ELSEVIER
Discrete Mathematics 199 (1999) 161–171
DISCRETE MATHEMATICS

How to draw a hexagon

Andreas E. Schroth*

Institut für Analysis, TU Braunschweig, Pockelsstr. 14, D-38106 Braunschweig, Germany

Received 13 October 1997; revised 16 March 1998; accepted 18 May 1998

Abstract

Pictures of the $G_2(2)$ hexagon and its dual are presented. A way to obtain these pictures is discussed. © 1999 Elsevier Science B.V. All rights reserved

Keywords: Generalised hexagon; Generalised polygon; Coordinatisation; Models of geometries

1. Introduction

This paper presents pictures of the two classical generalised hexagons over the field with two elements. Information on these generalised hexagons is employed so that the pictures make some properties of these hexagons obvious. Familiarity with the theory of generalised polygons is not assumed. Perhaps this paper will convince the uninitiated reader that even new and seemingly abstract geometrical structures sometimes have nice visual presentations.

Generalised polygons were introduced by Tits in [7]. They serve, among other things, as a geometric realisation of certain groups. Generalised triangles are essentially projective planes, generalised quadrangles are also known as polar spaces of rank 2. Classical (thick) generalised n-gons, $n>2$, exist only for $n \in \{3,4,6,8\}$. By [3] also thick finite generalised n-gons, $n>2$, exist only for $n \in \{3,4,6,8\}$. If the pictures and discussions presented here whets your appetite for generalised polygons, I recommend [6,9] for further reading.

There is a well-known picture of the smallest projective plane, also known as the Fano plane (Fig. 1). A picture of the smallest generalised quadrangle appeared first on the cover of a book on generalised triangles [4] and reminds of an ordinary pentagon (Fig. 2). The picture is due to S. Payne, who calls it *doily* for obvious reasons. More stunning pictures of geometries may be found in Polster's Geometrical Picture Book [5].

* E-mail: a.schroth@tu-bs.de. url:http://fb1.math.nat.tu-bs.de/~top/aschroth.

PII: S0012-365X(98)00294-5

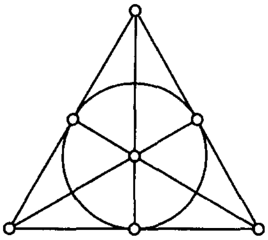

Fig. 1.

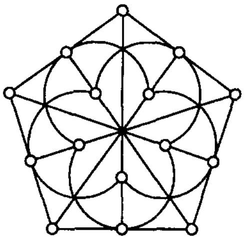

Fig. 2.

The classical generalised hexagons over GF(2) are the $G_2(2)$ hexagon and its dual. Both are hexagons of order 2. There are 63 points and 63 lines. Moreover, there are explicit descriptions of these generalised hexagons. So, in principle, it is possible to produce some pictures of these hexagons. However, just placing and labelling 63 dots on a sheet of paper and connecting collinear ones by a curve will most likely produce a mess. To obtain a nice picture one has to know and employ structural properties of the hexagons. The main property used here is the fact that the cyclic group of order 7 acts on the $G_2(2)$ hexagon and hence also on its dual. Thus, the 63 points split into nine orbits with seven points each. Likewise for lines. Also, three orbits of lines form different types of ordinary 7-gons. This takes care of 6 orbits of points. There are lines connecting vertices of one kind of 7-gons with midpoints of another type of 7-gons. This gives three more orbits of lines. In the $G_2(2)$ hexagon this also gives an additional orbit of points. In the $G_2(2)$ hexagon the remaining two orbits of points and three orbits of lines form the incidence graph of the Fano plane. The midpoints of the edges of this graph are glued to midpoints of the three types of ordinary 7-gons. In the dual hexagon the orbits not involved in the three ordinary 7-gons do not form easily recognisable figures.

To find a cyclic action of order seven and to work out the incidences two different coordinatisations of the $G_2(2)$ hexagon are used. The first coordinatisation, due to Tits [7], is good for calculations but less practical to detect incidences. The second coordinatisation, introduced by De Smet and Van Maldeghem in [2], is fine to detect incidences.

2. Preliminaries

A *point-line geometry* $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ consists of a *point set* $\mathcal{P}$, a *line set* $\mathcal{L}$ and an *incidence relation* I between points and lines. Often lines are considered as sets of points. Then incidence is given by inclusion and will not be mentioned explicitly. The elements of the union $\mathcal{V}=\mathcal{P}\cup\mathcal{L}$ are also called *vertices*.

A *chain* $(v_0,v_1,\dots,v_n), v_i\in\mathcal{V}$, is an ordered set of vertices such that $v_{i-1}\mathrm{I}v_i$ for $1\leqslant i\leqslant n$. The number of vertices in a chain diminished by 1 is called the *length* of the chain. A chain $(v_0,\dots,v_n)$ connects two vertices p and q if $p=v_0$ and $q=v_n$. Two vertices p and q are *at distance k,* if there is a chain of length k connecting p to q

and if the length of every chain connecting p to q is at least k. Two vertices that cannot be connected by a chain are at infinite distance.

A *generalised hexagon* is a point-line geometry $\mathscr{H} = (\mathscr{P}, \mathscr{L})$ satisfying the following axioms:

(1) The distance between two vertices is at most six.
(2) For any two vertices at distance $k < 6$ there exists a unique connecting chain of length k.
(3) Every vertex is incident with at least three vertices.

The *dual* of an incidence geometry is obtained by interchanging the rôles of points and lines. Since the definition of a generalised hexagon is given entirely in terms of vertices, the dual of a generalised hexagon is again a generalised hexagon.

An ordinary hexagon satisfies the first two axioms but not the third. Instead every vertex is incident with precisely two vertices.

Two points p, q are said to be *collinear*, if they can be joined by a line. The line connecting two collinear points p and q is denoted $p \vee q$. The set of lines incident with a point p is denoted $\mathscr{L}_p$.

A *finite* generalised hexagon is a generalised hexagon with only finitely many vertices. One readily verifies that for a finite generalised hexagon there are *parameters* s and t such that every line contains $s + 1$ points and every point lies on $t + 1$ lines. These parameters are also called the *order* of the generalised hexagon.

Suppose p is a point of a finite generalised hexagon of order (s, t). One sees that the set of points at distance 6, 4 or 2 from p consists of $s^3t^2, (t+1)ts^2$ or $(t+1)s$ points, respectively. Thus, all in all, there are $s^3t^2 + s^2t^2 + s^2t + st + s + 1$ points. A generalised hexagon with a minimal number of points has parameters $s = t = 2$. Thus there are 63 points. By duality there are also 63 lines.

3. The classical Hexagon

According to [1] there are only two generalised hexagons with parameters 2. They are dual to each other and are related to the Chevalley group $G_2(2)$. We will call one of these hexagons the *$G_2(2)$ hexagon.*

The $G_2(2)$ hexagon can be described explicitly in terms of coordinates in several ways. In this paper we will use two different coordinatisations. The first coordinatisation, introduced by Tits in [7], views the points of the $G_2(2)$ hexagon as the points of the quadric $Q(6,2)$ represented in the projective space $PG(6,2)$. Thus,

$$\mathscr{P} = \{(x_0, x_1, x_2, x_3, x_4, x_5, x_6)) \mid x_0x_4 + x_1x_5 + x_2x_6 = x_3^2\}.$$

Lines of the $G_2(2)$ hexagon are lines on this quadric whose Grassmanian coordinates satisfy

$$p_{12} = p_{34}, \quad p_{20} = p_{35}, \quad p_{01} = p_{36}, \quad p_{03} = p_{56}, \quad p_{13} = p_{64}, \quad p_{23} = p_{45}.$$

Table 1
Translation between the two coordinatisations

DS-VM	Tits
(∞)	$(1,0,0,0,0,0,0)$
(a)	$(a,0,0,0,0,0,1)$
(k,b)	$(b,0,0,0,0,1,k)$
(a,l,a')	$(l+aa',1,0,a,0,a,a')$
(k,b,k',b')	$(k'+bb',k,1,b,0,b',b+b'k)$
(a,l,a',l',a'')	$(al'+a'+a''l+aa'a'',a'',a,a'+aa'',1,l+aa'',l'+a'a'')$

The advantage of this coordinatisation is that automorphisms can be described in terms of matrices. The disadvantage is that collinearity is not easily detected.

The second coordinatisation is due to De Smet and Van Maldeghem and was introduced in [2]. We will restrict to the case where the field is GF(2). There is one special point denoted (∞). All other points are of the form $(a_0,\ldots,a_k)$, where $0\leqslant k\leqslant 5$ and $a_i\in \mathrm{GF}(2)$. Lines are denoted in the same way with the difference that square brackets are used instead of brackets. Incidence is given by

$$[k,b,k',b',k'']\,\mathrm{I}\,(k,b,k',b')\,\mathrm{I}\,[k,b,k']\,\mathrm{I}\,(k,b)\,\mathrm{I}\,[k]\,\mathrm{I}\,(\infty)\,\mathrm{I}\,[\infty]\,\mathrm{I}\,(a)\,\mathrm{I}\,[a,l]\,\mathrm{I}\,(a,l,a')$$
$$\mathrm{I}\,[a,l,a',l']\,\mathrm{I}\,(a,l,a',l',a'')$$

and

$$(a,l,a',l',a'')\,\mathrm{I}\,[k,b,k',b',k''] \Leftrightarrow \begin{cases} a'' = ak+b, \\ a' = ak+b', \\ l = ak+k''+aa''+aa', \\ l' = ak+k'+kk''+akb+bb'+ab, \end{cases}$$

$$\Leftrightarrow \begin{cases} b = ak+a'', \\ b' = ak+a', \\ k'' = ak+l+ab+ab', \\ k' = ak+l'+kl+aa''k+a'a''+aa'', \end{cases}$$

where all calculations are carried out in GF(2). In this coordinatisation it is not too difficult to determine whether two points are collinear. There is however no handy way to describe automorphisms.

Luckily there is a translation between the two coordinatisations. Table 1 states the translation for points if the field is GF(2). We will mainly work with points, so we do not need a translation table for lines. For more details on these coordinatisations see [2].

4. An automorphism of order seven

From now on we work entirely in the $G_2(2)$ hexagon. All indices of points and lines are taken in $\mathbb{Z}_7$. Our first aim is to find an automorphism of order seven in the hexagon. Then the 63 points split into nine orbits of seven points each. Analogously for the lines. In the Tits coordinatisation every automorphism can be represented by a matrix [7].

An ordinary ordered 7-gon is an ordered set of points $(a_0,\dots,a_6)$ such that a_k is collinear with but different from a_{k+1} (see Fig. 3). By [8] we know that the group of automorphisms acts transitively on ordinary ordered 7-gons. In particular, if $(a_0,\dots,a_6)$ forms an ordinary ordered 7-gon, then there is an automorphism Δ such that $a_i^{\Delta}=a_{i+1}$. As remarked in [8], the order of the group of automorphisms of the $G_2(2)$ hexagon equals the number of ordered 7-gons. Thus the group of automorphisms acts sharply transitive on the set of ordered 7-gons. This implies, that any such Δ has order seven. Our aim is to first present such a 7-gon and then to determine Δ.

Here is an ordinary 7-gon. It is obtained by gradually moving away from the point (∞) and then coming back via a different path.

$$
\begin{aligned}
a_0 &:= (\infty) && \sim (1,0,0,0,0,0,0),\\
a_1 &:= (0) && \sim (0,0,0,0,0,0,1),\\
a_2 &:= (0,0,0) && \sim (0,1,0,0,0,0,0),\\
a_3 &:= (0,0,0,0,0) && \sim (0,0,0,0,1,0,0),\\
a_4 &:= (1,1,1,1,1) && \sim (0,1,1,0,1,0,0),\\
a_5 &:= (0,1,1,1) && \sim (0,0,1,1,0,1,1),\\
a_6 &:= (0,1) && \sim (1,0,0,0,0,1,0).
\end{aligned}
$$

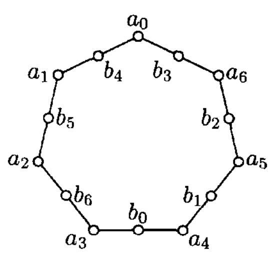

Fig. 3.

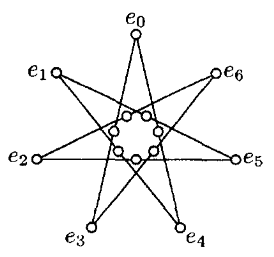

Fig. 4.

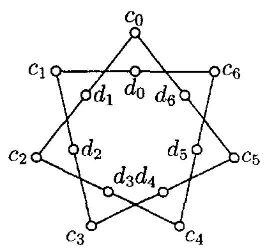

Fig. 5.

The connecting lines are:

$$A_4 := a_0 \vee a_1 = [\infty],$$
$$A_5 := a_1 \vee a_2 = [0,0],$$
$$A_6 := a_2 \vee a_3 = [0,0,0,0],$$
$$A_0 := a_3 \vee a_4 = [1,0,0,0,0],$$
$$A_1 := a_4 \vee a_5 = [0,1,1,1,1],$$
$$A_2 := a_5 \vee a_6 = [0,1,1],$$
$$A_3 := a_6 \vee a_0 = [0].$$

In the Tits coordinatisation the third point on every line is the sum of the other two points. Thus, the third point on A_k besides a_{k+3} and a_{k-3} is $b_k := a_{k-3} + a_{k+3}$. Table 3 lists the points b_i.

The points a_k, $0 \leqslant k \leqslant 6$, form a basis of the $\mathrm{GF}(2)^7$. Thus there is a unique linear map Δ of $\mathrm{GF}(2)^7$ with $a_k{}^{\Delta} = a_{k+1}$, or equivalently, $a_0{}^{\Delta^k} = a_k$. By linearity we also have $b_0{}^{\Delta^k} = b_k$ and $A_0{}^{\Delta^k} = A_k$. By elementary linear algebra, one obtains

$$\Delta = \begin{pmatrix} 0&0&0&0&0&1&0 \\ 0&0&1&0&1&0&1 \\ 0&0&0&0&1&0&0 \\ 0&0&1&1&0&0&0 \\ 0&1&0&0&1&0&0 \\ 0&0&1&0&0&0&0 \\ 1&0&1&0&0&0&0 \end{pmatrix}.$$

5. Working out the incidence

We now have to find the other seven orbits of points and eight orbits of lines. Indices of points and lines are such that $v_k = v_0{}^{\Delta^k}$.

To obtain the other orbits we gradually move away from the point (∞). The lines through $a_0 = (\infty)$ are $A_3 = [0]$, $A_4 = [\infty]$ and $B_0 := [1] = \{(\infty), (1,0), (1,1)\}$. Let

$$g_0 := (1,0) \sim (0,0,0,0,0,1,1),$$
$$f_0 := (1,1) \sim (1,0,0,0,0,1,1).$$

The orbits of g_0 and f_0 under Δ are listed in Table 3. By construction we have $B_k = \{a_k, f_k, g_k\}$.

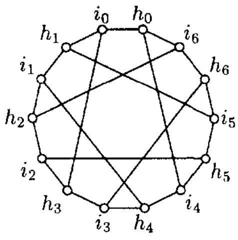

Fig. 6.

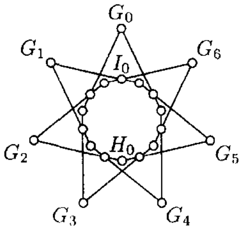

Fig. 7.

The lines intersecting the line B_0 besides A_4 and A_5 are:

$$D_0 := [1,0,0] \sim \{(1,0),(1,0,0,0),(1,0,0,1)\},$$

$$F_0 := [1,0,1] \sim \{(1,0),(1,0,1,0),(1,0,1,1)\},$$

$$I_0 := [1,1,0] \sim \{(1,1),(1,1,0,0),(1,1,0,1)\},$$

$$E_0 := [1,1,1] \sim \{(1,1),(1,1,1,0),(1,1,1,1)\}.$$

Incident with one of these lines but not with B_0 are the points:

$$(1,0,0,0) \sim (0,1,1,0,0,0,0) = b_0,$$

$$(1,0,0,1) \sim (0,1,1,0,0,1,1) =: c_0,$$

$$(1,0,1,0) \sim (1,1,1,0,0,0,0) =: e_0,$$

$$(1,0,1,1) \sim (1,1,1,0,0,1,1) =: d_0,$$

$$(1,1,0,0) \sim (0,1,1,1,0,0,1) =: x_0,$$

$$(1,1,0,1) \sim (1,1,1,1,0,1,0) =: y_0,$$

$$(1,1,1,0) \sim (0,1,1,1,0,1,0) =: u_0,$$

$$(1,1,1,1) \sim (1,1,1,1,0,0,1) =: v_0.6$$

Table 3 lists the orbits of c_0, d_0 and e_0 under Δ. We find $u_0 = e_5$ and $v_0 = e_2$. Hence $D_k = \{b_k, c_k, g_k\}$, $F_k = \{d_k, e_k, g_k\}$ and $E_k = \{e_{k+2}, e_{k-2}, f_k\}$. The orbit of e_0 forms an ordinary 7-gon where e_k is collinear with e_{k+3}. The connecting line is E_{k-2}. Fig. 4 depicts such a 7-gon. We say such a 7-gon is of type 3.

We now look at the lines through d_0. These are

$$F_0 = [1,0,1],$$

$$H_0 := [1,0,1,1,0] \sim \{(1,0,1,1),(1,0,0,0,1),(0,0,1,1,0)\},$$

$$C_0 := [1,0,1,1,1] \sim \{(1,0,1,1),(1,1,0,1,1),(0,1,1,0,0)\}.$$

Table 2
Lines and line pencils in the $G_2(2)$ hexagon

$A_k=\{a_{k+3},a_{k-3},b_k\}$	$\mathscr{L}_{a_k}=\{A_{k+3},A_{k-3},B_k\}$
$B_k=\{a_k,f_k,g_k\}$	$\mathscr{L}_{b_k}=\{A_k,D_k,G_k\}$
$C_k=\{c_{k+1},c_{k-1},d_k\}$	$\mathscr{L}_{c_k}=\{C_{k+1},C_{k-1},D_k\}$
$D_k=\{b_k,c_k,g_k\}$	$\mathscr{L}_{d_k}=\{C_k,F_k,H_k\}$
$E_k=\{e_{k+2},e_{k-2},f_k\}$	$\mathscr{L}_{e_k}=\{E_{k+2},E_{k-2},F_k6\}$
$F_k=\{d_k,e_k,g_k\}$	$\mathscr{L}_{f_k}=\{B_k,E_k,I_k\}$
$G_k=\{b_k,h_{k-2},i_{k+2}\}$	$\mathscr{L}_{g_k}=\{B_k,D_k,F_k\}$
$H_k=\{d_k,h_k,i_k\}$	$\mathscr{L}_{h_k}=\{G_{k+2},H_k,I_{k+3}\}$
$I_k=\{f_k,h_{k-3},i_{k+3}\}$	$\mathscr{L}_{i_k}=\{G_{k-2},H_k,I_{k-3}\}$

Table 3
Orbits of points under Δ

$a_0=(1,0,0,0,0,0,0)$	$b_0=(0,1,1,0,0,0,0)$	$c_0=(0,1,1,0,0,1,1)$
$a_1=(0,0,0,0,0,0,1)$	$b_1=(0,1,0,1,1,1,1)$	$c_1=(1,0,0,1,1,1,0)$
$a_2=(0,1,0,0,0,0,0)$	$b_2=(1,0,1,1,0,0,1)$	$c_2=(1,1,1,1,1,0,0)$
$a_3=(0,0,0,0,1,0,0)$	$b_3=(0,0,0,0,0,1,0)$	$c_3=(0,0,1,0,0,1,0)$
$a_4=(0,1,1,0,1,0,0)$	$b_4=(1,0,0,0,0,0,1)$	$c_4=(1,1,0,1,0,1,0)$
$a_5=(0,0,1,1,0,1,1)$	$b_5=(0,1,0,0,0,0,1)$	$c_5=(1,0,0,1,1,0,0)$
$a_6=(1,0,0,0,0,1,0)$	$b_6=(0,1,0,0,1,0,0)$	$c_6=(0,1,1,1,1,0,1)$
$d_0=(1,1,1,0,0,1,1)$	$e_0=(1,1,1,0,0,0,0)$	$f_0=(1,0,0,0,0,1,1)$
$d_1=(1,0,0,1,1,1,1)$	$e_1=(0,1,0,1,1,1,0)$	$f_1=(1,1,0,0,0,0,0)$
$d_2=(1,0,1,1,1,0,0)$	$e_2=(1,1,1,1,0,0,1)$	$f_2=(0,0,0,0,1,0,1)$
$d_3=(0,0,1,0,1,1,0)$	$e_3=(0,0,0,0,1,1,0)$	$f_3=(0,0,1,0,1,0,0)$
$d_4=(1,0,1,1,1,1,0)$	$e_4=(1,1,1,0,1,0,1)$	$f_4=(0,0,1,1,1,1,1)$
$d_5=(1,0,1,0,1,1,1)$	$e_5=(0,1,1,1,0,1,0)$	$f_5=(1,1,1,0,1,1,0)$
$d_6=(1,1,1,1,1,1,1)$	$e_6=(1,1,0,0,1,1,0)$	$f_6=(1,0,1,1,0,1,1)$
$g_0=(0,0,0,0,0,1,1)$	$h_0=(0,1,1,1,1,1,0)$	$i_0=(1,0,0,1,1,0,1)$
$g_1=(1,1,0,0,0,0,1)$	$h_1=(1,0,1,0,0,1,0)$	$i_1=(0,0,1,1,1,0,1)$
$g_2=(0,1,0,0,1,0,1)$	$h_2=(1,1,0,1,0,1,1)$	$i_2=(0,1,1,0,1,1,1)$
$g_3=(0,0,1,0,0,0,0)$	$h_3=(1,1,0,1,1,0,0)$	$i_3=(1,1,1,1,0,1,0)$
$g_4=(0,1,0,1,0,1,1)$	$h_4=(0,1,1,1,0,0,1)$	$i_4=(1,1,0,0,1,1,1)$
$g_5=(1,1,0,1,1,0,1)$	$h_5=(0,0,0,0,1,1,1)$	$i_5=(1,0,1,0,0,0,0)$
$g_6=(0,0,1,1,0,0,1)$	$h_6=(1,0,1,0,1,0,1)$	$i_6=(0,1,0,1,0,1,0)$

For the points on H_0 or C_0 different from g_0 we have

$$(1,0,0,0,1)\sim(0,1,1,1,1,1,0)=:h_0,$$
$$(1,0,1,1,0)\sim(1,0,0,1,1,0,1)=:i_0,$$
$$(1,1,0,1,1)\sim(0,1,1,1,1,0,1)=c_6,$$
$$(0,1,1,0,0)\sim(1,0,0,1,1,1,0)=c_1.$$

Table 3 lists the orbits of h_0 and i_0. We find $x_0=h_4$ and $y_0=i_3$. Thus $H_k=\{d_k,h_k,i_k\}$, $I_k=\{f_k,h_{k-3},i_{k+3}\}$ and $C_k=\{c_{k+1},c_{k-1},d_k\}$. The last equation shows that the orbit of c_0 forms an ordinary 7-gon where c_k is collinear with c_{k+2}. The connecting line is C_{k+1}. Fig. 5 depicts such a 7-gon. We say such a 7-gon is of type 2.

By now we have all points but one orbit of lines is still missing. One of the missing lines is incident with b_0. The lines through b_0 are $D_0 = [1,0,0]$, $A_0 = [1,0,0,0,0]$ and $G_0 := [1,0,0,0,1]$. The points on G_0 different from b_0 are

$$(1,0,1,0,1) \sim (0,1,1,0,1,1,1) = i_2,$$
$$(0,1,0,1,0) \sim (0,0,0,0,1,1,1) = h_5.$$

Thus $G_k = \{b_k, h_{k-2}, i_{k+2}\}$.

The equations for H_k, I_k and G_k show that the orbits of h_0 and i_0 may be represented as a 14-gon, where i_k is connected to h_k by the line H_k, to h_{k+1} via the line I_{k-3} and to h_{k+3} via the line G_{k-2}. This is the incidence graph of the Fano plane (Fig. 6). The dual of this configuration is depicted in Fig. 7.

Table 2 summarises all incidences between points and lines.

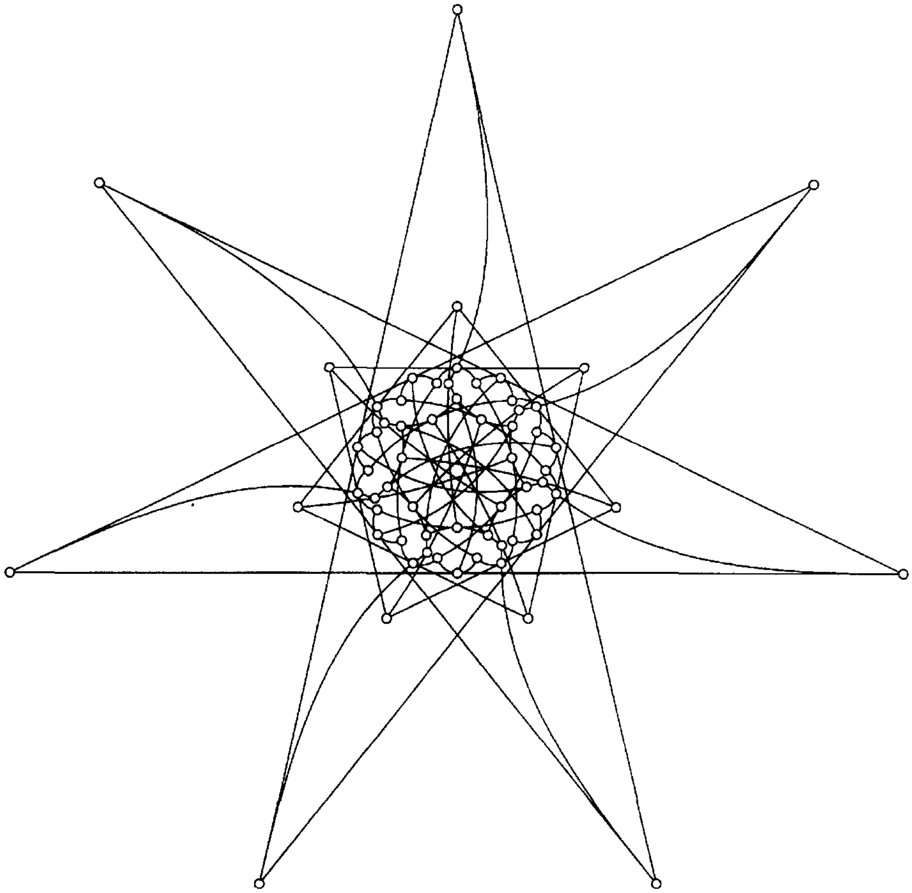

Fig. 8. The $G_2(2)$ hexagon

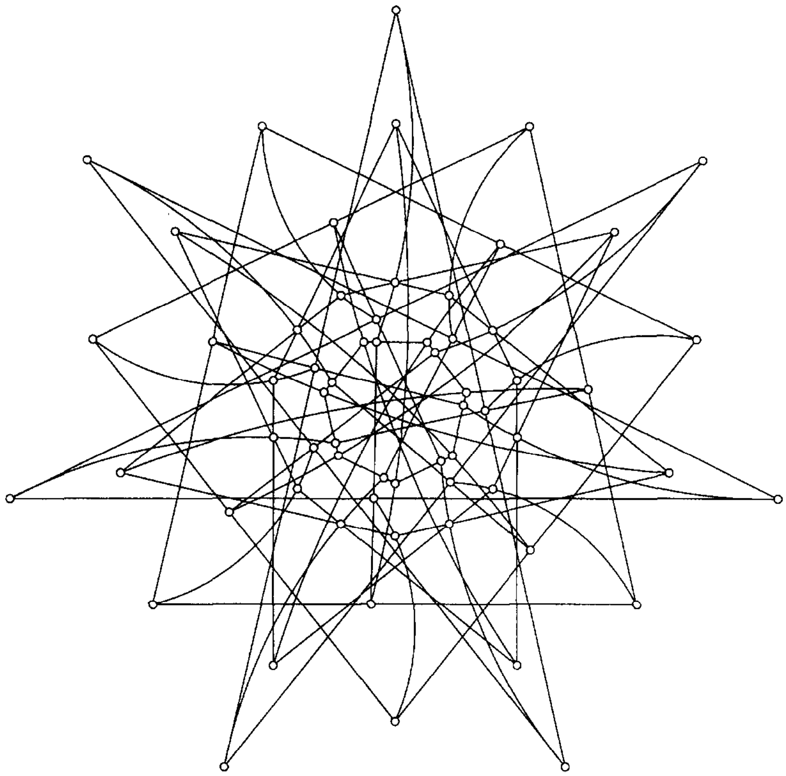

Fig. 9. The dual of the $G_2(2)$ hexagon

6. Drawing the hexagons

All information needed to draw the $G_2(2)$ hexagon and its dual is encoded in Table 2. We know already that both hexagons contain disjoint ordinary 7-gons of type 1, 2 and 3. Now, in the $G_2(2)$ hexagon every vertex of a 7-gon of type i is collinear with the opposite midpoint of the 7-gon of type $i+1$. (The types are enumerated modulo 3.) The connecting lines are B_k, D_k and F_k. In the dual hexagon every vertex of a 7-gon of type i is collinear with the opposite midpoint of the 7-gon of type $i-1$. The connecting lines are $\mathscr{L}_{b_k}$, $\mathscr{L}_{d_k}$ and $\mathscr{L}_{f_k}$. This indicates that the $G_2(2)$ hexagon is not self-dual.

But also these lines connecting vertices to midpoints behave differently in the $G_2(2)$ hexagon and its dual. In the $G_2(2)$ hexagon the lines B_k, D_k and F_k intersect. The common point is g_k. In the dual the corresponding lines $\mathscr{L}_{b_k}$, $\mathscr{L}_{d_k}$ and $\mathscr{L}_{f_k}$ do not intersect. However, the midpoints B_k, D_k and F_k are collinear.

The automorphism Δ together with the reflection σ where $\sigma(v_k)=v_{-k}$ if $k\in\{a,b,c,d,e,f,g\}$ and $\sigma(h_k)=i_{-k}$, $\sigma(i_k)=h_{-k}$ generate a subgroup of the automorphism group isomorphic to the dihedral group of order 14.

However, if one draws the points and lines considered so far in such a fashion that the dihedral group acts on the picture, then different lines intersect in intervals and can no longer be distinguished. More precisely, for any $0 \leqslant k \leqslant 6$ the lines B_k, D_k and F_k overlap. Thus, in the picture some of the 'midpoints' are slightly off the middle.

To complete the picture of the $G_2(2)$ hexagon the incidence graph of the Fano plane is added in such a fashion that the midpoints of the edges are glued to midpoints of the 7-gons. To avoid overlapping lines the edges are bent. The picture of the dual hexagon is completed analogously.

Fig. 8 depicts the $G_2(2)$ hexagon while Fig. 9 depicts the dual hexagon. If we use one more dimension, that is, if we construct spatial models of the hexagons, we can avoid overlapping lines by placing the different 7-gons in different but parallel planes. This produces models upon which the dihedral group acts. Stereograms of such models are presented on my homepage.

Acknowledgements

I acknowledge the contribution made by one of the referees. He or she made me realise the action of the dihedral group. This resulted in improved pictures.

References

[1] A.M. Cohen, J. Tits, A characterization by orders of the generalized hexagons and a near octagon where lines have length three, Eur. J. Combin. 6 (1985) 13–27.
[2] V. De Smet, H. Van Maldeghem, The finite Moufang hexagons coordinatized, Contrib. Algebra Geometry 34 (2) (1993) 217–232.
[3] W. Feit, G. Higman, The nonexistence of certain generalized polygons, J. Algebra 1 (1964) 114–131.
[4] S.E. Payne, Finite generalized quadrangles, a survey, in: Proc. Internat. Conf. on Projective Planes, Washington State University Press, Pullman, 1973, pp. 219–261.
[5] B. Polster, A Geometrical Picture Book, Springer, New York, 1998.
[6] J.A. Thas, Generalized polygons, in: F. Buekenhout (ed.), Handbook of Incidence Geometry: Buildings and Foundations, North-Holland, Amsterdam, 1995, pp. 383–432.
[7] J. Tits, Sur la trialité et certains groupes qui s'en déduisent, Publ. Math. I.H.E.S. 2 (1959) 13–60.
[8] H. Van Maldeghem, A finite generalized hexagon admitting a group acting transitively on ordered heptagons is classical, J. Combin. Theory Ser. A 75 (2) (1996) 254–269.
[9] H. Van Maldeghem, Generalized Polygons, Birkhäuser, Basel, 1998.