$, $\Sigma^2 \rightarrow \bar{0} = \bar{0}$, and so $\tau_n(\Sigma^2 \rightarrow \bar{0}) = 0$. Hence $\mu_n(\Gamma) = 0$.

(ii) Let $\Gamma = \{p\}$, then it follows from **Lemma 4.3** and **Example 3.3** that

$$\mu_n(\Gamma) = \tau_n(\Gamma^2 \rightarrow \bar{0}) = \tau_n(p^2 \rightarrow \bar{0}) = 1 - \tau_n(p^2) = \begin{cases} \frac{5n+1}{6n}, & \text{n is odd}, \\ \frac{5n-6}{8(n-1)}, & \text{n is even}. \end{cases}$$
(iii) Since \( \vdash (p_1 \& \cdots \& p_m^2) \rightarrow (p_1 \land \cdots \land p_m) \) holds in \( \mathcal{L}^* \), it also holds in the \( n \)-valued extension \( \mathcal{L}_n^* \) of \( \mathcal{L}^* \). Following from Lemma 3.4 and Example 3.3,

\[
\tau_n(p_1^2 \& \cdots \& p_m^2) \leq \tau_n(p_1 \land \cdots \land p_m) = \frac{1}{n^m(n-1)} \sum_{k=1}^{n-1} k^n \rightarrow 0 \quad (m \rightarrow \infty).
\]

Therefore

\[
\mu_n(S) = 1 - \inf \{ \tau_n(p_1^2 \& \cdots \& p_m^2) \mid p_1, \ldots, p_m \in S, m \in \mathbb{N} \} \\
\geq 1 - \inf \{ \tau_n(p_1 \land \cdots \land p_m) \mid p_1, \ldots, p_m \in S, m \in \mathbb{N} \} \\
= 1 - \inf \left\{ \frac{1}{n^m(n-1)} \sum_{k=1}^{n-1} k^n \mid m \in \mathbb{N} \right\} \\
= 1
\]

and so \( \mu_n(S) = 1 \).

But \( \Gamma = S \) is not inconsistent. In fact, suppose on the contrary that \( \Gamma \) is inconsistent, then \( \Gamma \vdash \neg \neg \bar{0} \). Hence it follows from the generalized deduction theorem of \( \mathcal{L}_n^* \) that there exist \( p_1, \ldots, p_m \in \Gamma \) such that \( \vdash \neg \neg p_1^2 \& \cdots \& p_m^2 \rightarrow \bar{0} \) holds. Choose \( v \in \Omega_n \) such that \( v(p_1) = \cdots = v(p_m) = 1 \), then

\[
v(p_1^2 \& \cdots \& p_m^2) = v(p_1) \otimes v(p_1) \otimes \cdots \otimes (v(p_m) \otimes v(p_m)) = 1.
\]

Thus

\[
v(p_1^2 \& \cdots \& p_m^2 \rightarrow \bar{0}) = 1 \rightarrow 0 = 0,
\]

contradicting the fact that \( p_1^2 \& \cdots \& p_m^2 \rightarrow \bar{0} \) is a theorem. This completes the proof.

**Theorem 4.5.** Let \( \Gamma \) be a theory in \( \mathcal{L}_n^* \). If \( \Gamma \) is inconsistent then \( \mu_n(\Gamma) = 1 \), but not vice versa.

**Proof.** The proof is straightforward and so omitted. For counterexample, please see Example 4.4(iii). \( \square \)

From the analysis at the beginning of this section, we know that \( \mu_n(\Gamma) \) is an ideal index to measure the inconsistency degree of \( \Gamma \). Perhaps this hints the idea that one may define the consistency degree \( \text{consist}_n(\Gamma) \) of \( \Gamma \) to be \( 1 - \mu_n(\Gamma) \), but this idea has a shortcoming that it could not distinguish theories with \( \mu_n(\Gamma) = 1 \) from inconsistent theories as shown in Example 4.4 and Theorem 4.5. Hence we have to revise the seemingly reasonable definition \( \text{consist}_n(\Gamma) = 1 - \mu_n(\Gamma) \).

**Definition 4.6.** Suppose that \( \Gamma \) is a theory of \( \mathcal{L}_n^* \), i.e., \( \Gamma \subset F(S) \), define

\[
i_n(\Gamma) = \max\{ [\tau_n(A_1^2 \& \cdots \& A_m^2 \rightarrow \bar{0})] \mid A_1, \ldots, A_m \in \Gamma, m \in \mathbb{N} \},
\]

and \( i_n(\Gamma) \) is called the polar index of \( \Gamma \) in \( \mathcal{L}_n^* \).

**Theorem 4.7.** Suppose that \( \Gamma \) is a theory of \( \mathcal{L}_n^* \). Then

(i) \( \Gamma \) is consistent iff \( i_n(\Gamma) = 0 \),

(ii) \( \Gamma \) is inconsistent iff \( i_n(\Gamma) = 1 \).
Proof. Since the concept of consistency of a theory is crisp rather than fuzzy, and $i_n(\Gamma) \in \{0,1\}$, it suffices to prove (ii). Suppose that $\Gamma$ is inconsistent, i.e., $\Gamma \vdash \emptyset$ then by the generalized deduction theorem of $\mathcal{L}_n^*$, there exist $A_1, \ldots, A_m \in \Gamma$ such that $A_1^2 \& \cdots \& A_m^2 \rightarrow \emptyset$ is a theorem, and it follows from the completeness theorem of $\mathcal{L}_n^*$ and Lemma 3.4 that $\tau_n(A_1^2 \& \cdots \& A_m^2 \rightarrow \emptyset) = 1$ and $i_n(\Gamma) = 1$. Conversely, if $i_n(\Gamma) = 1$, then there exist $A_1, \ldots, A_m \in \Gamma$ such that $\tau_n(A_1^2 \& \cdots \& A_m^2 \rightarrow \emptyset) = 1$. Again by completeness theorem of $\mathcal{L}_n^*$ and Lemma 3.4, $\vdash \tau_n(A_1^2 \& \cdots \& A_m^2 \rightarrow \emptyset)$ holds. Therefore $\Gamma \vdash \emptyset$ holds by MP 2m times, and hence $\Gamma$ is inconsistent. \(\square\)

Definition 4.8. Suppose that $\Gamma$ is a theory of $\mathcal{L}_n^*$, define

$$\text{consist}_n(\Gamma) = 1 - \frac{1}{2} \mu_n(\Gamma)(1 + i_n(\Gamma))$$

and call $\text{consist}_n(\Gamma)$ the consistency degree of $\Gamma$ in $\mathcal{L}_n^*$.

Theorem 4.9. Suppose that $\Gamma$ is a theory of $\mathcal{L}_n^*$, then

(i) $\Gamma$ is a useless theory, i.e., all members of $D(\Gamma)$ are tautologies, if and only if $\text{consist}_n(\Gamma) = 1$.

(ii) $\Gamma$ is consistent if and only if $\frac{1}{2} \leq \text{consist}_n(\Gamma) \leq 1$.

(iii) $\Gamma$ is consistent and $\mu_n(\Gamma) = 1$ if and only if $\text{consist}_n(\Gamma) = \frac{1}{2}$

(iv) $\Gamma$ is inconsistent if and only if $\text{consist}_n(\Gamma) = 0$.

Proof. (i) It is easy to check that if $\Gamma$ is a useless theory iff $\mu_n(\Gamma) = 0$, hence (i) holds.

(ii) On account of Theorem 4.7(i) $\Gamma$ is consistent iff $i_n(\Gamma) = 0$, and this is equivalent to $\frac{1}{2} \leq \text{consist}_n(\Gamma) = 1 - \frac{1}{2} \mu_n(\Gamma) \leq 1$.

(iii) It follows directly from Definition 4.8 and Theorem 4.7(i).

(iv) Suppose that $\Gamma$ is inconsistent, then $\mu_n(\Gamma) = 1$ by Theorem 4.5 and $i_n(\Gamma) = 1$ by Theorem 4.7(ii). Hence $\text{consist}_n(\Gamma) = 1 - \frac{1}{2} \times 1 \times (1 + 1) = 0$. Conversely, if $\Gamma$ is consistent, then by (ii) $\frac{1}{2} \leq \text{consist}_n(\Gamma) \leq 1$, a contradiction. \(\square\)

It must be stressed that item (i) says $\Gamma$ is a useless theory iff the elements of $D(\Gamma)$ are tautologies. This means that, passing through the completeness theorem, all the consequences of $\Gamma$ are logical theorems, or all the special axioms of $\Gamma$ are indeed logical theorems. In this sense each theory with $\text{consist}_n(\Gamma) = 1$ is quite useless.

Example 4.10. One has got in Example 4.4 that $\mu_n(\emptyset) = 0$, $\mu_n(\{p\}) = \frac{5n+1}{6n}$ if $n$ is odd, $\mu_n(\{p\}) = \frac{5n-6}{8(n-1)}$ if $n$ is even, and $\mu_n(S) = 1$. It is routine to check that these three theories are all consistent. This means that $i_n(\emptyset) = i_n(\{p\}) = i_n(S) = 0$. Hence one easily gets that

$$\text{consist}_n(\emptyset) = 1,$$

$$\text{consist}_n(\{p\}) = \begin{cases} \frac{11n-1}{16n}, & n \text{ is odd}, \\ \frac{11n-10}{16(n-1)}, & n \text{ is even}, \end{cases}$$

$$\text{consist}_n(S) = \frac{1}{2}.$$
Now calculate \( \text{consist}_n(\Gamma) \) for \( \Gamma = \{ p, \neg p \} \). Since \( p \& \neg p \) is a contradiction \( (v(p \& \neg p) = v(p) \otimes (1 - v(p)) = 0) \) and \( p \& \neg p \in D(\Gamma) \), \( \Gamma \) is inconsistent by the standard completeness theorem of \( L_n^* \). Hence \( \text{consist}_n(\Gamma) = 0 \).

**Remark 4.11.** Corresponding to the Limit Theorem (Theorem 3.5) in Section 3, here is a similar limit theorem for \( \text{consist}_n(\Gamma) \):

\[
\lim_{n \to \infty} \mu_n(\Gamma) = \mu(\Gamma).
\]

But \( \lim_{n \to \infty} i_n(\Gamma) = i(\Gamma) \) does not hold. As for the definitions of \( \mu(\Gamma) \) and \( i(\Gamma) \) in \( L_n^* \) we refer to [20].

5. Methods of graded reasoning in \( L_n^* \)

As has been pointed out in **Remark 4.2(iv)**, **Definition 4.1** indeed offers a method to evaluate the extent to which the contradiction \( 0 \) is a consequence of a theory \( \Gamma \).

If we replace \( 0 \) by a general formula \( A \), then the corresponding analysis at the beginning of Section 4 also holds in \( L_n^* \). Hence we get a method to evaluate the extent to which a formula \( A \) is a consequence of a given theory \( \Gamma \).

**Definition 5.1.** Suppose that \( \Gamma \) is a theory of \( L_n^* \), \( 2^{\{\Gamma\}} \) is the set of all finite subsets of \( \Gamma \), \( \Sigma = \{ A_1, \ldots, A_m \} \in 2^{\{\Gamma\}} \), and \( A \in F(S) \) is a formula. Let

\[
\Sigma \rightarrow A = \begin{cases} A_1 \& \cdots \& A_m \rightarrow A, & m > 0, \\ A, & m = 0, \end{cases}
\]

and define

\[
\mu_n(A, \Gamma) = \sup \{ \tau_n(\Sigma \rightarrow A) \mid \Sigma \in 2^{\{\Gamma\}} \}.
\]

Then \( \mu_n(A, \Gamma) \) is called the degree of entailment of \( A \) from \( \Gamma \), or say \( A \) is a consequence of \( \Gamma \) in the degree \( \mu_n(A, \Gamma) \).

Note that **Remark 4.2** also holds when \( 0 \) is replaced by \( A \), and clearly \( \mu_n(0, \Gamma) = \mu_n(\Gamma) \).

Similar to **Theorem 4.5**, we have the following theorem.

**Theorem 5.2.** Let \( A \) be not a theorem of \( L_n^* \) and let \( \Gamma \) be a theory of \( L_n^* \). If \( A \) is a consequence of \( \Gamma \) then \( \mu_n(A, \Gamma) = 1 \), but not vice versa.

Clearly **Theorem 4.5** is a special case of **Theorem 5.2**. It is not difficult to check that \( \sup \{ \tau_n(\Sigma \rightarrow A) \mid \Sigma \in 2^{\{\Gamma\}} \} = \sup \{ \tau_n(p_2 \& \cdots \& p_m \rightarrow p_1) \mid m \in \mathbb{N} \} = 1 \), where \( \Gamma = \{ p_2 \& \cdots \& p_m \mid m = 2, 3, \ldots \} \) and \( A = p_1 \), and it is routine to show that \( A \notin D(\Gamma) \). **Theorem 5.2** tells us that \( \mu_n(A, \Gamma) = 1 \) does not mean \( A \) is (100%) a consequence of \( \Gamma \).

**Definition 5.3.** Suppose that \( \Gamma \) is a theory of \( L_n^* \), \( A \in F(S) \). Define

\[
i_n(A, \Gamma) = \max \{ [\Sigma \rightarrow A] \mid \Sigma \in 2^{\{\Gamma\}} \},
\]

and call \( i_n(A, \Gamma) \) the polar index of \( \Gamma \) w.r.t. \( A \).
Theorem 5.4. Suppose that $\Gamma$ is a theory of $L_n^*$, $A \in F(S)$, then

(i) $A \in D(\Gamma)$, i.e., $A$ is a consequence of $\Gamma$ if and only if $i_n(A, \Gamma) = 1$.
(ii) $A \notin D(\Gamma)$ if and only if $i_n(A, \Gamma) = 0$.

Proof. The proof is trivial. □

Definition 5.5. Let $\Gamma$ be a theory of $L_n^*$, $A \in F(S)$, define

$$\mu_n(A, \Gamma) = \frac{1}{2} \mu_n(A, \Gamma)(1 + i_n(A, \Gamma)),$$

and also call $\mu_n(A, \Gamma)$ the degree of entailment of $A$ from $\Gamma$, or say $A$ is consequence of $\Gamma$ in the degree $\mu_n(A, \Gamma)$.

Theorem 5.6. Suppose that $\Gamma$ is a theory of $L_n^*$, $A \in F(S)$, then

(i) $A \in D(\Gamma)$ if and only if $\mu_n(A, \Gamma) = 1$.
(ii) $A \notin D(\Gamma)$ if and only if $\frac{1}{2} \tau_n(A) \leq \mu_n(A, \Gamma) \leq \frac{1}{2}$
(iii) $A \notin D(\Gamma)$ and $i_n(A, \Gamma) = 1$ if and only if $\mu_n(A, \Gamma) = \frac{1}{2}$.

Proof. The proof is analogous to that of Theorem 4.9 and so omitted. □

In the following, we give another method of graded reasoning in $L_n^*$ by evaluating the distance of $A$ to $D(\Gamma)$.

Definition 5.7. Let $\Gamma$ be a theory of $L_n^*$, $A \in F(S)$. Put

$$a = \rho_n(A, D(\Gamma)) = \inf\{\rho_n(A, B) \mid B \in D(\Gamma)\},$$

and call $A$ a consequence of $\Gamma$ with error $a$.

The following theorem reveals the relationship between $\rho_n$ and $\mu_n$.

Theorem 5.8. Let $\Gamma$ be a theory of $L_n^*$, $A \in F(S)$, then

$$\rho_n(A, D(\Gamma)) \geq 1 - \mu_n(A, \Gamma).$$

Proof

$$\rho_n(A, D(\Gamma)) = \inf\{\rho_n(A, B) \mid B \in D(\Gamma)\}$$
$$= \inf\{1 - \tau_n((A \rightarrow B) \land (B \rightarrow A)) \mid B \in D(\Gamma)\}$$
$$= 1 - \sup\{\tau_n((A \rightarrow B) \land (B \rightarrow A)) \mid B \in D(\Gamma)\}$$
$$\geq 1 - \sup\{\tau_n(B \rightarrow A) \mid B \in D(\Gamma)\}$$
$$\geq 1 - \sup\{\tau_n(A_1 \land \cdots \land A_m \rightarrow A) \mid A_1, \ldots, A_m \in \Gamma, m \in \mathbb{N}\}$$
$$= 1 - \mu_n(A, \Gamma).$$ □
Corollary 5.9. Suppose that $\Gamma$ is a theory of $L_n^*$, $A \in F(S)$, if $A \in \partial(D(\Gamma))$, i.e., $\rho_n(A, D(\Gamma)) = 0$ but $A \not\in D(\Gamma)$, then $\mu_n(A, \Gamma) = \frac{1}{2}$.

Proof. It follows directly from Theorem 5.8, Definition 5.5 and Theorem 5.6.

6. Concluding remarks

In the present paper we firstly extended the idea of [15] to $L_n^*$ and we proposed the concept of truth degrees of formulas, then based on deduction theorem, completeness theorem, we defined a consistency function measuring the extent to which a general theory is inconsistent. Two methods of graded reasoning in $L_n^*$ were given and the relationship between them was discussed.

So far, we have successfully established the theory of consistency degrees of theories in classical two-valued and fuzzy logic systems, and the $n$-valued extension $L_n^*$ of the formal deductive system $L^*$. It must be stressed that the presented results can be easily adapted to any standard complete finite-valued logic system such as $L_n$ (which is standard complete with respect to the standard $MV_n$-chain, see [2]). More precisely, in $L_n$ we only need to define $\Sigma^n \rightarrow 0 = A^n_1 \& \cdots \& A^n_m \rightarrow 0$ if $m > 0$, and $\Sigma^n \rightarrow 0 = 0$ otherwise, instead of $\Sigma^2 \rightarrow 0$ defined in $L_n^*$, because it is easy to check that the generalized deduction theorem: $\Gamma \cup \{A\} \vdash B$ iff $\Gamma \vdash A^n \rightarrow B$, holds in $L_n$. The following work is the same as in $L_n^*$. For example, we can also prove Theorems 4.5, 4.7, and 4.9. Maybe some computation of $\mu_n$ will be altered, such as $\mu_n(\{p\}) = 1 - \frac{1}{n}$ in $L_n$, which is not equal to the expression of $\mu_n(\{p\})$ in $L_n^*$.

Moreover, how to extend the concept of consistency degrees of theories and the method of graded reasoning to predicate logic systems and to the more general fuzzy logic system with graded syntax given in [7] would be a more attractive research.

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