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Generalizations of Young's Inequality

R. P. BOAS, JR. AND M. B. MARCUS

Department of Mathematics, Northwestern University, Evanston, Illinois 60201

1. Let f be a monotonic function whose domain contains the interval between the finite points a and b. (We do not assume that $a \leqslant b$.) Let f^{-1} be a generalized inverse of f; here a generalized inverse of the nondecreasing function f is any function f^{-1} satisfying

$$\inf\{x: f(x) > y\} \geqslant f^{-1}(y) \geqslant \sup\{x: f(x) < y\};$$

if f is nonincreasing, interchange inf and sup and reverse the $\geqslant$ signs. Note that $f(f^{-1}(t))$ is not necessarily equal to t.

For such an f, the equation for integration by parts

$$\int_a^b f(u)\, du = bf(b) - af(a) - \int_a^b u\, df(u)$$

can be written

$$bf(b) + \int_{f(b)}^{f(a)} f^{-1}(u)\, du = af(a) + \int_a^b f(u)\, du. \tag{1}$$

This is elementary when f is continuous and strictly monotonic and can be proved in the general case by approximating f by a continuous strictly monotonic function and using bounded convergence; for a somewhat different approach compare [5, p. 124].

Two generalizations of (1) provide interesting inequalities. They are obtained by replacing $f(a)$ or $f(b)$ by a variable t; the equality in (1) then becomes an inequality whose sense depends on whether f is nondecreasing or nonincreasing.

THEOREM. *With the preceding conventions, and with the $\leqslant$ signs for nonincreasing functions, the $\geqslant$ signs for nondecreasing functions,*

$$bf(b) + \int_{f(b)}^{t} f^{-1}(y)\, dy \lesseqqgtr at + \int_a^b f(u)\, du, \tag{2}$$

$$af(a) + \int_a^b f(u)\, du \lesseqqgtr bt + \int_t^{f(a)} f^{-1}(y)\, dy. \tag{3}$$

Here t is in the domain of f^{-1}, but not necessarily between a and b. There is equality in (2) *if and only if t is between $f(a^+)$ and $f(a^-)$, and in* (3) *if and only if t is between $f(b^-)$ and $f(b^+)$.*

In the first place, (2) and (3) are equivalent because we can interchange a and b; in the second place, the inequalities with decreasing functions are equivalent to those with increasing functions by the change of variable $u = a + b - v$. Hence, the content of the theorem is implicit in any of the four inequalities, but it is convenient to have them all. Inequality (3) for increasing functions is Young's inequality in the form originally given by Young [6] (who considered only strictly increasing differentiable functions); the usual statement ([3, p. 111]; [4, p. 48]) has $a = f(a) = 0$ and $b > a$, i.e.,

$$bt \leqslant \int_0^b f(u)\,du + \int_0^t f^{-1}(y)\,dy \tag{4}$$

and f strictly increasing. Generalized inverses were introduced in this context in [2].

The object of this note is to reintroduce Young's original inequality, to show that it is one of four generalizations of (1) when f is monotone, and to give some applications. Although all the inequalities can be obtained from Young's inequality, they have consequences that do not follow directly from Young's inequality itself. Of particular interest are the inequalities for decreasing f since we can let $b \to \infty$ in (2), or $a \to 0$ in (3), even when $f(0) = \infty$. Thus, for example, when $\int^\infty f(u)\,du$ converges, $bf(b) \to 0$ as $b \to \infty$ and consequently

$$\int_0^t f^{-1}(y)\,dy \leqslant at + \int_a^\infty f(u)\,du; \tag{5}$$

in particular, $\int_0 f^{-1}(y)\,dy$ converges if $\int^\infty f(u)\,du$ converges. The converse follows similarly from (3) (holding a fixed and letting $t \to 0$).

Both (2) and (3) are geometrically obvious from figures; the easiest case, the corollary of (5) just stated, simply says that the area under the graph of $y = f(x)$ is finite if and only if the area between the graph and the y-axis is finite. We shall give an analytic proof (which seems to us simpler than the usual proofs of Young's inequality). Finally, we shall illustrate the theorem by examples. Our experience is that such examples can be established by other methods, once found; but the generalizations of Young's inequality are useful for finding them in the first place.

2. *Proof of the Theorem.* We prove (2) for decreasing functions. The proof is similar for the other cases; or, as noted above, we can derive them all from this case by simple transformations.

When f is nonincreasing, we obviously have

$$(r - a) f(r) \leqslant \int_a^r f(u)\, du \tag{6}$$

whether $r > a$ or $r < a$, as long as the interval between a and r is in the domain of f; and there is equality when and only when $r = a$ or f is constant between a and r. We can write (6) in the form

$$rf(r) - \int_b^r f(s)\, ds \leqslant af(r) + \int_a^b f(u)\, du. \tag{7}$$

Now apply integration by parts, in the form (1), to the integral on the left of (7). We get

$$bf(b) - \int_{f(r)}^{f(b)} f^{-1}(u)\, du \leqslant af(r) + \int_a^b f(u)\, du. \tag{8}$$

If t is in the range of f, we can take r so that $f(r) = t$, and (8) becomes

$$bf(b) + \int_{f(b)}^{t} f^{-1}(u) \quad u \leqslant at + \int_a^b f(u)\, du, \tag{9}$$

with equality if and only if $t = f(a)$. In particular, we have established the "decreasing" case of (2) when f is strictly decreasing and, consequently, Young's inequality in the usual form.

We now turn to the general case when t is in the domain of f^{-1} but not necessarily in the range of f. Unless t is between $f(a^+)$ and $f(a^-)$, we can choose $r_0 \neq a$ so that r_0 is in the range of f and either $r_0 > a$, $f(r_0) > t$, or $r_0 < a$, $f(r_0) < t$. We consider the first case; the other is exactly parallel. We can then write (8) in the form

$$bf(b) + \int_{f(b)}^{t} f^{-1}(u)\, du + \int_t^{f(r_0)} f^{-1}(u)\, du \leqslant af(r_0) + \int_a^b f(u)\, du,$$

i.e.,

$$bf(b) + \int_{f(b)}^{t} f^{-1}(u)\, du \leqslant \int_a^b f(u)\, du + af(r_0) - \int_t^{f(r_0)} f^{-1}(u)\, du. \tag{10}$$

Since $t < f(r_0)$ and $f^{-1}(u) > a$ between $u = t$ and $u = f(r_0)$, we have

$$\int_t^{f(r_0)} f^{-1}(u)\, du > a(f(r_0) - t);$$

substituting this in (10), we obtain (9) with strict inequality.

When t is between $f(a^+)$ and $f(a^-)$, (9), with equality, follows from (1) since $f^{-1}(u) = a$ for u between $f(a)$ and t.

3. *Illustrations.* We give some illustrations to show the advantages of having all four inequalities (2), (3) instead of being restricted to Young's inequality in the usual form.

(a) From (5), we read off without calculation that $\int_0^1 \log(1/y)\, dy$ converges because $\int_1^\infty e^{-x}\, dx$ converges.

(b) If we apply (5) to $f(x) = x^{1/(p-1)}$, $0 < p < 1$, we get

$$ab \geqslant p^{-1}a^p + q^{-1}b^q, \qquad p^{-1} + q^{-1} = 1. \tag{11}$$

A standard application of (4) ([4, p. 50]) to $f(x) = x^{q-1}$, $q > 1$, is

$$ab \leqslant p^{-1}a^p + q^{-1}b^q, \qquad p^{-1} + q^{-1} = 1, \tag{12}$$

from which Hölder's inequality follows for $p > 1$. Similarly, (11) implies Hölder's inequality for $0 < p < 1$. Of course, (11) can be derived from (12), but (5) leads to (11) more directly.

(c) If we take $f(t) = e^t$ and use (2) for increasing functions we get

$$t - t\log t \leqslant e^a - at, \qquad -\infty < a < \infty, \quad t > 0. \tag{13}$$

This is given in [4, p. 49] (but only for $a > 1$) and in [3, p. 61] but with a more complicated proof. If we use (5) with $f(u) = e^{-u}$, we get

$$t - t\log t \leqslant e^{-a} + at, \qquad -\infty < a < \infty, \quad t > 0, \tag{14}$$

which is the same as (13) if we replace a by $-a$, something that we could not do on the basis of Young's inequality (4) alone.

Note that (14) says more than (13) when $t < a^{-1}\sinh a$. Note also that if we put $t = e^u$ and $a - u = x$, (13) becomes $e^x \geqslant 1 + x$, as was already noticed by Young [7].

(d) If we apply (2) and (5) to $f(x) = e^{-x^2}$, we get, after a change of variable, for $0 < u < \infty$,

$$2\int_u^\infty s^2 e^{-s^2}\, ds \leqslant ae^{-u^2} + \int_a^\infty e^{-x^2}\, dx, \qquad 0 < a < \infty;$$

$$\int_0^b e^{-u^2}\, du \leqslant be^{-u^2} + 2\int_0^u s^2 e^{-s^2}\, ds, \qquad 0 < b < \infty.$$

(e) A case of (5) in which generalized inverses are essential is

$$f(x) = \sum_{k=n}^{\infty} t_k\,, \quad n - 1 < x \leqslant n; \qquad f(0) = 1,$$

where

$$t_k \geqslant 0 \qquad \text{and} \qquad \sum_{k=1}^{\infty} t_k = 1.$$

Applications are given in [1].

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