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Discrete Mathematics 306 (2006) 918–922
DISCRETE MATHEMATICS
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Automorphism groups of graphs and edge-contraction

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Abstract

If a class $\mathscr{C}$ of finite graphs is closed under contraction and forming subgraphs, and if every finite abstract group occurs as the automorphism group of some graph in $\mathscr{C}$, then $\mathscr{C}$ contains all finite graphs (up to isomorphism). Also related results concerning automorphism groups of graphs on given surfaces are mentioned.

1.

The automorphism group $A(X)$ of the graph X consists of all permutations of the vertex set $V(X)$ preserving both edges and non-edges. As is well known, given any (finite) group G, there exists a (finite) graph X such that $A(X)$ and G are isomorphic [5]. Later, a number of papers showed that one could also specify that X belong to certain classes of finite graphs (graphs with given connectivity, chromatic number, graphs which are regular of given degree etc., see [1,6,8,10,11]). "It was becoming apparent that requiring X to have a given abstract group of automorphisms was not a severe restriction" [7, p. 170]. In this paper we are going to disprove this last assertion by showing that the given abstract group of automorphisms of X implies X to be contractible to a large complete graph (Theorem 3.1′). This result contains as a special case that for any compact surface, one can find a finite group which is not isomorphic to the automorphism group of any graph which can be drawn on that surface [3,4].

The characterization of the automorphism groups of such classes of finite graphs which are closed under contraction and forming subgraphs, seems to be an interesting question. For example, the class of finite graphs embeddable in a fixed surface has this property. The automorphism groups of finite planar graphs have been characterised in [3].

We mention some results concerning finite abelian groups. Given any finite abelian group G, there exists a finite connected planar graph X such that $A(X)$ and G are isomorphic [2]. 2-connectedness cannot, in general, be required; if the automorphism group of a 2-connected planar graph has odd order, then this group is cyclic [3]. But given any finite abelian group G, there exists a finite 2-connected graph X such that $A(X)$ and G are isomorphic and X is embeddable in any closed surface of Euler characteristic ≤ 0 (see [4]). However, for any closed surface, one can find a finite abelian group which is not isomorphic even to a subgroup of the automorphism group of any 3-connected graph embeddable in that surface [4].

DOI of original article: 10.1016/0012-365X(74)90104-6
The original article was published in Discrete Mathematics 8 (1974) 13–20

0012-365X/$ - see front matter © 1974 Published by Elsevier B.V.
doi:10.1016/j.disc.2006.03.012

Let $\mathscr{C}_n$ denote the class of those finite graphs which are not contractible to K_n, the complete graph on n vertices. We show that for any finite abelian group G, there are infinitely many $(n-4)$-connected graphs X in $\mathscr{C}_n$ such that $A(X)$ and G are isomorphic. This statement, compared with the preceding one, demonstrates that these classes of graphs are really "wider" (from the point of view of automorphism groups) than those of graphs embeddable in fixed surfaces.

2.

We recall some definitions. We deal only with graphs without loops and multiple edges. $V(X)$ denotes the set of vertices, $E(X)$ the set of edges of X.

Let X and Y be graphs. A mapping $f: V(X) \to V(Y)$ of the vertex set of X *onto* the vertex set of Y is called a *connected homomorphism* (or *contraction*) if

(a) $[y_1, y_2] \in E(Y) \Leftrightarrow y_1 \neq y_2$ and there is an edge $[x_1, x_2]$ in X for which $f(x_1) = y_1$, $f(x_2) = y_2$;
(b) for any $y \in V(Y)$, the full subgraph of X on the subset $f^{-1}(y)$ of $V(X)$ is connected.

If $f(x)=x$ holds for all but one vertex of X and for the exceptional one $f(x) \in V(X)$, then f is an *elementary contraction*. All connected homomorphisms of a finite graph are iterated elementary contractions, up to a graph isomorphism.

The graph Z is said to be *homomorphic* to the graph X, denoted by $X > Z$, if Z is a subgraph of some $f(X)$ where f is a connected homomorphism.

3.

The main result of this paper is the following:

Theorem 3.1. *Given a class $\mathscr{C}$ of (not necessarily finite) graphs, closed under connected homomorphisms and forming subgraphs, assume that for any finite group G, there exists a graph in $\mathscr{C}$, the automorphism group of which is isomorphic to G. Then $\mathscr{C}$ contains all finite graphs.*

Remark 3.2. Obviously, if $X > Z$, where Z is finite, then X contains a finite subgraph Y such that $Y > Z$. Hence in Theorem 3.1 we may restrict ourselves to requiring $\mathscr{C}$ to be closed under *forming finite subgraphs* and *contracting edges of finite graphs*.

Theorem 3.1′. *Let G be a finite simple group containing a subgroup isomorphic to Z_p^n, the direct product of n copies of the cyclic group of order p, where p is an odd prime number. Assume that $n \geq M$, $p \geq 4M^3$ where $M = 2^{N-3}$, $N \geq 3$. Then any graph, the automorphism group of which is isomorphic to G, is homomorphic to the complete graph on N vertices.*

We remark that G can be chosen, e.g., as the alternating group of degree np $(np \geq 5)$. Thus Theorem 3.1 is a trivial consequence of Theorem 3.1′. To prove the latter, we need the following:

Theorem 3.3. *Let G be a finite simple group, $G \supseteq Z_p^n$, p an odd prime, $n \geq M$, $p \geq 4M^3$, where $M = 2^{N-3}$. Assume that the automorphism group of the connected graph X contains a subgroup isomorphic to G, acting fixed-point-free on $V(X)$. Then X is homomorphic to the complete graph on N vertices. (By fixed-point-free we mean that no vertex is fixed under the action of the whole group.)*

Proof. (A) There exists a finite graph X' with the following properties:

$$(*) \quad \begin{cases} X' < X; \\ X' \text{ is connected;} \\ A(X') \text{ contains a subgroup } G' \text{ isomorphic to } G; \\ G' \text{ acts on } V(X') \text{ fixed-point-free.} \end{cases}$$

As a matter of fact, let us consider any vertex $x \in V(X)$; let Z be a connected finite subgraph of X containing the vertex orbit xG'; and put $X' = ZG'$ (the finite and obviously connected subgraph of X consisting of all edges of the form $e\pi(e \in E(Z), \pi \in G')$).

Let Y have the minimal number of edges among these finite graphs X'.

(B) We assert that G' acts transitively on the edges of Y. Let e denote an edge of Y and eG' the edge-orbit containing e.

Assume first that the graph eG' is disconnected. Let ϕ denote the connected homomorphism of Y which maps the components of eG' onto single points; let $\phi(Y) = Y_1$. Obviously, $Y_1 < X$, Y_1 is connected, and G' induces a permutation group $G'' \subseteq A(Y_1)$ on $V(Y_1)$ which acts fixed point-free. As G'' is a homomorphic image of the simple group G' and $|G''| \neq 1$ (because G'' acts transitively on the set of components of eG'), it follows that $G' \cong G''$. This contradicts the minimality of Y since $|E(Y_1)| < |E(Y)|$.

Thus we have shown that the graph Y_2, consisting of the edge set eG', is connected. Obviously, $Y_2 < X$, and G' induces a permutation group $G''' \subseteq A(Y_2)$ on $V(Y_2)$ which acts fixed-point-free. Consequently, $|G'''| \neq 1$, and hence (as G''' is a homomorphic image of the simple group G') $G' \cong G'''$. So Y_2 complies with the requirements $(*)$. By the minimality of Y, we have $|E(Y_2)| \geq |E(Y)|$. But Y_2 is a finite subgraph of the connected graph Y, hence $Y_2 = Y$. So we have proved that G' acts transitively on the edges of Y.

(C) The Cayley color-graph $X_{H,S}$ of the group H with respect to the subset S of H is defined as follows:

$$V(X_{H,S}) = H,$$

$$E(X_{H,S}) = \{[h, sh] : h \in H, s \in S, s \neq 1\}.$$

[4, Lemma 3] asserts the following:

Let Y be a connected graph, H a subgroup of $A(Y)$. Assume that none of the non-identity elements of H has a fixed point. Then Y is homomorphic to $X_{H,S}$ for some generating system S of H.

If H denotes a subgroup of G' isomorphic to Z_p^n, than the cardinality of any generating system S of H is not less than n, hence the valency of the vertices of $X_{H,S}$ is not less than $2n \geq 2M$. Hence

$$|E(X_{H,S})| \geq M \cdot |V(X_{H,S})| = 2^{N-3} \cdot |V(X_{H,S})|.$$

W. Mader proved the following theorem [9]:

If for the graph U, $|E(U)| \geq 2^{N-3}|V(U)|$, then $U > K_N$. (K_N denotes the complete graph on N vertices).

Hence by the above, if the non-identity elements of H have no fixed points, then we are through:

$$X > Y > X_{H,S} > K_N.$$

(D) In the remaining cases we have a permutation $\pi \in G'$ of order p which fixes some vertex of Y. As π is not the identity and Y is connected, we can find an edge $[x, y]$ in Y such that $x\pi = x$, $y\pi \neq y$. As p is prime, $y\pi \neq y$ implies that $y, y\pi, y\pi^2, \ldots, y\pi^{p-1}$ are different vertices, hence the valency of x is at least p.

If the valencies of all vertices of Y are $\geq 2M$, then using the theorem of Mader we are finished.

Assume now that the set

$$B = \{b_1, \ldots, b_s\}$$

of vertices of valency $< 2M$ is not empty. $p \geq 4M^3 > 2M$ implies by the above that $x \notin B$. Let $A = xG'$ denote the orbit of x. By the edge-transitivity of G', there are at most two orbits of G' in Y, hence A and B are these orbits.

Let us consider the graph W on the vertex set $V(W) = A$, where two vertices are adjacent if and only if they have a common neighbour in Y. (The common neighbour belongs to B because of the edge-transitivity.) W is connected. G' induces a group $G^{\mathrm{IV}} \subseteq A(W)$ which acts fixed-point-free on A. Hence $|G^{\mathrm{IV}}| \neq 1$, $G^{\mathrm{IV}} \cong G'$. Consequently, the restriction of π to A is not the identity permutation, but $x\pi = x$. From this, one can deduce – as at the beginning of this section – that there is a vertex in W the valency of which is not less than p. But G^{IV} acts transitively on A, hence all vertices of W have valency $\geq p$. Thus

$$|E(W)| \geq \tfrac{1}{2} p \cdot |A| \geq 2M^3 \cdot |A|.$$

Let

(i) $E(W) = E_1 \cup \ldots \cup E_s$

be the partition of $E(W)$ defined by:

$[a_1, a_2] \in E_i$ if and only if i is the least index for which b_i is a common neighbour of a_1 and a_2.

The sets E_i are pairwise disjoint.

Let m denote the valency of the vertices of B in Y. Then

$$|E_i| \leq \binom{m}{2} < \tfrac{1}{2}m^2 < 2M^2.$$

Hence if t denotes the number of non-empty sets on the right-hand side of (i), we have

$$t > \frac{|E(W)|}{2M^2} \geq \frac{2M^3 \cdot |A|}{2M^2} = M \cdot |A|.$$

Let us define a connected homomorphism ϕ of Y by:

$\phi(a) = a$ if $a \in A$,

$\phi(b_i)$ is a vertex of an arbitrary element of E_i if $E_i \neq \emptyset$,

$\phi(b_i)$ is an arbitrary neighbour of b_i if $E_i = \emptyset$.

(The vertex set of the obtained graph $\phi(Y) = Z$ is $V(Z) = A$.)

By definition, $|E(Z)| \geq t$. Hence

$$|E(Z)| > M \cdot |A| = M \cdot |V(Z)|.$$

Consequently, an application of Mader's theorem shows that

$$Z > K_N;$$

thus by

$$X > Y > Z > K_N,$$

the proof of Theorem 3.3 is complete.

Proof of Theorem 3.1′. Let X_0 be a graph with $A(X_0) \cong G$. Let V_0 be the set of all those vertices of X_0 which are fixed under all automorphisms of X_0. Let $\{X_i : i \in I\}$ be the set of connected components of the graph obtained by deleting V_0 (and the incident edges) from X. Let G_i denote the subgroup of $A(X_i)$ consisting of those automorphisms of X_i which are restrictions of automorphisms of X to $V(X_i)$. Clearly, the direct product D of the groups G_i is a normal subgroup of $A(X)$ (the graphs X_i can be moved independently), and the factor group $A(X)/D$ is isomorphic to the direct product of some symmetric groups (some of the X_i' can be permuted arbitrarily).[1] Hence as G is simple, some G_i is isomorphic to G, and all the others are trivial. Consequently, X_0 has a subgraph X which complies with the requirements of Theorem 3.3. An application of Theorem 3.3 proves Theorem 3.1′.

Remark 3.4. The same proof is valid for relational systems containing only unary and binary relations. The *shadow* of such a relational system means the graph defined on the same set by connecting a pair of distinct vertices iff it belongs to at least one of the relations (see [2]). The above proof yields:

Given a class $\mathscr{K}$ of relational systems having the property that for any finite group G, there is an object X in $\mathscr{K}$ such that $A(X) \cong G$, then for any finite graph Y, there is an object in $\mathscr{K}$ the shadow of which is homomorphic to Y.

Remark 3.5. Theorem 3.1 fails to be true if instead of forming subgraphs we restrict ourselves to forming full subgraphs. This is shown by the following counterexample:

Let $\mathscr{C}$ denote the class of those *finite graphs the complements of which are bipartite*.

Contraction and forming full subgraphs do not increase the chromatic number of the complement of a graph.

[1] We could exactly describe the group $A(X)$ in terms of the wreath product (called also composition of permutation groups [7, p. 164]), but there is no need to do this.

As the automorphism group of a graph coincides with that of its complement, we have to refer to a theorem in [11] which says that any finite group is isomorphic to the automorphism group of a finite bipartite graph.

4.

Now we give the simple construction for abelian groups.

Theorem 4.1. *Given any finite abelian group G and an integer $n \geq 5$, there are infinitely many finite $(n-4)$-connected graphs X such that $A(X)$ and G are isomorphic, and X is not homomorphic to the complete graph on n vertices.*

Proof. [2, Theorem 3] asserts that there is a connected finite planar graph X_0 such that $A(X_0) \cong G$. Moreover, X_0 has a vertex x_0 which is fixed under all automorphisms of X_0.

Let Y denote a 2-connected planar graph which is asymmetric (has no nontrivial automorphisms) such that $V(Y) \cap V(X_0) = \{x_0\}$, $|V(Y)| \geq n-5$, and the valency of the vertices of Y does not exceed 3. (E.g., a large circuit with two diagonals of lengths 2 and 3, respectively.)

Let S denote a set of $n-5$ elements. S is assumed to be disjoint from $V(X_0) \cup V(Y)$. We define the graph X by

$$V(X) = V(X_0) \cup V(Y) \cup S,$$

$$E(X) = E(X_0) \cup E(Y) \cup \{[a,b] : a \in V(X_0) \cup V(Y), b \in S, a \neq f(b)\} \cup \{[a,b] : a, b \in S, a \neq b\},$$

where f denotes a one-to-one function from S into $V(Y)$.

The n-connectedness of X is clear. $X \not\succ K_n$ because by deleting the $n-5$ vertices belonging to S we obtain a planar graph (which is not homomorphic to K_5).

The set S is invariant under $A(X)$ since S contains the only vertices of X of valency $|V(X)|-2$.

By the construction, the vertices of Y are fixed under all automorphisms of $X_0 \cup Y$, hence also under $A(X)$. As f is one-to-one, this implies that the vertices of S are likewise fixed. Hence $A(X)$ is isomorphic to its restriction to X_0, which is $A(X_0) \cong G$.

The various possible choices of Y result in an infinite collection of graphs X.

Added in proof. A strengthened form of Theorem 3.1′ based on the above proof will appear in Acta Math. Acad. Sci. Hungar.

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