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Asymptotic Methods for Peristaltic Transport of a Heat-Conducting Fluid

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Two asymptotic methods based upon Stokes and long-wave approximations are developed for the study of transporting a heat-conducting fluid through a flexible tube by peristaltic motion of the tube well. The asymptotic methods are justified rigorously and the existence of a unique generalized solution of the governing equations is proved if a condition in terms of the Reynolds number and other nondimensional parameters is satisfied. © 1987 Academic Press, Inc.

1. INTRODUCTION

The transport of a fluid through a flexible tube by peristaltic motion as a fundamental physiological process has found many applications in biomechanical and engineering sciences. The early mathematical models for peristaltic transport were based upon the Navier–Stokes equations for an incompressible viscous fluid subject to a periodic transverse displacement of the tube wall. A survey of the research results up to 1971 can be found in the article by Jaffrin and Shapiro [1], and a review of the recent work on the transport of physiological fluids has been given by Winet [2]. It became evident that the early models were inadequate to deal with transport problems when a fluid–particle mixture was considered, and some refined models taking into account the motion of solid particles in a viscous fluid were studied by Hung and Brown [3] and Kaimal [4], among others. The asymptotic methods developed for the solution of the mixture transport problem was justified and the existence of a unique solution to the exact equations was proved, in [5].

More recently there has been growing interest in the effect of heat transfer upon the transport of a fluid in a flexible tube due to various technical or physiological considerations, for example, the transport of noxious

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waste through a tube [6, 7]. In this paper, we shall adopt the so-called Oberbeck–Boussinesq equations [8] to investigate the peristaltic transport of a heat-conducting fluid. Two asymptotic methods will be developed for the study of this problem. One method is an extension of Stokes approximation, in which the Reynolds number is assumed small and the nondimensional thermal and solute diffusivity coefficients are assumed large. The other method is developed within the framework of long-wave approximation. In both cases the governing equations are reduced to a sequence of linear elliptical problems. The contributions of this paper are the following. We show that there exists a unique solution to the exact equations if a criterion in terms of the Reynolds number, thermal and solute diffusivities, and other coefficients together with a flux condition is satisfied. The solutions of the sequence of elliptical problems are shown to be successive asymptotic approximations to the exact solution. The existence proof given here is an extension of that due to Ladyzhenskaya [9]. We also note here that the mixed problem of stationary Oberbeck–Boussinesq equations in a bounded domain has been studied by Mulone and Rionero [10].

In Section 2, we formulate the problem and specify various functions spaces to be used later. In Section 3, we develop the formal asymptotic methods based upon Stokes and long-wave approximations, In Section 4, the asymptotic methods are justified and the error estimates are expressed in terms of L_2-norms. Finally in Section 5, we prove the existence and uniqueness of a generalized solution to the governing equations.

2. Formulation

We consider the motion of a heat-conducting fluid governed by the Oberbeck–Boussinesq equations in a flexible tube subject to a prescribed peristaltic motion of the tube wall in the form of a progressive wave of periodic P^* with constant axial speed λ and wavelength L. In reference to a coordinate system (x^*, y^*, z^*) moving with the wave, the boundary of the tube is stationary and the equations governing the steady flow of the fluid are the following:

$$\nabla \cdot \bar{q}^* = 0 \tag{1}$$

$$\rho(\bar{q}^* \cdot \nabla^* \bar{q}^*) = -\nabla^* p^* + \rho g[1 - \alpha_T^*(T^* - T_0^*) - \alpha_c^*(C^* - C_0^*)]\,\bar{k} + \mu(\nabla^*)^2 \bar{q}^*, \tag{2}$$

$$\bar{q}^* \cdot \nabla^* T^* = k_T^*(\nabla^*)^2 T^* + Q_T^*(y^*, z^*), \tag{3}$$

$$\bar{q}^* \cdot \nabla^* C^* = k_c^*(\nabla^*)^2 C^* + Q_c^*(y^*, z^*); \tag{4}$$

at the boundary $H^*(x^*, y^*, z^*) = 0$,

$$\bar{q}^* = [-\lambda, (\lambda/L) f^*(x^*, y^*, z^*), (\lambda/L) g^*(x^*, y^*, z^*)], \tag{5}$$

$$T^* = T_w^*(y^*, z^*), \qquad C^* = C_w^*(y^*, z^*). \tag{6}$$

Furthermore, we assume that the flux through a cross section D^* of the tube is given, and let u^* be the velocity component in the x^*-direction. Then

$$\iint_{D^*} u^* \, dA = \text{constant}. \tag{7}$$

Here $\bar{q}^*$ is the velocity, ρ is the constant density, p^* is the pressure, T^* is the temperature, C^* is the solute concentration, Q_T^* and Q_c^* are heat and solute sources respectively, f^* and g^* are prescribed functions periodic in x^*, T_w^* and C_w^* are prescribed temperature and solute concentration at the wall respectively, and the gravitational acceleration g, the viscosity coefficient μ, the temperature and solute concentration gradients α_T^* and α_c^*, the thermal and solute diffusivity coefficients k_T^* and k_c^* are all assumed to be positive constants. We now measure x^*, y^*, z^*, f^*, and g^* in units of d, the maximum radius of the tube, $\bar{q}^*$ in units of λ, p^* in units of $\rho\lambda^2$, T^* in units of T_0^*, C^* in units of C_0^*, and Q_T^* and Q_c^* in units of $\lambda T_0/d$ and $\lambda C_0/d$, respectively. We also define

$$\alpha_T = gdT_0\alpha_T^*/\lambda^2 \qquad \alpha_c = gdC_0\alpha_c^*/\lambda^2,$$

$$\tau_T = \lambda d/k_T^*, \qquad \tau_c = \lambda d/k_c^*,$$

$$R = \rho\lambda d/\mu, \qquad \bar{q} = \bar{q}^*/\lambda + \bar{\imath}$$

$$p = p^* + [1 - (\alpha_T - \alpha_c)]z, \qquad \alpha = d/L,$$

and $\bar{\imath}$ and $\bar{k}$ are unit vectors in the x^* and z^* directions, respectively. In terms of nondimensional variables and parameters, (1)–(7) become

$$\nabla \cdot \bar{q} = 0 \tag{8}$$

$$-\bar{q}_x + \bar{q} \cdot \nabla\bar{q} = -\nabla p + (\alpha_T T - \alpha_c C)\bar{k} + R^{-1}\nabla^2\bar{q}, \tag{9}$$

$$-T_x + \bar{q} \cdot \nabla T = \tau_T^{-1}\nabla^2 T + Q_T(y, z), \tag{10}$$

$$-C_x + \bar{q} \cdot \nabla C = \tau_c^{-1}\nabla^2 C + Q_c(y, z); \tag{11}$$

at the boundary $H^*(x, y, z) = 0$,

$$\bar{q} = \bar{q}_b = (0, \alpha f, \alpha g), \tag{12}$$

$$T = T_w, \qquad C = C_w; \tag{13}$$

$$\iint_D (u-1)\, dA = M, \tag{14}$$

where $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, D is an open cross section of the tube, and M is a given constant.

In the following we introduce some function spaces for later use. Let $\dot{J}(\Omega)$ be the space of C^∞ solenoidal vector functions with zero flux through a cross section, a support in $\Omega \cup \Gamma$ and a period $P = P^*/d$ in x, where

$$\Omega = \{(x, y, z) \mid 0 < x < P, (y, z) \in D\},$$

and Γ is the union of the two end cross sections of the tube at $x = 0$, P. We consider the completion of $\dot{J}(\Omega)$ with respect to the L_2-norm $\|\cdot\|$ associated with the scalar product

$$(\bar{u}, \bar{v}) = \int_\Omega \bar{u} \cdot \bar{v}\, d\Omega.$$

Let $\bar{H}(\Omega)$ be the completion of $\dot{J}(\Omega)$ with respect to the norm $\|\cdot\|_{\bar{H}}$ associated with the scalar product

$$(\bar{u}, \bar{v})_{\bar{H}} = \int_\Omega \nabla\bar{u} \cdot \nabla\bar{v}\, d\Omega.$$

Similarly we also consider the completion of the C^∞ scalar functions with a support in $\Omega \cup \Gamma$ and the period P in x with respect to the L_2-norm and the norm $\|\cdot\|_H$ associated with the scalar product

$$(u, v)_H = \int_\Omega \nabla u \cdot \nabla v\, d\Omega.$$

The latter space is denoted by H. The space $\mathscr{H} = \bar{H} \times H \times H$ is the direct product of $\bar{H}$, H, and H with the scalar product

$$(U, V)_{\mathscr{H}} = (\bar{u}_1, \bar{v}_1)_{\bar{H}} + (u_2, v_2)_H + (u_3, v_3)_H,$$

where $U = (\bar{u}_1, u_2, u_3)$, $V = (\bar{v}_1, v_2, v_3)$, $\bar{u}_1, \bar{v}_1 \in \bar{H}$, and $u_2, v_2, u_3, v_3 \in H$. A generalized solution of (7)–(12) is defined as a triplet of functions $(\bar{q}, T, C)$ satisfying the following integral equalities:

$$\int_\Omega [-\bar{q}_x + \bar{q} \cdot \nabla\bar{q} - (\alpha_T T - \alpha_c C)\bar{k}] \cdot \bar{\phi}\, d\Omega + R^{-1} \int_\Omega \nabla\bar{q} \cdot \nabla\bar{\phi}\, d\Omega = 0,$$

$$\int_\Omega (-T_x + \bar{q} \cdot \nabla T - Q_T) \cdot \phi_T\, d\Omega + \tau_T^{-1} \int_\Omega \nabla T \cdot \nabla\phi_T\, d\Omega = 0, \tag{15}$$

$$\int_\Omega (-C_x + \bar{q} \cdot \nabla C - Q_C) \cdot \phi_c\, d\Omega + \tau_H^{-1} \int_\Omega \nabla C \cdot \nabla\phi_c\, d\Omega = 0,$$

for any $(\bar{\phi}, \phi_T, \phi_c) \in \mathscr{H}$ and for sufficiently smooth periodic functions $\bar{a}$, b, and c such that $\bar{q} - \bar{a} \in \bar{H}$, $T - b \in H$, $C - c \in H$, $\bar{a} = \bar{q}_b$, $b = T_w$, and $c = C_w$ on the boundary $H^* = 0$, and $\iint_D (\bar{a} \cdot \bar{i} - 1)\, dA = M$.

3. Formal Asymptotic Expansions

3.1. *Stokes approximation*

Here we essentially extend the derivation of Stokes equations from the Navier–Stokes equations for low Reynolds number flow. Assume that there is a small parameter ε, $0 < \varepsilon \ll 1$ such that R, τ_T, $\tau_c = O(\varepsilon)$. Furthermore, in order to take fully into account the effects of pressure change, gravitational force and heat, and solute sources, we also assume $p = O(\varepsilon^{-1})$, α_T, α_c, Q_T, $Q_c = O(\varepsilon^{-1})$. Without loss of generality, we may set $\alpha = 1$. If no confusion arises, we replace R, τ_T, τ_c by εR, $\varepsilon\tau_T$, $\varepsilon\tau_c$ and p, α_T, α_c, Q_T, Q_c by $\varepsilon^{-1}p$, $\varepsilon^{-1}\alpha_T$, $\varepsilon^{-1}\alpha_c$, $\varepsilon^{-1}Q_T$, $\varepsilon^{-1}Q_c$, respectively. Equations (8)–(13) become

$$\nabla \cdot \bar{q} = 0, \tag{16}$$

$$\varepsilon(-\bar{q}_x + \bar{q} \cdot \nabla \bar{q}) = -\nabla p + (\alpha_T T - \alpha_c C)\bar{k} + R^{-1}\nabla^2 q, \tag{17}$$

$$\varepsilon(-T_x + \bar{q} \cdot \nabla T) = \tau_T^{-1}\nabla^2 T + Q_T(y, z). \tag{18}$$

$$\varepsilon(-C_x + \bar{q} \cdot \nabla C) = \tau_c^{-1}\nabla^2 C + Q_c(y, z); \tag{19}$$

at the boundary $H^*(x, y, z) = 0$,

$$\bar{q} = \bar{q}_b = (0, f, g) \tag{20}$$

$$T = T_w(y, z), \qquad C = C_w(y, z). \tag{21}$$

We assume that $\bar{q}$, T, C, and p may be expanded in power series of ε:

$$\bar{q} = \bar{q}_0 + \varepsilon\bar{q}_1 + \varepsilon^2\bar{q}_2 + \cdots,$$
$$T = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots,$$
$$C = C_0 + \varepsilon C_1 + \varepsilon^2 C_2 + \cdots,$$
$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_1 + \cdots.$$

Substitution of the series in (14) and (16)–(21) will yield a sequence of approximate equations, boundary and flux conditions. The equations for the zeroth approximations are

$$\nabla \cdot \bar{q}_0 = 0, \tag{22}$$

$$R^{-1}\nabla^2 \bar{q}_0 - \nabla p_0 + (\alpha_T T_0 - \alpha_c C_0)\bar{k} = 0, \tag{23}$$

$$\tau_T^{-1}\nabla^2 T_0 = -Q_T, \tag{24}$$

$$\tau_c^{-1}\nabla^2 C_0 = -Q_c; \tag{25}$$

at the boundary $H^*(x, y, z) = 0$,

$$\bar{q}_0 = \bar{q}_b = (0, f, g), \qquad T = T_w, \qquad C = C_w, \tag{26}$$

and the flux condition

$$\iint_D (u_0 - 1)\, dA = M.$$

If $\alpha_T = \alpha_c = 0$, (22) and (23) are just the well-known Stokes equations. The equations for the nth approximations can be written down as follows:

$$\nabla \cdot \bar{q}_n = 0, \tag{27}$$

$$R^{-1}\nabla^2 \bar{q}_n - \nabla p_n + (\alpha_T T_n - \alpha_c C_n) = -\bar{q}_{n-1,x} + \sum_{j=0}^{n-1} \bar{q}_j \cdot \nabla \bar{q}_{n-j-1}, \tag{28}$$

$$\tau_T^{-1}\nabla^2 T_n = -T_{n-1,k} + \sum_{j=0}^{n-1} \bar{q}_j \cdot \nabla T_{n-j-1}, \tag{29}$$

$$\tau_c^{-1}\nabla^2 C_n = -C_{n-1,x} + \sum_{j=0}^{n-1} \bar{q}_j \cdot \nabla T_{n-j-1}; \tag{30}$$

at the boundary $H^*(x, y, z) = 0$,

$$\bar{q}_n = (0, 0, 0), \qquad T_n = C_n = 0, \tag{31}$$

and the flux condition

$$\iint_D (u_n)\, dA = 0.$$

In principle, (24) and (25) subject to the boundary conditions (26) can be solved respectively first. Then (22) and (23) with T_0 and C_0 given can be dealt with next. Finally, (27)–(31) will yield solutions for the successive approximations. A discussion of the existence and uniqueness of the solutions of (22), (23), (26) and (27), (28), (31) may be found in [11, 12].

3.2. *Long-wave Approximation*

As seen from (22) and (31), the equations obtained from Stokes approximation are defined in Ω and we have to solve a sequence of three-dimensional problems. To simplify them further, we make use of the so-called long-wave approximation, that is, the radius to wave length ratio α is a small parameter in addition to the Stokes approximations. Without loss of generality, we let $\alpha = \varepsilon$. The sequence of three-dimensional problems can now be reduced to two-dimensional problems over each cross section of the tube. As a consequence of the long-wave approximation, we also assume that $\partial/\partial x = O(\varepsilon)$; $v, w = O(\varepsilon)$; $p = O(\varepsilon^{-2})$. However, to have a consistent and justifiable asymptotic scheme, we need to assume that both α_T and α_c are $O(1)$. If no confusion arises, we replace $\partial/\partial x$ by $\epsilon\ \partial/\partial\xi$; v, w by εv; εw, p by $\varepsilon^{-1}p$; and (16)–(21) become

$$\nabla \cdot q = 0, \tag{32}$$

$$\varepsilon^2(-u_\xi + uu_\xi + vu_y + wu_z) = -p_\xi + R^{-1}(\varepsilon^2 u_{\xi\xi} + \nabla_2^2 u) \tag{33}$$

$$\varepsilon^4(-v_\xi + uv_\xi + vv_y + wv_z) = -p_y + R^{-1}(\varepsilon^4 v_{\xi\xi} + \varepsilon^2\nabla_2^2 v), \tag{34}$$

$$\varepsilon^4(-w_\xi + uw_\xi + vw_y + ww_z) = -p_z + \varepsilon^2(a_T T - \alpha_c C) + R^{-1}(\varepsilon^4 w_{\xi\xi} + \varepsilon^2\nabla_2^2 w), \tag{35}$$

$$\varepsilon^2(-T_\xi + \bar{q}\cdot\nabla T) = \tau_T^{-1}(\varepsilon^2 T_{\xi\xi} + \nabla_2^2 T) + Q_T(y, z), \tag{36}$$

$$\varepsilon^2(-C_\xi + \bar{q}\cdot\nabla C) = \tau_c^{-1}(\varepsilon^2 C_{\xi\xi} + \nabla_2^2 C) + Q_c(y, z), \tag{37}$$

at the boundary $H^*(x, y, z) = 0$,

$$\bar{q} = \bar{q}_b = (0, f, g), \tag{38}$$

$$T = T_w(y, z), \qquad C = C_w(y, z), \tag{39}$$

where $\nabla_2^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$. As seen from (32)–(39), all powers of ε are even, and we may expand $\bar{q}$, p, T, and C in an asymptotic series with even powers of ε,

$$\begin{aligned} \bar{q} &= \bar{q}_0 + \varepsilon^2\bar{q}_2 + \varepsilon^4\bar{q}_4 + \cdots, \\ p &= p_0 + \varepsilon^2 p_2 + \varepsilon^4 p_4 + \cdots, \\ T &= T_0 + \varepsilon^2 T_2 + \varepsilon^4 T_4 + \cdots, \\ C &= C_0 + \varepsilon^2 C_2 + \varepsilon^4 C_4 + \cdots. \end{aligned}$$

where $\bar{q}_i = (u_i, v_i, w_i)$, $i = 0, 2, 4$.

As before, we substitute the series in (32)–(39) to obtain a sequence of equations and boundary conditions for the successive approximations. The equations for the zeroth approximation are

$$u_{0\xi} + v_{0y} + w_{0z} = 0, \tag{40}$$

$$-R^{-1}\nabla_2^2 u_0 = p_{0x}, \tag{41}$$

$$p_{0y}=0, \qquad p_{0z}=0, \tag{42}$$

$$-\tau_T^{-1}\nabla_2^2 T_0 = Q_T, \tag{43}$$

$$-\tau_c^{-1}\nabla_2^2 C_0 = Q_c; \tag{44}$$

at the boundary

$$u_0=0, \qquad v_0=f, \qquad w_0=g, \tag{45}$$

$$T_0=T_w, \qquad C_0=C_w. \tag{46}$$

It is seen from (42) that p_0 is a function of ξ only and may be expressed as

$$u_0 = u_{00}\, p_{0\xi}. \tag{47}$$

From (41) and (45), it follows that u_{00} satisfies the equations

$$-R^{-1}\nabla_2^2 u_{00} = 1 \qquad \text{in } D, \tag{48}$$

$$u_{00}=0 \qquad \text{at } \partial D, \tag{49}$$

where ∂D is the boundary of D. By integrating (47) over a cross section D and making use of (14), we have

$$p_{0\xi} = (M+A)\left(\iint_D u_{00}\, dA\right)^{-1}, \tag{50}$$

where A is the area of the cross section D. It is easy to show that $\iint_D u_{00}\, dA > 0$ if $A > 0$.

The equations for the second approximation are

$$u_{2\xi}+v_{2y}+w_{2z}=0, \tag{51}$$

$$R^{-1}\nabla_2^2 u_2 = p_{2\xi} - R^{-1}u_{0\xi\xi} - u_{0\xi} + u_0 u_{0\xi} + v_0 u_{0y} + w_0 u_{0z}, \tag{52}$$

$$R^{-1}\nabla_2^2 v_0 = p_{2y}, \tag{53}$$

$$R^{-1}\nabla_2^2 w_0 = p_{2z} - (\alpha_T T_0 - \alpha_c C_0), \tag{54}$$

$$\tau_T^{-1}\nabla_2^2 T_2 = -T_{0\xi} + u_0 T_{0\xi} + v_0 T_{0y} + w_0 T_{0z}, \tag{55}$$

$$\tau_c^{-1}\nabla_2^2 C_2 = -C_{0\xi} + u_0 C_{0\xi} + v_0 C_{0y} + w_0 C_{0z}; \tag{56}$$

at the boundary $H^*(\xi, y, z)=0$,

$$u_2=v_2=w_2=0, \qquad T_2=C_2=0. \tag{57}$$

First we introduce an auxiliary function Q_0, which satisfies

$$\nabla_2^2 Q_0 = -u_{0\xi} \qquad \text{in } D, \tag{58}$$

$$Q_{0n} = fn_1 + gn_2 \qquad \text{at } \partial D, \tag{59}$$

where (n_1, n_2) is the unit outward normal at ∂D. The Neumann problem posed by (58) and (59) is solvable because of (40) and (45). Hence from (40) and (58),

$$(v_0 - Q_{0y})_y + (w_0 - Q_{0z})_z = 0, \tag{60}$$

and (60) implies that we may define a stream functions Φ_0 such that

$$\Phi_{0z} = v_0 - Q_{0y}, \qquad \Phi_{0y} = -w_0 + Q_{0z}. \tag{61}$$

Upon using (61) in (53) and (54) to replace v_0 and w_0 in terms of Φ_0 and Q_0, cross-differentiating the resulting equations, and making use of (45), we obtain

$$\nabla_2^4 \Phi_0 = \alpha_T T_{0y} - \alpha_c C_{0y} \qquad \text{in } D, \tag{62}$$

$$\Phi_{0z} = f - Q_{0y}, \qquad P_{0y} = -g + Q_{0z} \qquad \text{at } \partial D. \tag{63}$$

Once Q_0 and Φ_0 are obtained form (58), (59), (62), and (63), we can determine v_0 and w_0 from (61). Furthermore, by integrating (53) and (54) along some smooth path Γ, say from (0, 0) to (y, z), in D, it follows that

$$p_2 = \int_\Gamma \nabla_2^2 v_0 \, dy + (\nabla_2^2 w_0 + \alpha_T T_0 - \alpha_c C_0) \, dz + p_{20}(\xi), \tag{64}$$

where $p_{20}(\xi)$ is a function of ξ only to be determined later. We may express u_2 as

$$u_2 = u_{00} \, p_{20}(\xi) + u_{21}. \tag{65}$$

Then from (52) and (57), we have

$$\begin{aligned} \nabla_2^2 u_{21} &= \int_\Gamma \nabla_2^2 v_{2\xi} \, dy + (\nabla_2^2 w_{0\xi} + \alpha_T T_{0\xi} - \alpha_c C_{0\xi}) \, dz \\ &\quad - R^{-1} u_{0\xi\xi} - u_{0\xi} + u_0 u_{0\xi} + v_0 u_{0y} + w_0 u_{0z} \qquad \text{in } D, \\ u_{21} &= 0 \qquad \text{at } \partial D. \end{aligned} \tag{66}$$

To determine $p_{20}(\xi)$, we make use of (14) which implies

$$\iint_D u_2 \, dA = 0, \tag{67}$$

and integrate (65) over D to obtain

$$p_{20\xi} = -\iint_D u_{21} \, dA \left(\iint_D u_{00} \, dA \right)^{-1}. \tag{68}$$

Finally, v_2, w_2 can be determined from (51), (57), and the equations for the fourth approximation

$$\nabla_2^2 v_2 = p_{4y} - v_{0\xi\xi} - v_{0\xi} + u_0 v_{0\xi} + v_0 v_{0y} + w_0 v_{0z},$$

$$\nabla_2^2 w_2 = p_{4z} - (\alpha_T T_2 - \alpha_c C_2) - w_{0\xi\xi} - w_{0\xi} + u_0 w_{0\xi} + v_0 w_{0y} + w_0 w_{0z}.$$

As before, we may introduce Q_2 satisfying

$$\nabla_2^2 Q_2 = -u_{2\xi} \qquad \text{in } D,$$

$$Q_{2n} = 0 \qquad \text{at } \partial D.$$

Then from (51), (57), and the equations for v_2 and w_2 we can find Φ_2 such that

$$\Phi_{2z} = v_2 - Q_{2y}, \qquad \Phi_{2y} = -w_2 + Q_{2y},$$

where Φ_2 satisfies

$$\nabla^4 \Phi_2 = (-v_{0\xi\xi} - v_{0\xi} + u_0 v_{0\xi} + v_0 v_{0y} + w_0 v_{0z})_z + (\alpha_c C_2)_y$$
$$+ (w_{0\xi\xi} + w_{0\xi} - u_0 w_{0\xi} - v_0 w_{0y} - v_0 w_{0y} - w_0 w_{0z})_z \qquad \text{in } D,$$
$$\Phi_{2z} = -Q_{2y}, \qquad \Phi_{2y} = Q_{2z} \qquad \text{at } \partial D.$$

Higher order approximations can be determined similarly and we shall not proceed any further.

4. Justification of the Asymptotic Expansions

In this section, assuming the existence of a generalized solution of (8) to (14), we show that both asymptotic expansions developed in Section 3 are asymptotic approximations to the exact solution in the L_2 sense.

4.1. *Stokes Expansion*

Let

$$\bar{q} = \bar{q}_0 + \varepsilon \bar{q}_1 + \bar{q}_*, \tag{69}$$

$$T = T_0 + \varepsilon T_1 + T_*, \tag{70}$$

$$C = C_0 - \varepsilon C_1 + C_*. \tag{71}$$

We substitute (69)–(71) in (15) and obtain

$$\int_\Omega [-\bar{q}_{*x} + (\bar{q}_0 + \varepsilon\bar{q}_1 + \bar{q}_*)\cdot\nabla\bar{q}_* + \bar{q}_x\cdot\nabla(\bar{q}_0 + \varepsilon\bar{q}_1)$$

$$+ (\varepsilon^{-1}\alpha_T T_* - \varepsilon^{-1}a_c C_*)\bar{k} - \bar{G}_1]\cdot\Phi\, d\Omega + (\varepsilon R)^{-1}\int_\Omega \nabla\bar{q}_*\cdot\nabla\Phi\, d\Omega = 0. \tag{72}$$

$$\int_\Omega [-T_{*x} + (\bar{q}_0 + \varepsilon\bar{q}_1 + \bar{q}_*)\cdot\nabla T_* + \bar{q}_*\cdot\nabla(T_0 + \varepsilon T_1) - G_2]$$

$$\times\phi_T\, d\Omega + (\varepsilon\tau_T)^{-1}\int_\Omega \nabla T_*\cdot\nabla\phi_T\, d\Omega = 0 \tag{73}$$

$$\int_\Omega [-C_{*x} + (\bar{q}_0 + \varepsilon\bar{q}_1 + \bar{q}_1 + \bar{q}_*)\cdot\nabla C_* + q_*\cdot\nabla(C_0 + \varepsilon C_1) - G_3]$$

$$\times\phi_c\, d\Omega + (\varepsilon\tau_c)^{-1}\int_\Omega \nabla C_*\cdot\nabla\phi_c\, d\Omega = 0, \tag{74}$$

where

$$\bar{G}_1 = (\bar{q}_0 + \varepsilon\bar{q}_1)_x - (\bar{q}_0 + \varepsilon\bar{q}_1)\cdot\nabla(\bar{q}_0 + \varepsilon\bar{q}_1)$$
$$+ \bar{k}[\varepsilon^{-1}\alpha_T(T_0 + \varepsilon T_1) - \varepsilon^{-1}\alpha_c(C_0 + \varepsilon C_1)] + (\varepsilon R)^{-1}\nabla^2(\bar{q}_0 + \varepsilon\bar{q}_1). \tag{75}$$

$$G_2 = (T_0 + \varepsilon T_1)_x - (\bar{q}_0 + \varepsilon\bar{q}_1)\cdot\nabla(T_0 + \varepsilon T_1)$$
$$+ (\varepsilon\tau_T)^{-1}\nabla^2(T_0 + \varepsilon T_1) + \varepsilon^{-1}Q_T, \tag{76}$$

$$G_3 = (C_0 + \varepsilon C_1)_x - (\bar{q}_0 + \varepsilon\bar{q}_1)\cdot\nabla(C_0 + \varepsilon C_1)$$
$$+ (\varepsilon\tau_c)^{-1}\nabla^2(C_0 + \varepsilon C_1) + \varepsilon^{-1}Q_c. \tag{77}$$

We choose $(\bar{\phi}, \phi_T, \phi_c) = (\bar{q}_*, T_*, C_*)$ in (72)–(74), performs integration by parts, and make use of the periodicity property of $\bar{q}_*$, T_*, and C_* to obtain

$$\int_\Omega \bar{q}_{*x}\, d\Omega = 0.$$

$$\int_\Omega (\bar{q}_0 + \varepsilon\bar{q}_1 + \bar{q}_*)\cdot\nabla\bar{\phi}\cdot\bar{\phi}\, d\Omega = 0,$$

$$\int_\Omega (\bar{q}_0 + \varepsilon\bar{q}_1 + \bar{q}_*)\cdot\nabla\phi\cdot\phi\, d\Omega = 0,$$

where ϕ stands for ϕ_c or ϕ_T. Hence (72)–(74) become

$$\|\bar{q}_*\|_{\bar{H}}^2 = \varepsilon R \left[\int_\Omega \bar{G}_1 \cdot \bar{q}_* \, d\Omega - \int_\Omega \bar{q}_* \cdot \nabla(\bar{q}_0 + \rho \bar{q}_1) \cdot \bar{q}_* \, d\Omega \right.$$

$$\left. - \int_\Omega \varepsilon^{-1}(\alpha_T T_* - \alpha_c C_*)\bar{k} \cdot \bar{q}_* \, d\Omega \right]. \tag{78}$$

$$\|T_*\|_H^2 = \varepsilon \tau_T \left[\int_\Omega G_2 T_* \, d\Omega - \int_\Omega \bar{q}_* \cdot \nabla(T_0 + \varepsilon T_1) T_* \, d\Omega \right], \tag{79}$$

$$\|C_*\|_H^2 = \varepsilon \tau_T \left[\int_\Omega G_3 C_* \, d\Omega - \int_\Omega \bar{q}_* \cdot \nabla(C_0 + \varepsilon C_1 \; C_* \, d\Omega \right], \tag{80}$$

In the following we establish some lemmas for the estimates of the right-hand sides of (78)–(80).

LEMMA 1.

$$\left| \int_\Omega \bar{G}_1 \cdot \bar{q}_* \, d\Omega \right| \leqslant \varepsilon K \|\bar{q}_*\|_{\bar{H}},$$

$$\left| \int_\Omega G_2 T_* \, d\Omega \right| \leqslant \varepsilon K \|T_*\|_H,$$

$$\left| \int_\Omega G_3 \, C_* \, d\Omega \right| \leqslant \varepsilon K \|C_*\|_H,$$

where K is a generic constant depending upon the zeroth and first approximations of $\bar{q}$, T, and C.

Proof. It is found by (22)–(25) and (27)–(30) that

$$\bar{G}_1 - \varepsilon^{-1}\nabla p_0 - \nabla p_1 = \varepsilon(\bar{q}_{1,x} - \bar{q}_1 \cdot \nabla \bar{q}_0 - \bar{q}_0 \cdot \nabla \bar{q}_1 - \varepsilon \bar{q}_1 \cdot \nabla \bar{q}_1),$$

$$G_2 = \varepsilon(T_{1,x} - \bar{q}_0 \cdot \nabla T_1 - \bar{q}_1 \cdot \nabla T_0 - \varepsilon \bar{q}_1 \cdot \nabla T_1),$$

$$G_3 = \varepsilon(C_{1,x} - \bar{q}_0 \cdot \nabla C_1 - \bar{q}_1 \cdot \nabla C_0 - \varepsilon \bar{q}_1 \cdot \nabla C_1).$$

We assume that the boundary conditions (26) and the heat and solute sources are sufficiently smooth so that $\bar{q}_0$, $\bar{q}_1 \in W_2^2(\Omega)$, ∇p_0, $\nabla p_1 \in L_2(\Omega)$, and C_0, C_1, T_0, $T_1 \in w_2^1(\Omega)$ [11]. First we estimate $|\int_\Omega \bar{G}_1 \cdot \bar{q}_* \, d\Omega|$. Note that ∇p_0 and ∇p are orthogonal to $\bar{q}_*$. Therefore,

$$\left| \int_\Omega \bar{G}_1 \cdot \bar{q}_* \, d\Omega \right| = \left| \int_\Omega (\bar{G}_1 - \bar{\varepsilon}^1 \nabla p_0 - \nabla p_1) \cdot \bar{q}_* \, d\Omega \right|$$

$$= \varepsilon \left| \int_\Omega (\bar{q}_{1,x} - \bar{q}_1 \cdot \nabla \bar{q}_0 - \bar{q}_0 \cdot \nabla \bar{q}_1 - \varepsilon \bar{q}_1 \cdot \nabla \bar{q}_1) \cdot \bar{q}_* \, d\Omega \right|$$

$$\leqslant \varepsilon k_1 (\|\bar{q}_1\|_{\bar{H}} + 2\|\bar{q}_1\|_{\bar{H}} \|\bar{q}_0\|_{\bar{H}} + \|\bar{q}_1\|_{\bar{H}}^2) \|\bar{q}_*\|_{\bar{H}}$$

$$\leqslant \in \bar{K} \|\bar{q}_*\|_{\bar{H}},$$

where we have used the Poincaré inequality, Sobolev inequalities, and integration by parts. Similarly,

$$\left|\int_\Omega G_2 T_* \, d\Omega\right| \leqslant \varepsilon k_2(\|T_1\|_H + \|\bar{q}_0\|_{\bar{H}} \|T_1\|_H + \|\bar{q}_1\|_{\bar{H}} \|T_0\|_H + \|\bar{q}_1\|_{\bar{H}} \|T_1\|_H) \|T_*\|_H \leqslant \varepsilon K \|T_*\|_H,$$

$$\left|\int_\Omega G_3 C_* \, d\Omega\right| \leqslant \varepsilon k_3(\|C_1\|_H + \|\bar{q}_0\|_{\bar{H}} \|C\|_H + \|\bar{q}_1\|_{\bar{H}} \|C_0\|_H + \|\bar{q}_1\|_{\bar{H}} \|C_1\|_H) \|C_1\|_H \leqslant \varepsilon K \|C_*\|_H.$$

LEMMA 2.

$$\left|\int_\Omega \bar{q}_* \cdot \nabla(\bar{q}_0 + \varepsilon \bar{q}_1) \cdot \bar{q}_* \, d\Omega\right| \leqslant k \|\bar{q}_0 + \varepsilon \bar{q}_1\|_{\bar{H}} \|\bar{q}_*\|_{\bar{H}}^2,$$

$$\left|\int_\Omega \bar{q}_* \cdot \nabla(T_0 + \varepsilon T_1)\, T_* \, d\Omega\right| \leqslant k \|T_0 + \varepsilon T_1\|_{\bar{H}} \|\bar{q}_*\|_{\bar{H}} \|T_*\|_H,$$

$$\left|\int_\Omega \bar{q}_* \cdot \nabla(C_0 + \varepsilon C_1)\, C_* \, d\Omega\right| \leqslant k \|C_0 + \varepsilon C_1\|_{\bar{H}} \|\bar{q}_*\|_{\bar{H}} C_*\|_H,$$

where k is a constant.

The proof of this lemma is similar to the proof of Lemma 1.

THEOREM 1. *If* $1 - K_*^2 R(\alpha_T + \alpha_c)/2 - \varepsilon k(R \|\bar{q}_0 + \varepsilon \bar{q}_1\|_{\bar{H}} + \tau_T \|T_0 + \varepsilon T_1\|_H + \tau_c \|C_0 + \varepsilon C_1\|_H) > 0$, *then*

$$\|Q_*\|_{\mathscr{H}} = O(\varepsilon^2),$$

where $Q_* = (\bar{q}_*, T_*, C_*)$ and $\|Q_*\|_{\mathscr{H}}^2 = \|\bar{q}_*\|_{\bar{H}}^2 + \|T_*\|_H^2 + \|C_*\|_H^2$.

Proof. From (78)–(80), Lemmas 1 and 2, and the Poincaré inequality, we have

$$\|\bar{q}_*\|_{\bar{H}}^2 \leqslant \varepsilon^2 RK \|\bar{q}_*\|_{\bar{H}} + \varepsilon RK \|\bar{q}_0 + \varepsilon \bar{q}_1\|_{\bar{H}} \|\bar{q}_*\|_{\bar{H}}^2 + K_*^2 (R\alpha_T \|T_*\|_H + R\alpha_c \|C_*\|_H) \|\bar{q}_*\|_{\bar{H}},$$

$$\|T_*\|_H^2 \leqslant \varepsilon^2 \tau_T K \|T_*\|_H + \varepsilon \tau_T k \|T_0 + \varepsilon T_1\|_H \|\bar{q}_*\|_{\bar{H}} T_*\|_H,$$

$$\|C_*\|_H^2 \leqslant \varepsilon^2 \tau_T K \|C_*\|_H + \varepsilon \tau_c k \|C_0 + \varepsilon C_1\|_H \|\bar{q}_*\|_{\bar{H}} \|C_*\|_H;$$

by adding, replacing $\|q_*\|_{\bar{H}}$, $\|T_*\|_H$, $\|C_*\|_H$ by $\|Q_*\|_{\mathscr{H}}$ on the right-hand side, and using $ab \leqslant (a^2+b^2)/2$,

$$\begin{aligned}\|Q_*\|^2_{\mathscr{H}} \leqslant{}& \varepsilon^2(R+\tau_T+\tau_c)\,K\|Q_*\|_{\mathscr{H}} \\ &+ (K_*^2 R(\alpha_T+\alpha_c)/2 + \varepsilon k(R\,\|\bar{q}_i+\varepsilon\bar{q}_1\|_{\bar{H}} + \tau_T\|T_0+\varepsilon T_1\|_H \\ &+ \tau_c\|C_0+\varepsilon C_1\|_H)\,\|Q_*\|^2_{\mathscr{H}}.\end{aligned}$$

This shows that if

$$\begin{aligned}1 - K_*^2 R(\alpha_T+\alpha_c)/2 - \varepsilon k(R\,\|\bar{q}_0+\varepsilon\bar{q}_1\|_{\bar{H}} + \tau_T\|T_0+\varepsilon T_1\|_H \\ + \tau_c\|C_0+\varepsilon C_1\|_H) > 0,\end{aligned}$$

$$\|Q_*\|_{\mathscr{H}} = O(\varepsilon^2).$$

We note that the constant K_* appears in the Poincaré inequality $\|\phi\| \leqslant K_*\,\|\phi\|_H$.

Remarks. Theorem 1 indeed has established that up to the first order

$$\begin{aligned}\|\bar{q}-\bar{q}_0+\varepsilon\bar{q}_1\| &= O(\varepsilon^2),\\ \|T-T_0-\varepsilon T_1\| &= O(\varepsilon^2),\\ \|C-C_0-\varepsilon C_1\| &= O(\varepsilon^2).\end{aligned}$$

By following the same argument, it can be shown that

$$\begin{aligned}\left\|\bar{q}-\sum_{k=0}^{n-1}\varepsilon^k\bar{q}_k\right\| &= O(\varepsilon^n)\\ \left\|T-\sum_{k=0}^{n-1}\varepsilon^k T_k\right\| &= O(\varepsilon^n),\\ \left\|C-\sum_{k=0}^{n-1}\varepsilon^k C_k\right\| &= O(\varepsilon^n),\end{aligned}$$

if $1-K_*^2 R\alpha_T+\alpha_c)/2>0$ and ε is sufficiently small.

4.2. *Long-wave Expansion*

The justification of long-wave approximation is similar to the one given in Section 4.1. However, there are also some differences between both cases. Here we assume that α_T and α_c are of $O(1)$ instead of $O(\varepsilon^{-1})$. Furthermore, the wave length of the peristaltic motion is long, that is, the nondimensional period $P = 1/\varepsilon$, and $v, w = O(\varepsilon)$ and $p = O(\varepsilon^{-2})$. As before, we express

$$\bar{q} = \bar{q}_0 + \varepsilon^2 \bar{q}_2 + \bar{q}_*,$$
$$T = T_0 + \varepsilon^2 T_2 + T_*,$$
$$C = C_0 + \varepsilon^2 C_2 + C_*,$$
$$\bar{q} = (u, \varepsilon v, \varepsilon w), \qquad \bar{q}_0 = (u_0, \varepsilon v_0, \varepsilon w_0),$$
$$\bar{q}_2 = (u_2, \varepsilon u_2, \varepsilon w_2), \qquad \bar{q}_* = (u_*, v_*, w_*).$$

By substituting (69)–(71) in (15) and using the assumptions in the long-wave approximation, the resulting equalities are

$$\int_{\Omega_*} [-\varepsilon \bar{q}_{*\xi} + (\bar{q}_0 + \varepsilon^2 \bar{q}_2 + \bar{q}_*) \cdot \nabla_* \bar{q}_* + \bar{q}_* \cdot \nabla_* (\bar{q}_0 + \varepsilon^2 \bar{q}_2)$$
$$+ (\alpha_T T_* - \alpha_c C_*) \bar{k} - \bar{G}_{1*}] \cdot \bar{\phi} \, d\Omega_* + (\varepsilon R)^{-1} \int_{\Omega_*} \nabla_* \bar{q}_* \cdot \nabla_* \bar{\phi} \, d\Omega_* = 0, \tag{81}$$

$$\int_{\Omega_*} [-\varepsilon T_{*\xi} + (\bar{q}_0 + \varepsilon^2 \bar{q}_2 + \bar{q}_*) \cdot \nabla_* T_* + \bar{q}_* \cdot \nabla (T_0 + \varepsilon^2 T_2) - G_{2*}] \cdot \phi_T \, d\Omega_*$$
$$+ (\varepsilon \tau_T)^{-1} \int_{\Omega_*} \nabla_* T_* \cdot \nabla_* \phi_T \, d\Omega_* = 0, \tag{82}$$

$$\int_{\Omega_*} [-\varepsilon C_{*\xi} + (\bar{q}_0 + \varepsilon^2 \bar{q}_2 + \bar{q}_*) \cdot \nabla_* C_* + \bar{q}_* \cdot \nabla (C_0 + \varepsilon^2 C_2) - G_{3*}] \cdot \phi_c \, d\Omega_*$$
$$+ (\varepsilon \tau_c)^{-1} \int_{\Omega_*} \nabla_* C_* \cdot \nabla_* \phi_c \, d\Omega_* = 0, \tag{83}$$

where

$$\Omega_* = \{(\xi, y, z) \mid 0 < \xi < 1, (y, z) \in D\},$$
$$\nabla_* = \varepsilon \frac{\partial}{\partial \xi} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$
$$\bar{G}_{1*} = \varepsilon(\bar{q}_0 + \varepsilon^2 \bar{q}_2)_\xi - (\bar{q}_0 + \varepsilon^2 \bar{q}_2) \cdot \nabla_* (\bar{q}_0 + \varepsilon^2 \bar{q}_2)$$
$$+ [\alpha_T (T_0 + \varepsilon^2 T_2) - \alpha_c (C_0 + \varepsilon^2 C_2)] + (\varepsilon R)^{-1} \nabla_*^2 (\bar{q}_0 + \varepsilon^2 \bar{q}_2),$$
$$G_{2*} = \varepsilon (T_0 + \varepsilon^2 T_2)_\xi - (\bar{q}_0 + \varepsilon^2 \bar{q}_2) \cdot \nabla_* (T_0 + \varepsilon^2 T_2)$$
$$+ (\varepsilon \tau_T)^{-1} \nabla_*^2 (T_0 + \varepsilon^2 T_2) + \varepsilon^{-1} Q_T,$$

$$G_{3*} = \varepsilon(C_0 + \varepsilon^2 C_2)_\xi - (\bar{q}_0 + \varepsilon^2 \bar{q}_2) \cdot \nabla_*(C_0 + \varepsilon^2 C_2)$$
$$+ (\varepsilon\tau_c)^{-1}\nabla_*^2(C_0 + \varepsilon^2 C_2) + \varepsilon^{-1} Q_c.$$

Let $(\bar{\phi}_1, \phi_T, \phi_2) = (\bar{q}_*, T_*, C_*)$ in (81)–(83), and we obtain

$$\|\bar{q}_*\|^2_{\bar{H}_*} = \varepsilon R\left[\int_{\Omega_*} \bar{G}_{1*} \cdot \bar{q}_*\, d\Omega_* - \int_{\Omega_*} \bar{q}_* \cdot \nabla_*(\bar{q}_0 + \varepsilon^2 \bar{q}_2) \cdot \bar{q}_*\, d\Omega_* - \int_{\Omega_*} (\alpha_T T_* - \alpha_c C_*)\bar{k} \cdot \bar{q}_*\, d\Omega_*\right], \tag{84}$$

$$\|T_*\|^2_{H_*} = \varepsilon\tau_T\left[\int_{\Omega_*} G_{2*} T_*\, d\Omega_* - \int_{\Omega_*} \bar{q}_* {}^*\nabla_*(T_0 + \varepsilon^2 T_2)\, T_*\, d\Omega_*\right], \tag{85}$$

$$\|C_*\|^2_{H^*} = \varepsilon\tau_c\left[\int_{\Omega_*} G_{3*} C_*\, d\Omega_* - \int_{\Omega_*} \bar{q}_* \cdot \nabla_*(C_0 + \varepsilon^2 C_2)\, C_*\, d\Omega_*\right], \tag{86}$$

where $\bar{H}_*$, H_* denote spaces of functions defined on Ω_*, and

$$\|\bar{q}_*\|^2_{\bar{H}_*} = \int_{\Omega_*} \nabla_* \bar{q}_* \cdot \nabla_* \bar{q}_*\, d\Omega_*.$$

$$\|T_*\|^2_{H_*} = \int_{\Omega_*} \nabla_* T_* \cdot \nabla_* T_*\, d\Omega_*,$$

$$\|C_*\|^2_{H_*} = \int_{\Omega_*} \nabla_* C_* \cdot \nabla_* C_*\, d\Omega_*.$$

LEMMA 3.

$$\left|\int_{\Omega_*} \bar{G}_{1*} \cdot \bar{q}_*\, d\Omega_*\right| \leqslant K\varepsilon^2 \|\bar{q}_*\|_{\bar{H}_*},$$

$$\left|\int_{\Omega_*} G_{2*} \cdot T_*\, d\Omega_*\right| \leqslant K\varepsilon^2 \|T_*\|_{H_*},$$

$$\left|\int_{\Omega_*} G_{3*} \cdot C_*\, d\Omega_*\right| \leqslant K\varepsilon^2 \|C_*\|_{H_*}.$$

Proof. As in Lemma 1, we replace $\bar{G}_{1*}$ by $\bar{G}_{1*} - \varepsilon^{-2}\nabla_* p_0 - \nabla_* p_2$ and obtain from (40)–(44 and (51)–(56) that

$$\bar{G}_{1*} = \varepsilon^2[(\bar{q}_0 - \bar{i}u_0 + \varepsilon^2 \bar{q}_2)_\xi - (\bar{q}_0 + \varepsilon^2 \bar{q}_2) \cdot \nabla_* \bar{q}_2$$
$$- \bar{q}_2 \cdot \nabla \bar{q}_0 + (\alpha_T T_2 - \alpha_c C_2) + R^{-1}(\bar{q}_{2\xi\xi} + \nabla_2^2 \bar{q}_2)$$
$$+ (\bar{q}_0 - \bar{i}u_0)_{\xi\xi}],$$

$$G_{2*} = \varepsilon^2[-\bar{q}_* \cdot \nabla_*(T_0 + \varepsilon^2 T_2) - \bar{q}_0 \cdot \nabla_* T_2 + \varepsilon T_{2\xi}],$$
$$G_{3*} = \varepsilon^2[-\bar{q}_* \cdot \nabla_*(C_0 + \varepsilon^2 C_2) - \bar{q}_i \cdot \nabla_* C_2 + \varepsilon C_{2\xi}],$$

where $\bar{i}$ is the unit vector in the ξ-direction. We make the same assumption as in Section 4.1 that $\bar{q}_0, \bar{q}_2 \in W_2^2(\Omega)$, ∇p_0, $\nabla p_2 \in L_2(\Omega_*)$, T_0, T_2, C_0, and $C_2 \in W_2^1(\Omega_*)$. Then by integration by parts, periodicity, and the Poincaré inequality $\|\bar{\phi}\|_* \leqslant K_* \|\bar{\phi}\|_{\bar{H}^*}$, where $\|\bar{\phi}\|_*$ is the norm for $L_2(\Omega_*)$, it is shown that

$$\left|\int_{\Omega^*} \bar{G}_{1*} \cdot \bar{q}_* \, d\Omega_*\right| \leqslant \left|\int_{\Omega^*} (\bar{G}_{1*} - \varepsilon^{\ 2}\nabla p_0 - \nabla p_2) \cdot \bar{q}_* \, d\Omega_*\right|$$
$$\leqslant K\varepsilon^2 \|\bar{q}_*\|_{\bar{H}^*}.$$

Similarly, we also obtain

$$\left|\int_{\Omega^*} G_{2*} \cdot T_* \, d\Omega_*\right| \leqslant K\varepsilon^2 \|T_*\|_{H^*},$$
$$\left|\int_{\Omega^*} G_{3*} C_* \, d\Omega_*\right| \leqslant K\varepsilon^2 \|C_*\|_{H^*},$$

where K depends upon $\bar{q}_0$, $\bar{q}_2$, T_0, T_2, C_0, and C_1.

LEMMA 4.

$$\left|\int_{\Omega_*} \bar{q}_* \cdot \nabla_*(\bar{q}_0 + \varepsilon^2 \bar{q}_2) \cdot \bar{q}_* \, d\Omega_*\right| \leqslant \varepsilon^{-1/2} K \|\bar{q}_0 + \varepsilon^2 \bar{q}_2\|_{\bar{H}^*} \|\bar{q}_*\|_{\bar{H}^*},$$
$$\left|\int_{\Omega_*} \bar{q}_* \cdot \nabla_*(T_0 + \varepsilon^2 T_2)\, T_* \, d\Omega_*\right| \leqslant \varepsilon^{-1/2} K \|T_0 + \varepsilon^2 T_2\|_{\bar{H}_*} \|\bar{q}_*\|_{\bar{H}_*} \|T_*\|_{H_*},$$
$$\left|\int_{\Omega_*} \bar{q}_* \cdot \nabla_*(C_0 + \varepsilon^2 C_2)\, C_* \, d\Omega_*\right| \leqslant \varepsilon^{-1/2} K \|T_0 + \varepsilon^2 T_2\|_{\bar{H}_*} \|\bar{q}_*\|_{\bar{H}_*} \|C_*\|_{H_*}.$$

Proof.

$$\left|\int_{\Omega_*} \bar{q}_* \cdot \nabla_*(\bar{q}_0 + \varepsilon^2 \bar{q}_2) \cdot \bar{q}_* \, d\Omega_*\right|$$
$$= \left|\int_{\Omega_*} \bar{q}_* \cdot [(\bar{q}_0 + \varepsilon^2 \bar{q}_2) \cdot \nabla_* \bar{q}_*] \, d\Omega_*\right.$$
$$\leqslant \|\bar{q}_*\|_{L_4^*} \|\bar{q}_0 + \varepsilon^2 \bar{q}_2\|_{L_4^*} \|\bar{q}_*\|_{\bar{H}_*}$$
$$\leqslant K\varepsilon^{-1/2} \|\bar{q}_0 + \varepsilon^2 \bar{q}_2\|_{\bar{H}_*} \|\bar{q}_*\|_{\bar{H}_*}^2,$$

where $\|\bar{q}_*\|_{L_4^*} \leqslant K_1 \varepsilon^{-1/4} \|\bar{q}_*\|_{\bar{H}^*}$ as a consequence of the inequality $\|q_*\|_{L_4} \leqslant K_1 \|\bar{q}_*\|_{\bar{H}}$ by changing x to ξ/ε and L_4^* is the space of L_4-functions on Ω_*. The proof of the remaining two inequalities are similar.

THEOREM 2. *If*

$$1 - \varepsilon^{1/2} K(R \|\bar{q}_0 + \varepsilon^2 \bar{q}_2\|_{\bar{H}_*} + \tau_T \|T_0 + \varepsilon^2 T_2\|_{H_*} + \tau_c \|C_0 + \varepsilon^2 C_2\|_{H_*} - \varepsilon K_* R[\alpha_T + \alpha_\tau)/2] > 0,$$

then, for $Q_* = (\bar{q}_*, T_*, C_*) \in \mathscr{H}_* = \bar{H}_* \times H_* \times H_*$

$$\|Q_*\|_{\mathscr{H}_*} = (\|\bar{q}_*\|^2_{\bar{H}_*} + \|T_*\|_{H_*} + \|C_*\|^2_{H_*})^{1/2} = O(\varepsilon^3).$$

Proof. By adding (84) to (86) and making use of Lemmas 3 and 4 it is obtained that

$$\begin{aligned}\|Q_*\|^2_{\mathscr{H}_*} \leqslant \varepsilon^3 K(R + \tau_T + \tau_c) \|Q_*\|_{\mathscr{H}_*} &+ [\varepsilon^{1/2} K(R \|\bar{q}_0 + \varepsilon^2 \bar{q}_2\|_{\bar{H}_*} + \tau_T \|T_0 + \varepsilon^2 T_2\|_{H_*} + \tau_c \|C_0 + \varepsilon^2 C_2\|_{H_*}) \\ &+ \varepsilon K_* R(\alpha_T + \alpha_\tau)/2] \|Q_*\|^2_{\mathscr{H}_*}.\end{aligned}$$

This shows that if ε is sufficiently small, then

$$\|Q_*\|_{\mathscr{H}_*} = O(\varepsilon^3).$$

COROLLARY 1.

$$\|u - u_0\|_* = O(\varepsilon^2), \qquad \|v - \varepsilon v_0\|_*, \|w - \varepsilon w_0\|_* = O(\varepsilon^3),$$
$$\|T - T_0\|_*, \|C - C_0\|_* = O(\varepsilon^2).$$

Proof. Since

$$\begin{aligned}&\|u - u_0 - \varepsilon^2 u_2\|_*, \|v - v_0 - \varepsilon^2 v_2\|_*, \|w - w_0 - \varepsilon^2 w_2\|_* \\ &\quad \leqslant \|Q_*\|_* \leqslant K_* \|Q_*\|_{\mathscr{H}_*} = O(\varepsilon^3). \\ &\|u - u_0\|_* \leqslant \|u - u_0 - \varepsilon^2 u_2\|_* + \|\varepsilon^2 u_2\|_* = O(\varepsilon^2).\end{aligned}$$

The proof of the remaining estimates is the same.

Remark. By following the same argument, it can be shown that

$$\left\| u - \sum_{k=0}^{n-1} \varepsilon^{2k} u_{2k} \right\|_* = O(\varepsilon^{2n}),$$

$$\left\| v - \sum_{k=0}^{n-1} \varepsilon^{2k+1} u_{2k} \right\|_*, \quad \left\| w = \sum_{k=0}^{n-1} \varepsilon^{2k+1} w_{2k} \right\|_* = O(\varepsilon^{2n+1}),$$

$$\left\| T - \sum_{k=0}^{n-1} \varepsilon^{2k} T_{2k} \right\|_*, \quad \left\| C - \sum_{k=0}^{n-1} \varepsilon^{2k} C_{2k} \right\|_* = O(\varepsilon^{2n}).$$

5. Existence of a Generalized Solution

In this section we prove that there exists a unique generalized solution satisfying the integral relations (15). We choose $a = \bar{q}_0 + \varepsilon\bar{q}_1$, $b = T_0 + \varepsilon T_1$, $c = C_0 + C_1$, the solutions obtained for Stokes approximation in Section 3.1, and replace $\bar{q}$, T and C by $\bar{a} + \bar{q}_*$, $b + T_*$, and $c + C_*$ in (15),

$$\int_\Omega [-\bar{q}_{*x} + (\bar{a} + \bar{q}_*) \cdot \nabla\bar{q}_* + \bar{q}_* \cdot \nabla\bar{a} + (\alpha_{\mathrm{T}} T_* - \alpha_{\mathrm{c}} C_*)\bar{k} - \bar{F}_1] \cdot \bar{\phi}\alpha\Omega$$

$$+ R^{-1} \int_\Omega \nabla\bar{q}_* \cdot \nabla\bar{\phi}\, d\Omega = 0; \tag{87}$$

$$\int_\Omega [-T_{*x} + (\bar{a} + \bar{q}_*) \cdot \nabla T_* + \bar{q}_* \cdot \nabla b - F_2] \cdot \phi_{\mathrm{T}}\, d\Omega$$

$$+ \tau_{\mathrm{T}}^{-1} \int_\Omega \nabla T_* \nabla\phi_{\mathrm{T}}\, d\Omega = 0; \tag{88}$$

$$\int_\Omega [-C_{*x} + (a + q_*) \cdot \nabla C_* + q_* \cdot \nabla c - F_3] \cdot \phi_{\mathrm{c}}\, d\Omega$$

$$+ \tau_{\mathrm{c}}^{-1} \int_\Omega \nabla C_* \cdot \nabla\phi_{\mathrm{c}}\, d\Omega = 0, \tag{89}$$

where

$$\bar{F}_1 = \bar{a}_x - \bar{a} \cdot \nabla\bar{a} + [\alpha_{\mathrm{T}} b - \alpha_{\mathrm{c}} c]\bar{k} + R^{-1}\nabla^2\bar{a},$$

$$F_2 = b_x - \bar{a} \cdot \nabla b + \tau_{\mathrm{T}}^{-1}\nabla^2 b + Q_{\mathrm{T}}.$$

$$F_3 = C_x - \bar{a} \cdot \nabla c + \tau_{\mathrm{c}}^{-1}\nabla^2 c + Q_{\mathrm{c}}.$$

We show that the integrals in (87)–(89) are bounded functionals on $\bar{H}$ or H. We take up (87) first,

$$R^{-1} \int_\Omega \nabla q_* \cdot \nabla\phi\, d\Omega = R(\bar{q}_*, \bar{\phi})_{\bar{H}},$$

$$\left|\int_\Omega [-q_{*x} + (a+q_*)\cdot\nabla q_* + q_*\cdot\nabla a + (\alpha_T T_* - \alpha_c C_*)k]\cdot\phi\, d\Omega\right|$$

$$\leqslant K(\|\bar{q}_*\|_H + 2\|\bar{a}\|_H \|\bar{q}_*\|_H + \|\bar{q}_*\|_H^2 + \alpha_T \|T_*\|_H + \alpha_c \|C_*\|_H)\|\phi\|_{\bar{H}}$$

$$\left|\int_\Omega \bar{G}_1 \cdot \bar{\phi}\, d\Omega\right| \leqslant K(\|\bar{a}\|_{\bar{H}} + \|\bar{\alpha}\|_{\bar{H}}^2 + \alpha_T\|b\|_H + \alpha_c\|C\|_H + R^{-1}\|\bar{a}\|_H)\|\bar{\phi}\|_{\bar{H}}.$$

Therefore, all three integrals above are bounded functional on $\bar{H}$, and by Reisz's representation theorem, there exist operators M_1on $\bar{H}$, N_1, and S_1 on H and a fixed element $\bar{f}_1 \in \bar{H}$ such that, for any $\bar{\phi} \in \bar{H}$,

$$(R^{-1}\bar{q}_*, \phi)_H = (M_1\bar{q}_* + N_1 T_* + S_1 C_* + \bar{f}_1, \phi)_{\bar{H}}.$$

and we have

$$\bar{q}_* = R(M_1\bar{q}_* + N_1 T_* + S_1 C_* + \bar{f}_1). \tag{90}$$

Similarly, there exist operators M_2, M_3 on $\bar{H}$, N_2, and S_3 on H and fixed elements $f_2, f_3 \in H$ such that

$$T_* = \tau_T(M_2\bar{q}_* + N_2 T_* + f_2), \tag{91}$$

$$C_* = \tau_c(M_3\bar{q}_* + S_3 C_* + f_3). \tag{92}$$

We may express (90)–(92) in matrix form

$$Q = MQ + F, \tag{93}$$

where $Q = (\bar{q}_*, T_*, C_*)^T$, $F = (\bar{f}_1, f_2, f_3)^T$, and

$$M = \begin{bmatrix} RM_1 & RN_1 & RS_1 \\ \tau_T M_2 & \tau_T N_2 & 0 \\ \tau_c M_3 & 0 & \tau_c S_3 \end{bmatrix}.$$

First we show M is completely continuous on $\mathscr{H}$. Consider a weakly convergent sequence $\{Q_n\}_{n=0}^\infty = \{\bar{q}_n, T_n, C_n\}$ in $\mathscr{H}$, and make use of the following results:

(1) The embedding

$$W_2^1(\Omega) \to L^q(\Omega), \qquad 1 \leqslant q < 6$$

is compact.

(2) A weakly convergent sequence in $W_2^1(\Omega)$ is bounded.

(3) For a bounded Ω,

$$\|\phi\| \leqslant C\|\phi\|_{L_4}.$$

It is not difficult to show, for any $\Phi \in \mathscr{H}$,

$$|(MQ_m - MQ_n, \Phi)_{\mathscr{H}}| \leqslant K(\|\bar{q}_m - \bar{q}_n\|_{L_4} + \|T_m - T_n\|_{L_4} + \|C_m - C_n\|_{L_4}) \|\Phi\|_{\mathscr{H}}.$$

Let $\Phi = MQ_m - MQ_n$. This proves that $\{MQ_n\}_{n=0}^{\infty}$ converges strongly in $\mathscr{H}$ and M is completely continuous. Next we show that if

$$1 - K_*^2 R(\alpha_T + \alpha_c)/2 - k(R\|\bar{a}\|_{\bar{H}} + \tau_T \|b\|_H + \tau_c \|c\|_H) > 0, \tag{94}$$

then all solutions of

$$Q - \sigma(MQ + F) = 0 \tag{95}$$

are uniformly bounded for $\sigma \in [0, 1]$. We consider the dot product of both sides of (95) with Q itself and obtain

$$(\bar{Q}, \bar{Q})_{\mathscr{H}} = \sigma(MQ, Q)_{\mathscr{H}} + \sigma(F, Q)_{\mathscr{H}}. \tag{96}$$

By (87)–(89), (96) is equivalent to

$$\|\bar{q}_*\|_{\bar{H}}^2 = \sigma R \left[\int_\Omega \bar{F}_1 \cdot \bar{q}_* \, d\Omega - \int_\Omega \bar{q}^* \cdot \nabla(a) \cdot \bar{q}_* \, d\Omega \right.$$
$$\left. - \int_\Omega (\alpha_T T_* - \alpha_c C_*) \bar{k} \cdot \bar{q}_* \, d\Omega \right] \tag{97}$$

$$\|T_*\|_H^2 = \sigma\tau_T \left[\int_\Omega F_2 T_* \alpha\Omega - \int_\Omega \bar{q}_* \cdot \nabla b T_* \, d\Omega \right], \tag{98}$$

$$\|C_*\|_H^2 = \alpha\tau_c \left[\int_\Omega F_3 C_{*d\Omega} - \int_\Omega \bar{q}_* \cdot \nabla c C_* \, d\Omega \right]. \tag{99}$$

As before, we estimate the right-hand sides of (97)–(99) and obtain

$$\begin{aligned} \|Q\|_{\mathscr{H}}^2 \leqslant\; & K\sigma(R + \tau_T + \tau_c) \|Q\|_{\mathscr{H}} \\ & + k\sigma(R\|a\|_{\bar{H}} + \tau_T \|b\|_H + \tau_c \|c\|_H) \|Q\|_{\mathscr{H}}^1 \\ & + K_*^2 \sigma [R(\alpha_T + \alpha_c)/2] \|Q\|_{\mathscr{H}}^2. \end{aligned} \tag{100}$$

This shows that $\|Q\|_{\mathscr{H}}$ is uniformly bounded for $\sigma \in [0, 1]$ if (94) holds. By the Leray–Schauder fixed point theorem [9], there exists a solution $Q \in \mathscr{H}$.

Finally we show that under the same condition (94) the solution is unique. Let $Q_{(1)} = (\bar{q}_{(1)}, T_{(1)}, C_{(1)})$ and $Q_{(2)} = (q_{(2)}, T_{(2)}, C_{(2)})$ be two solutions of (15), and set

$$Q_{(1)} = Q_{(2)} + Q_*,$$

where $Q_* = (\bar{q}_*, T_*, C_*)$. Then Q_* satisfies (87)–(89) with a, b, c replaced by $\bar{q}_{(2)}, T_{(2)}, C_{(2)}$, respectively, and $\bar{F}=0$, $F_2 = F_3 = 0$, since $Q_{(2)}$ satisfies (15). Therefore,

$$\|Q_*\|^2_{\mathscr{H}} \leqslant [k(R\|\bar{q}_{(2)}\|_{\bar{H}} + \tau_T \|T_{(2)}\|_H + \tau_c \|C_{(2)}\|_H) + K^2_* R(\alpha_\tau + \alpha_c)/2] \|Q_*\|^2_{\mathscr{H}}. \tag{101}$$

For Stokes approximation,

$$\|\bar{q}_{(2)} - \bar{q}_0 - \varepsilon\bar{q}_1\|, \ \|T_{(2)} - T_0 - \varepsilon T_1\|, \ \|C_{(2)} - C_0 - \varepsilon C_1\| = O(\varepsilon^2);$$

if (94) holds and ε is sufficiently small, then

$$1 - K(R\|\bar{q}_{(2)}\|_{\bar{H}} + \tau_T \|T_{(2)}\|_H + \tau_c \|C_{(2)}\|_H) + K^2_* R(\alpha_\tau + \alpha_c)/2 > 0,$$

where R, τ_T and $\tau_c = O(\varepsilon)$. Hence $\|Q_*\|^2_{\mathscr{H}} = 0$ and $Q_{(1)} = Q_{(2)}$.

For long-wave approximation, we just replace $\bar{a}$, b, c in (87)–(89) by $\bar{q} + \varepsilon^2 \bar{q}_2$, $T_0 + \varepsilon^2 q_2$, and $C_0 + \varepsilon^2 C$, obtained in Section 3.2. The existence of a solution follows, and the uniqueness of a solution is established by use of the error estimates given in Theorem 2. We summarize our results as

THEOREM 3. *The systems of Eqs.* (16)–(21) and (32)–(39) *possess unique generalized solutions if* ε *is sufficiently small and* $1 - K^2_* R(\alpha_T + \alpha_c)/2 > 0$ *for Stokes approximation.*

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