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Generalized intersection bodies ☆

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Abstract

We study the structures of two types of generalizations of intersection-bodies and the problem of whether they are in fact equivalent. Intersection-bodies were introduced by Lutwak and played a key role in the solution of the Busemann–Petty problem. A natural geometric generalization of this problem considered by Zhang, led him to introduce one type of generalized intersection-bodies. A second type was introduced by Koldobsky, who studied a different analytic generalization of this problem. Koldobsky also studied the connection between these two types of bodies, and noted that an equivalence between these two notions would completely settle the unresolved cases in the generalized Busemann–Petty problem. We show that these classes share many identical structural properties, proving the same results using integral geometry techniques for Zhang's class and Fourier transform techniques for Koldobsky's class. Using a functional analytic approach, we give several surprising equivalent formulations for the equivalence problem, which reveal a deep connection to several fundamental problems in the integral geometry of the Grassmann manifold.

Keywords: Generalized intersection bodies; Equivalence problem; Generalized Busemann–Petty problem; Radon transform; Fourier transform of distributions; Grassmann geometry; Generalized Blaschke–Petkantschin formula

1. Introduction

Let $\mathrm{Vol}(L)$ denote the Lebesgue measure of a set $L \subset \mathbb{R}^n$ in its affine hull, and let $G(n,k)$ denote the Grassmann manifold of k-dimensional subspaces of $\mathbb{R}^n$. Let D_n denote the Euclidean unit ball, and S^{n-1} the Euclidean sphere. All of the bodies considered in this note will be

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assumed to be centrally-symmetric star-bodies, defined by a continuous radial function $\rho_K(\theta) = \max\{r > 0 \mid r\theta \in K\}$ for $\theta \in S^{n-1}$ and a star-body K. We shall deal with two generalizations of the notion of an *intersection body*, first introduced by Lutwak in [24] (see also [25]). A star-body K is said to be an intersection body of a star-body L, if $\rho_K(\theta) = \mathrm{Vol}(L \cap \theta^\perp)$ for every $\theta \in S^{n-1}$, where $\theta^\perp$ is the hyperplane perpendicular to θ. K is said to be an intersection body, if it is the limit in the radial metric d_r of intersection bodies $\{K_i\}$ of star-bodies $\{L_i\}$, where $d_r(K_1, K_2) = \sup_{\theta \in S^{n-1}} |\rho_{K_1}(\theta) - \rho_{K_2}(\theta)|$. This is equivalent (e.g. [6,25]) to $\rho_K = R^*(d\mu)$, where μ is a non-negative Borel measure on S^{n-1}, R^* is the dual transform (as in (1.3)) to the spherical Radon transform $R : C(S^{n-1}) \to C(S^{n-1})$, which is defined for $f \in C(S^{n-1})$ as

$$R(f)(\theta) = \int_{S^{n-1} \cap \theta^\perp} f(\xi)\, d\sigma_\theta(\xi), \tag{1.1}$$

where σ_θ the Haar probability measure on $S^{n-1} \cap \theta^\perp$.

The notion of an intersection body has been shown to be fundamentally connected to the Busemann–Petty problem (first posed in [5]), which asks whether two centrally-symmetric convex bodies K and L in $\mathbb{R}^n$ satisfying:

$$\mathrm{Vol}(K \cap H) \leqslant \mathrm{Vol}(L \cap H) \quad \forall H \in G(n, n-1) \tag{1.2}$$

necessarily satisfy $\mathrm{Vol}(K) \leqslant \mathrm{Vol}(L)$. It was shown in [6,25] that the answer is equivalent to whether all convex bodies in $\mathbb{R}^n$ are intersection bodies, and in a series of results [1,3,6–8,11,18, 23,27,37] that this is true for $n \leqslant 4$, but false for $n \geqslant 5$.

In [36], Zhang considered a generalization of the Busemann–Petty problem, in which $G(n, n-1)$ in (1.2) is replaced by $G(n, n-k)$, where k is some integer between 1 and $n-1$. Zhang showed that the *generalized k-codimensional Busemann–Petty problem* is also naturally associated to another class of bodies, which will be referred to as *k-Busemann–Petty bodies* (note that these bodies are referred to as *$(n-k)$-intersection bodies* in [36] and *generalized k-intersection bodies* in [21]), and that the generalized k-codimensional problem is equivalent to whether all convex bodies in $\mathbb{R}^n$ are k-Busemann–Petty bodies. It was shown in [4] (see also a correction in [29]), and later in [21], that the answer is negative for $k < n-3$, but the cases $k = n-3$ and $k = n-2$ still remain open (the case $k = n-1$ is obviously true). Several partial answers to these cases are known. It was shown in [36] (see also [29]) that when K is a centrally-symmetric convex body of revolution then the answer is positive for the pair K, L with $k = n-2, n-3$ and any star-body L. When $k = n-2$, it was shown in [4] that the answer is positive if L is a Euclidean ball and K is convex and sufficiently close to L. Several other generalizations of the Busemann–Petty problem were treated in [29,34,35,38].

Before defining the class of k-Busemann–Petty bodies we shall need to introduce the m-dimensional spherical Radon transform, acting on spaces of continuous functions as follows:

$$R_m : C(S^{n-1}) \to C\big(G(n,m)\big)$$
$$R_m(f)(E) = \int_{S^{n-1} \cap E} f(\theta)\, d\sigma_E(\theta),$$

where σ_E is the Haar probability measure on $S^{n-1} \cap E$. It is well known (e.g. [17]) that as an operator on *even* continuous functions, R_m is injective. The dual transform is defined on spaces of *signed* Borel measures $\mathcal{M}$ by

$$R_m^*: \mathcal{M}\big(G(n,m)\big) \to \mathcal{M}(S^{n-1})$$
$$\int\limits_{S^{n-1}} f R_m^*(d\mu) = \int\limits_{G(n,m)} R_m(f)\, d\mu \quad \forall f \in C(S^{n-1}), \tag{1.3}$$

and for a measure μ with continuous density g, the transform may be explicitly written in terms of g (see [36]):

$$R_m^* g(\theta) = \int\limits_{\theta \in E \in G(n,m)} g(E)\, d\nu_{m,\theta}(E),$$

where $\nu_{m,\theta}$ is the Haar probability measure on the homogeneous space $\{E \in G(n,m) \mid \theta \in E\}$.

We shall say that a body K is a k-Busemann–Petty body if $\rho_K^k = R_{n-k}^*(d\mu)$ as measures in $\mathcal{M}(S^{n-1})$, where μ is a non-negative Borel measure on $G(n,n-k)$. We shall denote the class of such bodies by $\mathcal{BP}_k^n$. Choosing $k=1$, for which $G(n,n-1)$ is isometric to S^{n-1}/Z_2 by mapping H to $S^{n-1} \cap H^\perp$, and noticing that R is equivalent to R_{n-1} under this map, we see that $\mathcal{BP}_1^n$ is exactly the class of intersection bodies.

Another generalization of the notion of an intersection body, which was considered by Koldobsky in [21], is that of a *k-intersection body*. A star-body K is said to be a k-intersection body of a star-body L, if $\mathrm{Vol}(K \cap H^\perp) = \mathrm{Vol}(L \cap H)$ for every $H \in G(n,n-k)$. K is said to be a k-intersection body, if it is the limit in the radial metric of k-intersection bodies $\{K_i\}$ of star-bodies $\{L_i\}$. We shall denote the class of such bodies by $\mathcal{I}_k^n$. Again, choosing $k=1$, we see that $\mathcal{I}_1^n$ is exactly the class of intersection bodies.

In [21], Koldobsky considered the relationship between these two types of generalizations, $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$, and proved that $\mathcal{BP}_k^n \subset \mathcal{I}_k^n$ (hence our reluctance to use the term "generalized $(n-k)$-intersection bodies" for $\mathcal{BP}_k^n$). Koldobsky also asked whether the opposite inclusion is equally true for all k between 2 and $n-2$ (for 1 and $n-1$ this is true). If this were true, as remarked by Koldobsky, a positive answer to the generalized k-codimensional Busemann–Petty problem for $k \geqslant n-3$ would follow, since for those values of k any centrally-symmetric convex body in $\mathbb{R}^n$ is known to be a k-intersection body [19–21].

Our first remark in this note is that the two classes $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$ share many identical structural properties, suggesting that it is indeed reasonable to believe that $\mathcal{BP}_k^n = \mathcal{I}_k^n$. Some previously known characterizations of these classes and associated tools are outlined in Section 2, providing some intuitive motivation and common ground to start from. Some of these previously known results are also given simplified proofs in this section. It turns out that the natural language for handling the class $\mathcal{I}_k^n$ is the language of Fourier transforms of homogeneous distributions, developed extensively by Koldobsky, while the natural language for the class $\mathcal{BP}_k^n$ is the language of integral geometry and Radon transforms. In Section 3 we show that both classes share a common structure, by proving the same results for $\mathcal{BP}_k^n$ (using Grassmann geometry techniques) and for $\mathcal{I}_k^n$ (using Fourier transform techniques). We define the *k-radial sum* of two star-bodies L_1, L_2 as the star-body L satisfying $\rho_L^k = \rho_{L_1}^k + \rho_{L_2}^k$. For each of these classes $\mathcal{C}_k^n$, where $\mathcal{C} = \mathcal{I}$ or $\mathcal{C} = \mathcal{BP}$ and $k, l = 1, \ldots, n-1$, we show the following.

Structure Theorem.

(1) *$\mathcal{C}_k^n$ is closed under full-rank linear transformations, k-radial sums and taking limit in the radial metric.*
(2) *$\mathcal{C}_1^n$ is the class of intersection-bodies in $\mathbb{R}^n$, and $\mathcal{C}_{n-1}^n$ is the class of all symmetric star-bodies in $\mathbb{R}^n$.*
(3) *Let $K_1 \in \mathcal{C}_{k_1}^n$, $K_2 \in \mathcal{C}_{k_2}^n$ and $l = k_1 + k_2 \leqslant n-1$. Then the star-body L defined by $\rho_L^l = \rho_{K_1}^{k_1} \rho_{K_2}^{k_2}$ satisfies $L \in \mathcal{C}_l^n$. As corollaries:*
   (a) *$\mathcal{C}_{k_1}^n \cap \mathcal{C}_{k_2}^n \subset \mathcal{C}_{k_1+k_2}^n$ if $k_1 + k_2 \leqslant n-1$.*
   (b) *$\mathcal{C}_k^n \subset \mathcal{C}_l^n$ if k divides l.*
   (c) *If $K \in \mathcal{C}_k^n$ then the star-body L defined by $\rho_L = \rho_K^{k/l}$ satisfies $L \in \mathcal{C}_l^n$ for $l \geqslant k$.*
(4) *If $K \in \mathcal{C}_k^n$ then any m-dimensional central section L of K (for $m > k$) satisfies $L \in \mathcal{C}_k^m$.*

(1) and (2) above are well known and basically follow from the definitions (or from the characterizations in Section 2), but we mention them here for completeness. It should also be clear that (3) implies the three corollaries following it: (3a) by using $K_1 = K_2$, (3b) by successively applying (3a), and (3c) by using $K_2 = D_n$. (3) for $\mathcal{I}_k^n$ was also noticed independently by Koldobsky, but never published. For $\mathcal{BP}_k^n$, (4) and (3b) for $k = 1$ were proved by Grinberg and Zhang in [16]. In the same paper, a very useful characterization of the class $\mathcal{BP}_k^n$ was given (see Section 2). Combining it with (3) and (3c), we get as a corollary the following non-trivial result, which is of independent interest.

Ellipsoid Corollary. *For any $1 \leqslant k \leqslant n-1$ and k ellipsoids $\{\mathcal{E}_i\}_{i=1}^k$ in $\mathbb{R}^n$, define the body L by*

$$\rho_L = \rho_{\mathcal{E}_1} \cdot \cdots \cdot \rho_{\mathcal{E}_k},$$

and let $k \leqslant l \leqslant n-1$. Then there exists a sequence of star-bodies $\{L_i\}$ which tends to L in the radial metric and satisfies:

$$\rho_{L_i} = \rho_{\mathcal{E}_1^i}^l + \cdots + \rho_{\mathcal{E}_{m_i}^i}^l,$$

where $\{\mathcal{E}_j^i\}$ are ellipsoids.

Naturally, the case $\mathcal{E}_1 = \cdots = \mathcal{E}_k$ is of particular interest. In the same spirit, we give a strengthened version of Grinberg and Zhang's characterization of $\mathcal{BP}_k^n$ in Section 3. We remark that (3) from the Structure Theorem may in fact be a characterization of the classes $\mathcal{I}_k^n$ or $\mathcal{BP}_k^n$ for $k > 1$. In other words, it may be that for $\mathcal{C} = \mathcal{BP}$ or $\mathcal{C} = \mathcal{I}$, $L \in \mathcal{C}_k^n$ iff there exist $\{K_i\}_{i=1}^k \subset \mathcal{C}_1^n$, such that $\rho_L^k = \rho_{K_1} \cdot \cdots \cdot \rho_{K_k}$. Since in either case $\mathcal{C}_1^n$ is the class of intersection bodies in $\mathbb{R}^n$, a proof of such a characterization for $\mathcal{C} = \mathcal{I}$ and a fixed k would imply that $\mathcal{BP}_k^n = \mathcal{I}_k^n$ for that k.

In order to prove (3) for $\mathcal{C} = \mathcal{BP}$, we derive (what seems to be) a new formula for integration on products of Grassmann manifolds. The complete formulation and proof are given in Appendix A. A very similar formulation of the case $k_1, \ldots, k_r = 1$ was given by Blaschke and Petkantschin (see [26,30] for an easy derivation), and used by Grinberg and Zhang in [16] to deduce that $\mathcal{BP}_1^n \subset \mathcal{BP}_l^n$ for all $1 \leqslant l \leqslant n-1$. For $F \in G(n, n-l)$ and $1 \leqslant k < l \leqslant n-1$, we denote by $G_F(n, n-k)$ the manifold $\{E \in G(n, n-k) \mid F \subset E\}$. The volume of the parallelepiped mentioned in the statement below is defined in Appendix A. A simplified formulation then reads as follows:

Integration on products of Grassmann manifolds. *Let $n > 1$. For $i = 1, \dots, r$, let $k_i \geqslant 1$ denote integers whose sum l satisfies $l \leqslant n-1$. For $a = 1, \dots, n$ denote by $G^a = G(n, n-a)$, and by μ^a the Haar probability measure on G^a. For $F \in G^l$ and $a = 1, \dots, l-1$, denote by μ_F^a the Haar probability measure on G_F^a. Denote by $\bar{E} = (E_1, \dots, E_r)$ an ordered set with $E_i \in G^{k_i}$. Then for any continuous function $f(\bar{E}) = f(E_1, \dots, E_r)$ on $G^{k_1} \times \dots \times G^{k_r}$:*

$$\begin{aligned}&\int_{E_1 \in G^{k_1}} \dots \int_{E_r \in G^{k_r}} f(\bar{E})\, d\mu^{k_1}(E_1) \dots d\mu^{k_r}(E_r)\\&\quad = \int_{F \in G^l} \int_{E_1 \in G_F^{k_1}} \dots \int_{E_r \in G_F^{k_r}} f(\bar{E}) \Delta(\bar{E})\, d\mu_F^{k_1}(E_1) \dots d\mu_F^{k_r}(E_r)\, d\mu^l(F),\end{aligned}$$

where $\Delta(\bar{E}) = C_{n,\{k_i\},l}\, \Omega(\bar{E})^{n-l}$, $C_{n,\{k_i\},l}$ is a constant depending only on $n, \{k_i\}, l$, and $\Omega(\bar{E})$ denotes the l-dimensional volume of the parallelepiped spanned by unit volume elements of $E_1^{\perp}, \dots, E_r^{\perp}$.

In Section 4 we attempt to bridge the gap between the languages of integral geometry and Fourier transforms, by establishing several new identities. As a by-product, we show, for instance, that $\operatorname{Ker} R_{n-k}^* = \operatorname{Ker}(I \circ R_k)^*$, where $I : C(G(n,k)) \to C(G(n,n-k))$ denotes the operator defined as $I(f)(E) = f(E^{\perp})$. Essentially using the latter result, we show the following equivalence:

1.1. Equivalence between k and $n-k$

$$\mathcal{BP}_k^n = \mathcal{I}_k^n \quad \text{iff} \quad \mathcal{BP}_{n-k}^n = \mathcal{I}_{n-k}^n.$$

In Section 5 we try to attack the $\mathcal{BP}_k^n = \mathcal{I}_k^n$ question using the results of the previous sections together with a functional analytic approach. Our results indicate that this question is deeply connected to several fundamental questions in integral geometry concerning the structure of the Grassmann manifold. Let $C_+(S^{n-1})$ denote the set of non-negative continuous functions on the sphere, and let $R_{n-k}(C(S^{n-1}))_+$ denote the set of non-negative functions in the image of R_{n-k}. Let $\overline{A}$ denote the closure of a set A in the corresponding normed space. If $\mu \in \mathcal{M}(G(n,n-k))$, let $\mu^{\perp} \in \mathcal{M}(G(n,k))$ denote the measure defined by $\mu^{\perp}(A) = \mu(A^{\perp})$ for any Borel set $A \subset G(n,k)$, where $A^{\perp} = \{E^{\perp} | E \in A\}$.

Fixing n and $1 \leqslant k \leqslant n-1$, the main result of Section 5 is the following:

Equivalence Theorem. *The following statements are equivalent*:

(1) *Equivalence of generalizations of intersection-bodies.*

$$\mathcal{BP}_k^n = \mathcal{I}_k^n.$$

(2) *Characterization of non-negative range of R_{n-k}.*

$$\overline{R_{n-k}\big(C(S^{n-1})\big)_+} = \overline{R_{n-k}\big(C_+(S^{n-1})\big) + I \circ R_k\big(C_+(S^{n-1})\big)}. \tag{1.4}$$

(3) *A Negation statement.*
There does not exist a non-negative measure $\mu \in \mathcal{M}(G(n, n-k))$ such that $R^*_{n-k}(d\mu) \geqslant 1$ and $R^*_k(d\mu^\perp) \geqslant 1$ (where "$\nu \geqslant 1$" means that $\nu - 1$ is a non-negative measure), and such that:

$$\inf\big\{\langle \mu, f\rangle \;\big|\; f \in R_{n-k}\big(C(S^{n-1})\big)_+ \text{ and } \langle 1, f\rangle = 1\big\} = 0.$$

The approach developed in Section 4 easily shows (once again) that $\mathcal{BP}^n_k \subset \mathcal{I}^n_k$. Analogously, it will be evident that the right-hand side of (1.4) is a subset of the left-hand side.

We will say that a set $Z \subset G(n, n-k)$ satisfies the *covering property* if

$$\bigcup_{E \in Z} E \cap S^{n-1} = S^{n-1} \quad \text{and} \quad \bigcup_{E \in Z} E^\perp \cap S^{n-1} = S^{n-1}. \tag{1.5}$$

The following natural conjecture is given in Section 5 (see Lemma 5.10 and Remark 5.12).

Covering Property Conjecture. *For any $n > 0$, $1 \leqslant k \leqslant n-1$, if $Z \subset G(n, n-k)$ is a closed set satisfying $\bigcup_{E \in Z} E \cap S^{n-1} = S^{n-1}$, then there exists a non-negative measure $\mu \in \mathcal{M}(G(n, n-k))$ supported in Z, such that $R^*_{n-k}(d\mu) \geqslant 1$.*

Using this conjecture, we extend formulations (1)–(3) from the Equivalence Theorem in the following.

Weak Equivalence Theorem. *The following statements are equivalent to each other*:

(4) "*Injectivity*" *of the restricted Radon transform.*
For any $g \in \overline{R_{n-k}(C(S^{n-1}))_+}$, if $Z = g^{-1}(0)$ satisfies the covering property then $g = 0$.

(5) *Existence of barely balanced measures.*
For any closed $Z \subset G(n, n-k)$ with the covering property, there exists a measure $\mu \in \mathcal{M}(G(n, n-k))$ such that $\mu|_{Z^C} \geqslant 1$ and $R^*_{n-k}(d\mu) = 0$.

Assuming the Covering Property Conjecture, formulations (1)–(3) *imply* (4), (5).

For us, the formulation in (5) seems to have the most potential for understanding this problem, although we have not been able to advance in this direction. Without a doubt, (2) is the most elegant formulation, and perhaps the most natural for integral geometrists.

We conclude by proposing another natural problem in integral geometry. Consider the operator $V_k : C(G(n,k)) \to C(G(n,k))$ defined as $V_k = I \circ R_{n-k} \circ R^*_k$. It is easy to see from general principles of functional analysis that $\operatorname{Ker} V_k$ is orthogonal to $\overline{\operatorname{Im} V_k}$, and therefore as an operator from $\overline{\operatorname{Im} V_k}$ to itself, V_k is injective and onto a dense set. We show in Section 4 that in addition, V_k is self-adjoint. In the case $k = 1$, $C(G(n,1))$ may be identified with the class of even continuous functions on the sphere $C_e(S^{n-1})$, in which case $V_1 : C_e(S^{n-1}) \to C_e(S^{n-1})$ becomes the classical spherical Radon transform R given by (1.1). Elegant inversion formulas for V_1 have been developed by many authors (see [17] and also [14,15,28,31,33]). Is it possible to do the same for the general V_k?

2. Additional notations and previous results

In this section we present some previously known results which will be useful for us later on. For completeness, we try to at least sketch the proofs of the main results, and on some occasions, provide alternative proofs. We also add several useful notations along the way.

2.1. Additional notations

Let G denote any locally compact topological space. The spaces of continuous and non-negative continuous real-valued functions on G will be denoted by $C(G)$ and $C_+(G)$, respectively. When G has a natural involution operator "$-$", we will denote by $C_e(G)$ the space of continuous even functions on G. Whenever it makes sense, we will denote by $C^\infty(G)$ the space of infinitely smooth real-valued functions on G, and define $C_+^\infty(G)$ and $C_{+,e}^\infty(G)$ accordingly. Similarly, the spaces of signed and non-negative finite Borel measures on G will be denoted $\mathcal{M}(G)$ and $\mathcal{M}_+(G)$, respectively. When a natural involution operator "$-$" exists, the spaces $\mathcal{M}_e(G)$ and $\mathcal{M}_{+,e}(G)$ will denote the corresponding spaces of even measures. A measure μ is called even if $\mu(A) = \mu(-A)$ for every Borel set $A \subset G$. For $\mu \in \mathcal{M}(G)$ and $f \in C(G)$, we denote by $\langle \mu, f \rangle_G$ the action of the measure μ on f as a linear functional. Whenever it is clear from the context what the underlying space G is, we will write $\langle \mu, f \rangle$ instead of $\langle \mu, f \rangle_G$.

We will always assume that a fixed Euclidean structure is given on $\mathbb{R}^n$, and denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^n$. We will denote by $O(n)$ the group of orthogonal rotations in $\mathbb{R}^n$. The group of volume-preserving linear transformations in $\mathbb{R}^n$ will denoted by $SL(n)$. For $T \in SL(n)$, we denote $T^{-*} = (T^{-1})^*$.

We will always use σ to denote the Haar probability measure on S^{n-1}. $G(n,0)$ and $G(n,n)$ will denote the trivial atomic manifolds, and these are equipped of course with the trivial Haar probability measure.

For a star-body K (not necessarily convex), we define its Minkowski functional as $\|x\|_K = \min\{t \geqslant 0 \mid x \in tK\}$. When K is a centrally-symmetric convex body, this, of course, coincides with the natural norm associated with it. Obviously $\rho_K(\theta) = \|\theta\|_K^{-1}$ for $\theta \in S^{n-1}$.

2.2. Closure under basic operations

It is not hard to check *from the definitions* that the classes $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$ are closed under k-radial sums, full-rank linear transformations and limit in the radial metric. Indeed, the closure under limit in the radial metric follows from the definition of $\mathcal{I}_k^n$ and from the w^*-compactness of the unit ball of $\mathcal{M}(G(n,n-k))$ for $\mathcal{BP}_k^n$. The closure under k-radial sums is also immediate for $\mathcal{BP}_k^n$, but for $\mathcal{I}_k^n$ this requires a little more thought. Indeed, by polar integration, if K_i is a k-intersection body of a star-body L_i, for $i = 1,2$, then the body K which is the k-radial sum of K_1 and K_2 is a k-intersection body of the $n-k$-radial sum of L_1 and L_2, and the general case follows by passing to a limit. The closure under full-rank linear-transformations requires a little more ingenuity. It is not so hard to check that if K is a k-intersection body of a star-body L then $T(K)$ is a k-intersection body of $T^{-*}(L)$ for $T \in SL(n)$, which settles the case of $\mathcal{I}_k^n$. For $\mathcal{BP}_k^n$, this requires additional work, and is actually a good exercise to show directly. Instead, we prefer to trivially deduce this from Theorem 2.1.

2.3. The class $\mathcal{BP}_k^n$

The following characterization of $\mathcal{BP}_k^n$, first proved by Goodey and Weil in [12] for intersection-bodies (the case $k = 1$), and extended to general k by Grinberg and Zhang in [16], is extremely useful.

Theorem 2.1. (Grinberg and Zhang [16]) *A star-body K is a k-Busemann–Petty body iff it is the limit of $\{K_i\}$ in the radial metric, where each K_i is a finite k-radial sums of ellipsoids $\{\mathcal{E}_j^i\}$:*

$$\rho_{K_i}^k = \rho_{\mathcal{E}_1^i}^k + \cdots + \rho_{\mathcal{E}_{m_i}^i}^k .$$

Before commenting on the proof of this theorem, we introduce the following useful notion used by Grinberg and Zhang. For any G, a homogeneous space of $O(n)$, and measures $\mu \in \mathcal{M}(G)$ and $\eta \in \mathcal{M}(O(n))$, we define their convolution $\eta * \mu \in \mathcal{M}(G)$ as the measure satisfying $\eta * \mu(A) = \int_{O(n)} \mu(u^{-1}(A))\, d\eta(u)$ for every Borel subset $A \subset G$. The definition is essentially the same when $\eta \in \mathcal{M}(H)$, where H is another homogeneous space of $O(n)$, by identifying between η and its lifting $\tilde{\eta} \in \mathcal{M}(O(n))$ defined as $\tilde{\eta}(A) = \eta(\pi(A))$ for any Borel subset $A \subset O(n)$, where $\pi : O(n) \to H$ is the canonical projection.

Let σ_F denote the Haar probability measure on $S^{n-1} \cap F$, so that as a linear functional, for any $f \in C(S^{n-1})$, $\sigma_F(f) = R_{n-k}(f)(F)$. The key idea underlying Theorem 2.1 is an important observation: for any $F \in G(n, n-k)$, one may explicitly construct a family of ellipsoids $\{\mathcal{E}_i(F, \epsilon)\}$, such that $\rho_{\mathcal{E}_i(F,\epsilon)}^k$ tends to σ_F in the w^*-topology (as $\epsilon \to 0$). The ellipsoid $\mathcal{E}_i(F, \epsilon)$ is defined by

$$\|x\|_{\mathcal{E}_i(F,\epsilon)}^2 = \frac{|\operatorname{Proj}_F(x)|^2}{a(\epsilon)^2} + \frac{|\operatorname{Proj}_{F^\perp}(x)t|^2}{b(\epsilon)^2},$$

where Proj_E denotes the orthogonal projection onto E, and $a(\epsilon), b(\epsilon)$ are chosen appropriately. As observed by Grinberg and Zhang, one may write $R_{n-k}^*(d\mu)$ as $\mu * \sigma_{F_0}$, where $F_0 = \pi(e)$, e is the identity element in $O(n)$ and π is the canonical projection as above. Since in the w^*-topology, σ_{F_0} may be approximated by $\rho_{\mathcal{E}_i(F_0,\epsilon)}^k$, and μ by a discrete measure, the theorem follows after several technicalities are treated.

We mention a different way to conclude the theorem. It is easy to verify that

$$R_{n-k}\big(\rho_{\mathcal{E}(F,\epsilon)}^k\big)(E) = R_{n-k}\big(\rho_{\mathcal{E}(E,\epsilon)}^k\big)(F) \quad \forall E, F \in G(n, n-k).$$

Denoting $G = G(n, n-k)$ for short, if $\rho_K^k = R_{n-k}^*(d\mu)$ we have:

$$\begin{aligned} R_{n-k}\big(\rho_K^k\big)(F) &= \int_{S^{n-1}} \rho_K^k(\theta)\, d\sigma_F(\theta) = \lim_{\epsilon\to 0} \int_{S^{n-1}} \rho_{\mathcal{E}(F,\epsilon)}^k(\theta) \rho_K^k(\theta)\, d\sigma(\theta) \\ &= \lim_{\epsilon\to 0} \int_{S^{n-1}} \rho_{\mathcal{E}(F,\epsilon)}^k(\theta) R_{n-k}^*(d\mu)(\theta)\, d\sigma(\theta) = \lim_{\epsilon\to 0} \int_G R_{n-k}\big(\rho_{\mathcal{E}(F,\epsilon)}^k\big)(E)\, d\mu(E) \\ &= \lim_{\epsilon\to 0} \int_G R_{n-k}\big(\rho_{\mathcal{E}(E,\epsilon)}^k\big)(F)\, d\mu(E) = R_{n-k}\bigg(\lim_{\epsilon\to 0} \int_G \rho_{\mathcal{E}(E,\epsilon)}^k\, d\mu(E)\bigg)(F), \end{aligned}$$

where we have used the uniform convergence of all the limits involved and that R_{n-k} is a continuous operator with respect to the maximum-norm. The result then follows from the injectivity of R_{n-k} on $C_e(S^{n-1})$.

Grinberg and Zhang's characterization of the class $\mathcal{BP}_k^n$ implies that it is actually generated from D_n, the Euclidean unit ball, by taking full-rank linear transformations, k-radial sums, and limit in the radial metric. By starting from any other star-body L and performing these operations, it is obvious that D_n may be constructed, and therefore we see that $\mathcal{BP}_k^n$ is the minimal non-empty class which is closed under these three operations. Since $\mathcal{I}_k^n$ trivially contains D_n and is also closed under these operations, it immediately implies the following corollary.

Corollary 2.2. $\mathcal{BP}_k^n \subset \mathcal{I}_k^n$.

This was first observed by Koldobsky in [21] using a different approach. We will give another proof of this in Corollary 4.4, which is in a sense more concrete.

We conclude this preliminary discussion of the class $\mathcal{BP}_k^n$ by elaborating a little more on the operation of convolution between measures on homogeneous spaces of $O(n)$. Let G, H denote homogeneous spaces of $O(n)$. We identify between a function $f \in C(G)$ and the measure on $C(G)$ whose density with respect to the Haar probability measure on G is given by f, and consider expressions of the form $f * \mu$ and $\mu * f$ for $\mu \in \mathcal{M}(H)$. With the same notations, if $f \in C^\infty(G)$ then a standard argument shows that $f * \mu \in C^\infty(H)$ and that $\mu * f \in C^\infty(G)$. If $\eta \in \mathcal{M}(O(n))$, it is immediate to check that $\langle \mu, \eta * f\rangle_G = \langle \eta^{-1} * \mu, f\rangle_G$, where $\eta^{-1} \in \mathcal{M}(O(n))$ is the measure defined by $\eta^{-1}(A) = \eta(A^{-1})$ and $A^{-1} = \{u^{-1} \mid u \in A\}$ for a Borel set $A \subset O(n)$. If $\mu_i \in \mathcal{M}(G_i)$ for $i = 1, 2, 3$, one may verify that this operation is associative:

$$(\mu_1 * \mu_2) * \mu_3 = \mu_1 * (\mu_2 * \mu_3).$$

We conclude with the following lemma from [16] which will be useful later on.

Lemma 2.3. *There exists a sequence of functions* $\{u_i\} \subset C_+^\infty(O(n))$ *called an* approximate identity*, such that for any homogeneous space* G *of* $O(n)$:

1. *For any* $\mu \in \mathcal{M}(G)$, $u_i * \mu \in C^\infty(G)$ *tends to* μ *in the* w^**-topology.*
2. *For any* $g \in C(G)$, $u_i * g \in C^\infty(G)$ *tends to* g *uniformly.*

2.4. The class $\mathcal{I}_k^n$

In order to handle the class $\mathcal{I}_k^n$, we shall need to adopt a technique extensively used by Koldobsky: Fourier transforms of homogeneous distributions. We will only outline the main ideas here, usually omitting the technical details—we refer the reader to [22] for those. We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of rapidly decreasing infinitely differentiable test functions in $\mathbb{R}^n$, and by $\mathcal{S}'(\mathbb{R}^n)$ the space of distributions over $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform $\hat{f}$ of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by $\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle$ for every test function ϕ, where

$$\hat{\phi}(y) = \int \phi(x) \exp\bigl(-i\langle x, y\rangle\bigr)\, dx.$$

A distribution f is called homogeneous of degree $p \in \mathbb{R}$ if $\langle f, \phi(\cdot/t)\rangle = |t|^{n+p}\langle f, \phi\rangle$ for every $t > 0$, and it is called even if the same is true for $t = -1$. An even distribution f always satisfies

$(\hat{f})^\wedge = (2\pi)^n f$. The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$. A distribution f is called positive if $\langle f, \phi\rangle \geqslant 0$ for every $\phi \geqslant 0$, implying that f is necessarily a non-negative Borel measure on $\mathbb{R}^n$. We use Schwartz's generalization of Bochner's theorem [9] as a definition, and call a homogeneous distribution positive-definite if its Fourier transform is a positive distribution.

Before proceeding, let us give some intuition about how the Fourier transform of a homogeneous continuous function looks like. Because of the homogeneity, it is enough to consider a continuous function on the sphere $f \in C(S^{n-1})$, and take its homogeneous extension of degree $p \in \mathbb{R}$, denoted $E_p(f)$, to the entire $\mathbb{R}^n$ (formally excluding $\{0\}$ if $p < 0$). When $p > -n$, the function $E_p(f)$ is locally integrable, and its action as a distribution on a test function ϕ is simply by integration. Passing to polar coordinates, we have

$$\langle E_p(f), \phi\rangle = \int_{S^{n-1}} f(\theta) \int_0^\infty r^{p+n-1}\phi(r\theta)\, dr\, d\theta.$$

When $p \leqslant -n$, we can no longer interpret the action of $E_p(f)$ as an integral. Fortunately, we will mainly be concerned with Fourier transforms of continuous functions which are homogeneous of degree $p \in (-n, 0)$. This ensures that the Fourier transform is a homogeneous distribution of degree $-p-n$, which is in the same range $(-n, 0)$. Note that the resulting distribution need not necessarily be a continuous function on $\mathbb{R}^n \setminus \{0\}$, nor even a measure on $\mathbb{R}^n$ (although this will not occur in our context). We will denote by $E_p^\wedge(f)$ the Fourier transform of $E_p(f)$. In order to ensure that $E_p^\wedge(f)$ is a continuous function, we need to add some smoothness assumptions on f [22]. We remark that for a continuous function $f \in C(S^{n-1})$, $E_p^\wedge(f)$ is always continuous for $p \in (-n, n+1]$, and that for an infinitely smooth $f \in C^\infty(S^{n-1})$, $E_p^\wedge(f)$ is infinitely smooth for any $p \in (-n, 0)$. Whenever $E_p^\wedge(f)$ is continuous on $\mathbb{R}^n \setminus \{0\}$, it is uniquely determined by its value on S^{n-1} (by homogeneity). In that case, by abuse of notation, we identify between $E_p^\wedge(f)$ and its restriction to S^{n-1}, and in particular, consider $E_p^\wedge$ as an operator from $C^\infty(S^{n-1})$ to $C^\infty(S^{n-1})$.

When $f = 1$, it is easy to verify that $E_p^\wedge(1)$ is rotational invariant, so by the homogeneity, it must be a multiple of $E_{-n-p}(1)$. For a rigorous proof we refer to [9, p. 192], and state this for future reference as:

Lemma 2.4. *Fix n and let $p \in (0, n)$. Then*

$$E_{-p}^\wedge(1) = c(n,p) E_{-n+p}(1) \quad \textit{where } c(n,p) = \pi^{n/2} 2^{n-p} \frac{\Gamma((n-p)/2)}{\Gamma(p/2)}.$$

Since $(E_{-p}^\wedge(1))^\wedge = (2\pi)^n E_{-p}(1)$, it is clear that

$$c(n,p)c(n,n-p) = (2\pi)^n.$$

The following characterization was given by Koldobsky.

Theorem 2.5. (Koldobsky [21]) *The following are equivalent for a centrally-symmetric star-body K in $\mathbb{R}^n$:*

(1) *K is a k-intersection body.*
(2) *$\|x\|_K^{-k}$ is a positive definite distribution on $\mathbb{R}^n$, meaning that its Fourier-transform $(\|\cdot\|_K^{-k})^\wedge$ is a non-negative Borel measure on $\mathbb{R}^n$.*
(3) *The space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{-k}.*

For completeness, we briefly give the definition of embedding in L_{-k}, although we will not use this later on. Let us denote the class of centrally-symmetric star bodies K in $\mathbb{R}^n$ for which $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_p by SL_p^n. For $p > 0$, it is well known (e.g. [21]) that $K \in SL_p^n$ iff:

$$\|x\|_K^p = \int_{S^{n-1}} \left|\langle x, \theta\rangle\right|^p d\mu_K(\theta), \tag{2.1}$$

for some $\mu_K \in \mathcal{M}_+(S^{n-1})$. Unfortunately, this characterization breaks down at $p = -1$ since the above integral no longer converges. However, Koldobsky showed that it is possible to regularize this integral by using Fourier-transforms of distributions, and gave the following definition: $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{-p} for $0 < p < n$ iff there exists a measure $\mu_K \in \mathcal{M}_+(S^{n-1})$ such that for any even test-function ϕ

$$\int_{\mathbb{R}^n} \|x\|_K^{-p} \phi(x)\, dx = \int_{S^{n-1}} \int_0^\infty t^{p-1} \hat{\phi}(t\theta)\, dt\, d\mu_K(\theta). \tag{2.2}$$

Let us review the statements of Theorem 2.5. (2) is an extremely useful characterization of k-intersection bodies, and immediately implies the closure of $\mathcal{I}_k^n$ under the standard three operations. Characterization (3) provides additional motivation for why it is reasonable to believe that $\mathcal{BP}_k^n = \mathcal{I}_k^n$. For $p \neq 0$, the p-norm sum of two bodies L_1, L_2 is defined as the body L satisfying $\|\cdot\|_L^p = \|\cdot\|_{L_1}^p + \|\cdot\|_{L_2}^p$. We will denote by D_p^n, the class of bodies created from D_n by applying full-rank linear-transformations, p-norm sums, and taking the limit in the radial metric. Using the characterization in (2.1), it is easy to show (e.g. [16, Theorem 6.13]) that for $p > 0$, the class SL_p^n coincides with D_p^n. Although this characterization breaks down at $p = -1$, it is still reasonable to expect that the property $SL_p^n = D_p^n$ should pass over to negative values of p when SL_p^n is (in some sense) extended to this range and becomes $SL_{-k}^n = \mathcal{I}_k^n$. But by Grinberg and Zhang's characterization (Theorem 2.1), this is exactly satisfied by k-Busemann–Petty bodies: $\mathcal{BP}_k^n = D_{-k}^n$. This suggests that indeed $\mathcal{BP}_k^n = \mathcal{I}_k^n$.

In addition to the characterization (3) of $\mathcal{I}_k^n$ as the class of unit-balls of subspaces of *scalar* L_{-k} spaces, a functional analytic characterization of $\mathcal{BP}_k^n$ as the class of unit-balls of subspaces of *vector-valued* L_{-k} spaces (in a manner similar to (2.2)), was given in [21]. This provides additional motivation for believing that $\mathcal{BP}_k^n = \mathcal{I}_k^n$, as this would be an extension to negative values of p of the fact that every separable vector valued L_p space is isometric to a subspace of a scalar L_p space and vice-versa.

We proceed to explain why (1) and (2) in Theorem 2.5 are equivalent. To this end, we will need the following spherical Parseval identity, due to Koldobsky.

Spherical Parseval. (Koldobsky [22]) *Let* $f, g \in C_e^\infty(S^{n-1})$, *and* $p \in (0, n)$. *Then*

$$\int_{S^{n-1}} E_{-p}^\wedge(f)(\theta) E_{-n+p}^\wedge(g)(\theta)\, d\sigma(\theta) = (2\pi)^n \int_{S^{n-1}} f(\theta) g(\theta)\, d\sigma(\theta).$$

We prefer to present a self-contained proof of this identity, which seems simpler than the previous approaches in [22].

Proof. Let $f = \sum_{k=0}^\infty f_k$ and $g = \sum_{k=0}^\infty g_k$ be the canonical decompositions into spherical harmonics, where $f_k, g_k \in H_k$ and H_k is the space of spherical harmonics of degree k. Since f and g are even, it follows that $f_{2k+1} = g_{2k+1} = 0$. It is well known [32] that for $q \in (-n, 0)$, the linear operator $E_q^\wedge : C^\infty(S^{n-1}) \to C^\infty(S^{n-1})$ decomposes into a direct sum of scalar operators acting on H_k. Indeed, one only needs to check that the H_k's are eigenspaces of $E_q^\wedge$, and by Schur's representation lemma and the fact that the Fourier transform commutes with the action of the orthogonal group, it follows that $E_q^\wedge$ must act as a scalar on these spaces. Denote by $c_k^{(q)}$ the eigenvalue satisfying $E_q^\wedge(h_k) = c_k^{(q)} h_k$ for any $h_k \in H_k$. The exact value of $c_k^{(q)}$ is well known [32, Theorem 4.1], but is irrelevant to our proof. It remains to notice that since:

$$E_{-n+p}^\wedge\big(E_{-p}^\wedge(f)\big) = \big(E_{-p}(f)^\wedge\big)^\wedge\big|_{S^{n-1}} = (2\pi)^n f,$$

for any $f \in C_e^\infty(S^{n-1})$, we must have $c_k^{(-n+p)} c_k^{(-p)} = (2\pi)^n$ for all *even* k's. Using the fact that spherical harmonics of different degrees are orthogonal to each other in $L_2(S^{n-1})$, and that $f, g, E_{-p}^\wedge(f)$ and $E_{-n+p}^\wedge(g)$ are all in $L_2(S^{n-1})$, we conclude:

$$\begin{aligned}
&\int_{S^{n-1}} E_{-p}^\wedge(f)(\theta) E_{-n+p}^\wedge(g)(\theta)\, d\sigma(\theta) \\
&\quad = \int_{S^{n-1}} \sum_{k=0}^\infty c_k^{(-p)} f_k(\theta) \sum_{l=0}^\infty c_l^{(-n+p)} g_l(\theta)\, d\sigma(\theta) \\
&\quad = \int_{S^{n-1}} \sum_{k=0}^\infty c_k^{(-p)} c_k^{(-n+p)} f_k(\theta) g_k(\theta)\, d\sigma(\theta) = (2\pi)^n \int_{S^{n-1}} \sum_{k=0}^\infty f_k(\theta) g_k(\theta)\, d\sigma(\theta) \\
&\quad = (2\pi)^n \int_{S^{n-1}} \sum_{k=0}^\infty f_k(\theta) \sum_{l=0}^\infty g_l(\theta)\, d\sigma(\theta) = (2\pi)^n \int_{S^{n-1}} f(\theta) g(\theta)\, d\sigma(\theta). \qquad \square
\end{aligned}$$

Note that the above argument actually shows that the spherical Parseval identity is also valid when $f, g, E_{-p}^\wedge(f), E_{-n+p}^\wedge(g) \in L_2(S^{n-1})$.

Remark 2.6. Applying the theorem to $g = E^{\wedge}_{-p}(g')$ for $g' \in C^\infty_e(S^{n-1})$ and using that $E^{\wedge}_{-n+p}(g) = (2\pi)^n g'$, we note that the spherical Parseval identity has the following equivalent form, which we will sometimes use:

$$\int_{S^{n-1}} E^{\wedge}_{-p}(f)(\theta) g(\theta)\, d\sigma(\theta) = \int_{S^{n-1}} f(\theta) E^{\wedge}_{-p}(g)(\theta)\, d\sigma(\theta).$$

Another useful result due to Koldobsky, which looks very similar to the spherical Parseval identity, is the following.

Theorem 2.7. (Koldobsky) *Let $f \in C^\infty_e(S^{n-1})$, and let $k = 1, \ldots, n-1$. Then for any $H \in G(n,k)$*

$$\int_{S^{n-1}\cap H^\perp} E^{\wedge}_{-k}(f)(\theta)\, d\sigma_{H^\perp}(\theta) = c(n,k) \int_{S^{n-1}\cap H} f(\theta)\, d\sigma_H(\theta),$$

where $c(n,k)$ is the constant from Lemma 2.4.

Informally, the latter theorem may be considered as a special case of the spherical Parseval identity, by setting $g = d\sigma_H$ and verifying that in the appropriate sense $E^{\wedge}_{-n+k}(d\sigma_H) = c(n,k) d\sigma_{H^\perp}$. The constant in front of the right-hand integral is verified by choosing $f = 1$ and using Lemma 2.4. One way to make this argument work is to use Grinberg and Zhang's approximation of $d\sigma_H$ by the functions $\rho^{n-k}_{\mathcal{E}_i}$, which when written as $\|\cdot\|^{-n+k}_{\mathcal{E}_i}$ are seen to be already homogeneous of degree $-n+k$. Computing the Fourier transform is particularly easy, since $\mathcal{E}_i = T_i(D_n)$, and therefore

$$\begin{aligned}\big(\|\cdot\|^{-n+k}_{T_i(D_n)}\big)^\wedge(x) &= \big(\big\|T_i^{-1}(\cdot)\big\|^{-n+k}_{D_n}\big)^\wedge(x) = \det(T_i)\big(\|\cdot\|^{-n+k}_{D_n}\big)^\wedge\big(T_i^*(x)\big) \\ &= \det(T_i) d(n,k)\big\|T_i^*(x)\big\|^{-k}_{D_n} = \det(T_i) d(n,k)\|x\|^{-k}_{T_i^{-*}(D_n)}.\end{aligned}$$

Using Grinberg and Zhang's approximation again, it turns out that $\det(T_i) d(n,k) \rho^k_{T_i^{-*}(D_n)}$ tends in the w^*-topology to $c(n,k) d\sigma_{H^\perp}$.

We can now sketch a proof of Koldobsky's Fourier transform characterization of k-intersection bodies. By abuse of notation, when $(\|\cdot\|^{-k}_K)^\wedge$ is continuous, we will often use $\|\cdot\|^{-k}_K$, $(\|\cdot\|^{-k}_K)^\wedge$ to indicate both locally integrable functions on $\mathbb{R}^n$ and continuous functions on S^{n-1}. By definition, an *infinitely smooth* star-body K which is a k-intersection body of a star-body L, satisfies

$$\mathrm{Vol}(K \cap H^\perp) = \mathrm{Vol}(L \cap H) \quad \text{for all } H \in G(n, n-k).$$

Passing to polar coordinates, this is equivalent to

$$R_k\big(\|\cdot\|^{-k}_K\big)(H^\perp) = \frac{\mathrm{Vol}(D_{n-k})}{\mathrm{Vol}(D_k)} R_{n-k}\big(\|\cdot\|^{-n+k}_L\big)(H) \quad \forall H \in G(n, n-k).$$

But using Theorem 2.7, we see that

$$R_k\big(\|\cdot\|_K^{-k}\big)(H^\perp) = c(n,k)^{-1} R_{n-k}\big(\big(\|\cdot\|_K^{-k}\big)^\wedge\big)(H) \quad \forall H \in G(n, n-k).$$

From the injectivity of R_{n-k} on $C_e(S^{n-1})$, it follows that

$$\big(\|\cdot\|_K^{-k}\big)^\wedge = c(n,k)\frac{\mathrm{Vol}(D_{n-k})}{\mathrm{Vol}(D_k)}\|\cdot\|_L^{-n+k}$$

on S^{n-1}, and hence on all $\mathbb{R}^n$ by homogeneity. We conclude that $(\|\cdot\|_K^{-k})^\wedge$ is a non-negative continuous function on $\mathbb{R}^n \setminus \{0\}$, and hence positive as a distribution. For an arbitrary star-body K which is a k-intersection body of a star-body L, the same conclusion holds by approximation ($(\|\cdot\|_K^{-k})^\wedge$ is still continuous by the continuity of $\|\cdot\|_L^{-n+k}$). One may also invert the argument, proving that for a star-body K, if $(\|\cdot\|_K^{-k})^\wedge$ is a *continuous* function which is non-negative, then K is a k-intersection body of a star-body L (defined as above). Taking the limit in the radial metric, $(\|\cdot\|_K^{-k})^\wedge$ need not necessarily be a continuous function for a general k-intersection body K which is the limit of the bodies $\{K_i\}$ (which are k-intersection bodies of star-bodies). Nevertheless, the non-negative continuous functions $(\|\cdot\|_{K_i}^{-k})^\wedge$ must satisfy:

$$\int_{S^{n-1}} \big|\big(\|\cdot\|_{K_i}^{-k}\big)^\wedge(\theta)\big|\, d\sigma(\theta) = c(n,k)\int_{S^{n-1}} \|\theta\|_{K_i}^{-k}\, d\sigma(\theta),$$

by the spherical Parseval identity with $g = 1$ and Lemma 2.4, and therefore the integral on the left-hand side is bounded. Using the compactness of the unit-ball of $\mathcal{M}(S^{n-1})$ in the w^*-topology, there must be an accumulation point of $\{(\|\cdot\|_{K_i}^{-k})^\wedge\}$, which is a non-negative Borel measure on S^{n-1}. This argument is the main idea in the proof that for a star-body K, $K \in \mathcal{I}_k^n$ iff $(\|\cdot\|_K^{-k})^\wedge$ is a non-negative Borel measure on $\mathbb{R}^n$.

When K is infinitely smooth, we summarize this in the following alternative definition for $\mathcal{I}_k^n$, and use it instead of the original one.

Alternative Definition of $\mathcal{I}_k^n$. For an infinitely smooth star-body K, $K \in \mathcal{I}_k^n$ iff $(\|\cdot\|_K^{-k})^\wedge \geqslant 0$ as a C^∞ function on S^{n-1}.

For a general star-body K, we will use Koldobsky's characterization in the following spherical version, which is an immediate consequence of the above reasoning (a rigorous proof is given in [22, Corollary 3.23]).

Proposition 2.8. *For a star-body K, $K \in \mathcal{I}_k^n$ iff there exists a non-negative Borel measure μ on S^{n-1}, such that for any $f \in C_e^\infty(S^{n-1})$*

$$\int_{S^{n-1}} f(\theta)\rho_K^k(\theta)\, d\sigma(\theta) = \int_{S^{n-1}} E_{-n+k}^\wedge(f)(\theta)\, d\mu(\theta).$$

3. The identical structures of $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$

In this section we will prove the Structure Theorem, which was formulated in Section 1.

We will skip over item (1) which basically follows from the definitions, and was already explained in detail in Section 2.

Item (2) also follows immediately: by definition, $\mathcal{I}_1^n = \mathcal{BP}_1^n$ is exactly the class of intersection bodies in $\mathbb{R}^n$; any star-body K in $\mathbb{R}^n$ is an $(n-1)$-intersection body of a star-body L, defined by $\rho_L(\theta) = 1/2\mathrm{Vol}(K \cap \theta^\perp)$; and by definition, R_1^* acts as the identity on $C_e(S^{n-1})$, hence $\rho_K^{n-1} = R_1^*(\rho_K^{n-1})$ for any star-body K, implying that $K \in \mathcal{BP}_{n-1}^n$.

We therefore commence the proof from item (3). We will prove the theorem for $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$ separately, because of the different techniques involved in the proof.

Before we start, we will need the following useful lemma, which appears implicitly in [16]. We denote by $\mathcal{BP}_k^{n,\infty}$ the class of star-bodies K such that $\rho_K^k = R_{n-k}^*(g)$, where $g \in C_+^\infty(G(n,n-k))$. Obviously $\mathcal{BP}_k^{n,\infty} \subset \mathcal{BP}_k^n$.

Lemma 3.1. [16] *$\mathcal{BP}_k^{n,\infty}$ is dense in $\mathcal{BP}_k^n$. In particular, the class of infinitely smooth bodies in $\mathcal{BP}_k^n$ is dense in $\mathcal{BP}_k^n$.*

Proof. Let $K \in \mathcal{BP}_k^n$, and assume that $\rho_K^k = R_{n-k}^*(d\mu)$ where $d\mu \in \mathcal{M}_+(G(n,n-k))$. Let $\{u_i\} \subset C^\infty(O(n))$ be an approximate identity as in Lemma 2.3. Let K_i be the star-body for which $\rho_{K_i}^k = u_i * \rho_K^k$. Then by Lemma 2.3, $\{K_i\}$ is a sequence of infinitely smooth star-bodies which tend to K in the radial metric. As in the proof of Theorem 2.1, we write $\rho_K^k = \mu * \sigma_{H_0}$, and therefore

$$\rho_{K_i}^k = u_i * (\mu * \sigma_{H_0}) = (u_i * \mu) * \sigma_{H_0} = R_{n-k}^*(u_i * \mu).$$

Since $u_i * \mu \in C_+^\infty(G(n,n-k))$, this concludes the proof of the lemma. $\square$

Remark 3.2. By the lemma and the closure of $\mathcal{BP}_k^n$ (for any $k = 1, \ldots, n-1$) under limit in the radial metric, it is enough to prove all the remaining items for the classes $\mathcal{BP}_k^{n,\infty}$.

We will also require the following notations. Given $F \in G(n,m)$ and $k \geqslant m$, we denote by $G_F(n,k)$ the manifold $\{E \in G(n,k) | F \subset E\}$. For $\theta \in S^{n-1}$ we identify between θ and the one-dimensional subspace spanned by it. $G_F(n,k)$ is a homogeneous space of $O(n)$, therefore there exists a unique Haar probability measure on $G_F(n,k)$, which is invariant to orthogonal rotations in $O(n)$ which preserve F. Thus, if we denote by ν_σ the Haar probability measure on $G_\sigma(n,m)$ for $\sigma \in S^{n-1}$, then for any $g \in C(G(n,m))$ we may write:

$$R_m^*(g)(\theta) = \int_{G_\theta(n,m)} g(E)\, d\nu_\sigma(E).$$

We will need the following fact, which is an immediate corollary of Proposition A.1. We postpone the formulation and proof of Proposition A.1 for Appendix A, as the technique involved is different in spirit to the rest of this note.

Corollary 3.3. *Let $n > 1$ and let $k_1, k_2 \geqslant 1$ denote integers such that $l = k_1 + k_2 \leqslant n - 1$. Let $\theta \in S^{n-1}$. For $a = k_1, k_2, l$, denote by $G^a = G(n, n-a)$ and by μ^a_θ the Haar probability measure on G^a_θ. For $F \in G^l$ and $a = k_1, k_2$, denote by μ^a_F the Haar probability measure on G^a_F. Then for any continuous function $f(E_1, E_2)$ on $G^{k_1} \times G^{k_2}$:*

$$\int\limits_{E_1 \in G^{k_1}_\theta} \int\limits_{E_2 \in G^{k_2}_\theta} f(E_1, E_2)\, d\mu^{k_1}_\theta(E_1)\, d\mu^{k_2}_\theta(E_2)$$
$$= \int\limits_{F \in G^l_\theta} \int\limits_{E_1 \in G^{k_1}_F} \int\limits_{E_2 \in G^{k_2}_F} f(E_1, E_2) \Delta(E_1, E_2)\, d\mu^{k_1}_F(E_1)\, d\mu^{k_2}_F(E_2)\, d\mu^l_\theta(F),$$

where $\Delta(E_1, E_2)$ is some (known) non-negative continuous function on $G^{k_1} \times G^{k_2}$.

We will show the following basic property of k-Busemann–Petty bodies, and immediately deduce (3a), (3b) and (3c) from the Structure Theorem in Section 1.

Proposition 3.4. *Let $K_1 \in \mathcal{BP}^n_{k_1}$ and $K_2 \in \mathcal{BP}^n_{k_2}$ for $k_1, k_2 \geqslant 1$ such that $l = k_1 + k_2 \leqslant n - 1$. Then the star-body L defined by $\rho^l_L = \rho^{k_1}_{K_1} \rho^{k_2}_{K_2}$ satisfies $L \in \mathcal{BP}^n_l$.*

Proof. First, assume that $K_i \in \mathcal{BP}^{n,\infty}_{k_i}$ for $i = 1, 2$, so that $\rho^{k_i}_K = R^*_{n-k_i}(g_i)$ with $g_i \in C^\infty_+(G(n, n-k_i))$. Using the notations and result of Corollary 3.3, we have:

$$\rho^l_L(\theta) = \rho^{k_1}_{K_1}(\theta) \rho^{k_2}_{K_2}(\theta) = \int\limits_{E_1 \in G^{k_1}_\theta} g_1(E_1)\, d\mu^{k_1}_\theta(E_1) \int\limits_{E_2 \in G^{k_2}_\theta} g_2(E_2)\, d\mu^{k_2}_\theta(E_2)$$
$$= \int\limits_{F \in G^l_\theta} \int\limits_{E_1 \in G^{k_1}_F} \int\limits_{E_2 \in G^{k_2}_F} g(E_1) g(E_2) \Delta(E_1, E_2)\, d\mu^{k_1}_F(E_1)\, d\mu^{k_2}_F(E_2)\, d\mu^l_\theta(F).$$

Denoting

$$h(F) = \int\limits_{E_1 \in G^{k_1}_F} \int\limits_{E_2 \in G^{k_2}_F} g(E_1) g(E_2) \Delta(E_1, E_2)\, d\mu^{k_1}_F(E_1)\, d\mu^{k_2}_F(E_2),$$

we see that $h(F)$ is a non-negative continuous function on $G(n, n-l)$. Therefore

$$\rho^l_L(\theta) = \int\limits_{F \in G^l_\theta} h(F)\, d\mu^l_\theta(F),$$

implying that $L \in \mathcal{BP}^n_l$. The general case, when $K_i \in \mathcal{BP}^n_{k_i}$ without any smoothness assumptions, follows from Remark 3.2. Indeed, by approximating each K_i in the radial metric by smooth bodies $\{K^m_i\} \subset \mathcal{BP}^n_{k_i}$, the bodies $\{L^m\}$ defined by $\rho^l_{L^m} = \rho^{k_1}_{K^m_1} \rho^{k_2}_{K^m_2}$ satisfy that $L^m \in \mathcal{BP}^n_l$ and obviously L^m approximate L in the radial metric, implying that $L \in \mathcal{BP}^n_l$. □

Applying Proposition 3.4 with $K_1 = K_2$, we have:

Corollary 3.5. *$\mathcal{BP}^n_{k_1} \cap \mathcal{BP}^n_{k_2} \subset \mathcal{BP}^n_{k_1+k_2}$ for $k_1, k_2 \geqslant 1$ such that $k_1 + k_2 \leqslant n-1$.*

By successively applying Corollary 3.5, we see that $\mathcal{BP}^n_k \subset \mathcal{BP}^n_l$ if k divides l. The question whether $\mathcal{BP}^n_k \subset \mathcal{BP}^n_l$ for general $1 \leqslant k < l \leqslant n-1$ remains open. Nevertheless, we are able to show the following "non-linear" embedding of $\mathcal{BP}^n_k$ into $\mathcal{BP}^n_l$, which is again an immediate corollary of Proposition 3.4 (using $K_2 = D_n \in \mathcal{BP}^n_{l-k}$).

Proposition 3.6. *If $K \in \mathcal{BP}^n_k$ then the star-body L defined by $\rho_L = \rho_K^{k/l}$ satisfies $L \in \mathcal{BP}^n_l$ for $1 \leqslant k \leqslant l \leqslant n-1$.*

We prefer to give another proof of this statement, one which does not rely on Proposition A.1.

Proof. Assume that $K \in \mathcal{BP}^{n,\infty}_k$, so that $\rho_K^k = R^*_{n-k}(g_K)$ and $g_K \in C^\infty_+(G(n, n-k))$, and define the star-body L by $\rho_L = \rho_K^{k/l}$. For $\theta \in S^{n-1}$ and $a = k, l$ denote by μ^a_θ the Haar probability measure on $G_\theta(n, n-a)$. For $F \in G(n, n-l)$, denote by μ^k_F the Haar probability measure on $G_F(n, n-k)$. Then:

$$\rho_L^l(\theta) = \rho_K^k(\theta) = \int\limits_{G_\theta(n,n-k)} g_K(E)\, d\mu^k_\theta(E)$$

$$= \int\limits_{G_\theta(n,n-l)} \int\limits_{G_F(n,n-k)} g_K(E)\, d\mu^k_F(E)\, d\mu^l_\theta(F).$$

The last transition is justified by the fact that the probability measure $d\mu^k_F(E)\, d\mu^l_\theta(F)$ on $G_\theta(n, n-k)$ is invariant under orthogonal rotations in $O(n)$ which preserve θ, and therefore coincides with $d\mu^k_\theta(E)$, the Haar probability measure on $G_\theta(n, n-k)$. Defining $g_L \in C_+(G(n, n-l))$ by $g_L(F) = \int_{G_F(n,n-k)} g(E)\, d\mu^k_F(E)$ for $F \in G(n, n-l)$, we see that

$$\rho_L^l(\theta) = R^*_{n-l}(g_L)(\theta).$$

Together with Remark 3.2, this concludes the proof. □

The Ellipsoid Corollary from Section 1 should now be clear. We repeat it here for convenience.

Corollary 3.7. *For any $1 \leqslant k \leqslant n-1$ and k ellipsoids $\{\mathcal{E}_i\}_{i=1}^k$ in $\mathbb{R}^n$, define the body L by*

$$\rho_L = \rho_{\mathcal{E}_1} \cdot \cdots \cdot \rho_{\mathcal{E}_k},$$

and let $k \leqslant l \leqslant n-1$. Then there exists a sequence of star-bodies $\{L_i\}$ which tends to L in the radial metric and satisfies

$$\rho_{L_i} = \rho^l_{\mathcal{E}^i_1} + \cdots + \rho^l_{\mathcal{E}^i_{m_i}},$$

where $\{\mathcal{E}^i_j\}$ are ellipsoids.

Proof. The body L_2 defined by $\rho_{L_2}^k = \rho_L$ is in $\mathcal{BP}_k^n$ by Proposition 3.4 (applied successively to the ellipsoids $\{\mathcal{E}_i\}$, which are in $\mathcal{BP}_1^n$). For $l > k$, Proposition 3.6 implies that the body L_3 defined by $\rho_{L_3}^l = \rho_{L_2}^k = \rho_L$ is in $\mathcal{BP}_l^n$, otherwise this is trivial. Using Grinberg and Zhang's characterization of $\mathcal{BP}_l^n$ (Theorem 2.1), the claim is established. $\square$

Incidentally, Proposition 3.4 also enables us to give the following strengthened version of Theorem 2.1.

Corollary 3.8. *A star-body K is a k-Busemann–Petty body iff it is the limit of $\{K_i\}$ in the radial metric, where each K_i is of the following form*:

$$\rho_{K_i}^k = \rho_{\mathcal{E}_{1,1}^i} \cdot \dots \cdot \rho_{\mathcal{E}_{1,k}^i} + \dots + \rho_{\mathcal{E}_{m_i,1}^i} \cdot \dots \cdot \rho_{\mathcal{E}_{m_i,k}^i},$$

where $\{\mathcal{E}_{j,l}^i\}$ are ellipsoids.

Proof. Obviously this representation generalizes the one given by Grinberg and Zhang in Theorem 2.1, so it is enough to show the "if" part. But this follows from the closure of $\mathcal{BP}_k^n$ under limit in the radial metric, k-radial sums, and Proposition 3.4 (which as above shows that the body L defined by $\rho_L^k = \rho_{\mathcal{E}_1} \cdot \dots \cdot \rho_{\mathcal{E}_k}$ is in $\mathcal{BP}_k^n$). $\square$

For completeness, we conclude our investigation of the structure of $\mathcal{BP}_k^n$ with the following result of Grinberg and Zhang from [16]. Their argument is the same one used by Goodey and Weil for intersection bodies ($\mathcal{BP}_1^n$), and is an immediate corollary of Theorem 2.1.

Corollary 3.9. (Grinberg and Zhang [16]) *If $K \in \mathcal{BP}_k^n$ then any m-dimensional central section L of K (for $m > k$) satisfies $L \in \mathcal{BP}_k^m$.*

Proof. Since and central section of an ellipsoid is again an ellipsoid, the claim follows immediately from Theorem 2.1. $\square$

We now turn to prove the Structure Theorem from Section 1 for $\mathcal{I}_k^n$. As will be evident, the techniques involved are totally different from those which were used for $\mathcal{BP}_k^n$. The only point of similarity is Lemma 3.11. We denote by $\mathcal{I}_k^{n,\infty}$ the class of infinitely smooth k-intersection bodies in $\mathbb{R}^n$. As mentioned in Section 2, this implies for $K \in \mathcal{I}_k^{n,\infty}$ that $\|\cdot\|_K^{-k}, (\|\cdot\|_K^{-k})^\wedge \in C^\infty(\mathbb{R}^n \setminus \{0\})$. We begin with the following useful lemma.

Lemma 3.10. *For any $p \in (-n, 0)$, $g \in C^\infty(S^{n-1})$ and $\mu \in \mathcal{M}(O(n))$, $E_p^\wedge(\mu * g) = \mu * E_p^\wedge(g)$ as functions on $\mathbb{R}^n \setminus \{0\}$.*

Proof. First, let us extend the definition of $\mu * f$ to any function $f \in C(\mathbb{R}^n)$, as follows:

$$(\mu * f)(x) = \int_{O(n)} f\bigl(u(x)\bigr)\, d\mu(u) \quad \text{for every } x \in \mathbb{R}^n.$$

Next, notice that for a test function ϕ, $(\mu * \phi)^\wedge = \mu * \hat{\phi}$. Indeed, when μ is a delta function at $u \in O(n)$, $(\phi(u(\cdot)))^\wedge(x) = \hat{\phi}(u(x))$ because the Fourier transform commutes with the action of $O(n)$. And for a general $\mu \in \mathcal{M}(O(n))$, by Fubini's theorem:

$$
\begin{aligned}
(\mu * \phi)^\wedge(x) &= \int_{\mathbb{R}^n} \int_{O(n)} \phi\big(u(y)\big)\, d\mu(u) \exp\big(-i\langle y, x\rangle\big)\, dy \\
&= \int_{O(n)} \int_{\mathbb{R}^n} \phi\big(u(y)\big) \exp\big(-i\langle y, x\rangle\big)\, dy\, d\mu(u) \\
&= \int_{O(n)} \big(\phi\big(u(\cdot)\big)\big)^\wedge(x)\, d\mu(u) = \int_{O(n)} \hat{\phi}\big(u(x)\big)\, d\mu(u) = \mu * \hat{\phi}.
\end{aligned}
$$

Since $g, \mu * g \in C^\infty(S^{n-1})$, it follows that $E_p^\wedge(\mu * g), \mu * E_p^\wedge(g) \in C^\infty(\mathbb{R}^n \setminus \{0\})$, and for any test function ϕ:

$$
\begin{aligned}
\big\langle E_p^\wedge(\mu * g), \phi\big\rangle &= \big\langle E_p(\mu * g), \hat{\phi}\big\rangle = \big\langle \mu * E_p(g), \hat{\phi}\big\rangle = \big\langle E_p(g), \mu^{-1} * \hat{\phi}\big\rangle \\
&= \big\langle E_p(g), (\mu^{-1} * \phi)^\wedge\big\rangle = \big\langle E_p^\wedge(g), \mu^{-1} * \phi\big\rangle = \big\langle \mu * E_p^\wedge(g), \phi\big\rangle.
\end{aligned}
$$

Therefore $E_p^\wedge(\mu * g) = \mu * E_p^\wedge(g)$ as functions. □

Lemma 3.11. *$\mathcal{I}_k^{n,\infty}$ is dense in $\mathcal{I}_k^n$.*

Proof. Let $K \in \mathcal{I}_k^n$, and let $\mu \in \mathcal{M}_+(S^{n-1})$ be the measure from Proposition 2.8 satisfying for every $f \in C^\infty(S^{n-1})$

$$
\int_{S^{n-1}} f(\theta)\rho_K^k(\theta)\, d\sigma(\theta) = \int_{S^{n-1}} E_{-n+k}^\wedge(f)(\theta)\, d\mu(\theta).
$$

Let $\{u_i\} \subset C^\infty(O(n))$ be an approximate identity as in Lemma 2.3, and let K_i be the star-body for which $\rho_{K_i}^k = u_i * \rho_K^k$. Then by Lemma 2.3, $\{K_i\}$ is a sequence of infinitely smooth star-bodies which tend to K in the radial metric. It remains to check that each K_i is a k-intersection body. Indeed, using the notations of Section 2 and Lemma 3.10, for any $f \in C^\infty(S^{n-1})$:

$$
\begin{aligned}
\big\langle f, \rho_{K_i}^k\big\rangle &= \big\langle f, u_i * \rho_K^k\big\rangle = \big\langle u_i^{-1} * f, \rho_K^k\big\rangle = \big\langle E_{-n+k}^\wedge(u_i^{-1} * f), \mu\big\rangle \\
&= \big\langle u_i^{-1} * E_{-n+k}^\wedge(f), \mu\big\rangle = \big\langle E_{-n+k}^\wedge(f), u_i * \mu\big\rangle.
\end{aligned}
$$

Since $u_i * \mu \in C_+^\infty(S^{n-1})$, again by Proposition 2.8 this implies that $K_i \in \mathcal{I}_k^n$. □

Remark 3.12. By the lemma and the closure of $\mathcal{I}_k^n$ (for any $k = 1, \ldots, n-1$) under limit in the radial metric, it is enough to prove all the remaining items for the classes $\mathcal{I}_k^{n,\infty}$.

For the next fundamental proposition, we will need the following observation. It is classical that for two test functions ϕ_1, ϕ_2, $(\phi_1\phi_2)^\wedge = \hat{\phi}_1 \star \hat{\phi}_1$ where $\star$ denotes the standard convolution on $\mathbb{R}^n$. In general, the convolution of two distributions does not exist. Nevertheless, when the two distributions f_1, f_2 are locally integrable homogeneous functions with the right degrees, their convolution may be defined as usual. Assume that f_i is even homogeneous of degree $-n + p_i$ for

$p_i > 0$ and that $p_1 + p_2 < n$. Since f_i are locally integrable and at infinity their product decays faster than $|x|^{-n}$, the following integral converges for $x \in \mathbb{R}^n \setminus \{0\}$:

$$f_1 \star f_2(x) = \int f_1(x-y) f_2(y)\, dy. \tag{3.1}$$

It is easy to check that with this definition, $f_1 \star f_2$ is homogeneous of degree $-n + p_1 + p_2$, hence again locally integrable. Now assume in addition that f_i are infinitely smooth functions on $\mathbb{R}^n \setminus \{0\}$, and therefore so are $\hat{f}_i$. We claim that as distributions $(f_1 \star f_2)^\wedge = \hat{f}_1 \hat{f}_2$. To see this, we define the product and convolution of an even distribution f with an even test-function ϕ, as the distributions denoted ϕf and $\phi \star f$, respectively, satisfying for any test function φ that:

$$\langle \phi f, \varphi \rangle = \langle f, \phi\varphi \rangle \text{ and } \langle \phi \star f, \varphi \rangle = \langle f, \phi \star \varphi \rangle.$$

When f is a locally integrable function, it is clear that ϕf and $\phi \star f$ as distributions coincide with the usual product and convolution as functions. The same reasoning shows that when f_1, f_2 are locally integrable even functions such that $f_1 f_2$ is integrable at infinity (as before the definition in (3.1)), we have:

$$\langle f_1 \star f_2, \phi \rangle = \langle f_1, \phi \star f_2 \rangle, \tag{3.2}$$

where the action $\langle \cdot, \cdot \rangle$ is interpreted here and henceforth as integration in $\mathbb{R}^n$. Similarly, when $f_1 f_2$ is locally integrable, we have:

$$\langle f_1 f_2, \phi \rangle = \langle f_1, \phi f_2 \rangle. \tag{3.3}$$

With the above definitions, we see that $(\phi \star f)^\wedge = \hat{\phi} \hat{f}$ because for any test function φ:

$$\langle \phi \star f, \hat{\varphi} \rangle = \langle f, \phi \star \hat{\varphi} \rangle = \langle \hat{f}, \hat{\phi}\varphi \rangle = \langle \hat{\phi} \hat{f}, \varphi \rangle. \tag{3.4}$$

Now when f, g are two locally integrable infinitely smooth functions on $\mathbb{R}^n \setminus \{0\}$, such that $\hat{f} g$ is locally integrable, it is easy to see that we may replace φ in (3.4) with g. The reason is that we may weakly approximate g with test functions g_i such that $\int h g_i \to \int hg$ and $\int h \hat{g_i} \to \int h\hat{g}$, for any locally integrable continuous function h on $\mathbb{R}^n \setminus \{0\}$ such that $\int hg$ exists. For instance, we may use $g_i = (g \star \delta_i)\hat{\delta_i}$, where δ_i are Gaussians tending to a delta-function at 0; by (3.4) it is clear that $\hat{g_i} = (\hat{g}\hat{\delta_i}) \star \delta_i$, which weakly tends to $\hat{g}$ (by testing against a test-function). We summarize this by writing:

$$\langle \phi \star f, \hat{g} \rangle = \langle \hat{\phi} \hat{f}, g \rangle. \tag{3.5}$$

Combining (3.2), (3.3) and (3.5) and using the fact that $f_i, \hat{f}_i, \hat{f}_1 \hat{f}_2$ are infinitely smooth and locally integrable, we see that for any even test function ϕ:

$$\big\langle (f_1 \star f_2)^\wedge, \phi \big\rangle = \langle f_1 \star f_2, \hat{\phi} \rangle = \langle f_1, \hat{\phi} \star f_2 \rangle = \langle \hat{f}_1, \phi \hat{f}_2 \rangle = \langle \hat{f}_1 \hat{f}_2, \phi \rangle.$$

This proves that under the above conditions:

$$(f_1 \star f_2)^\wedge = \hat{f}_1 \hat{f}_2. \tag{3.6}$$

Remark 3.13. Note that the homogeneity of f_1, f_2 was not used, we only needed the appropriate asymptotic behavior at 0 and infinity. Using the homogeneity, a different approach to derive (3.6) was suggested to us by A. Koldobsky, by applying [13, Lemma 1]. With this approach, the smoothness assumptions on f_1, f_2 may be omitted, and (3.6) is understood as equality between distributions.

Using this notion of convolution, we can now show the following basic property of k-intersection bodies, and immediately deduce (3a), (3b) and (3c) from the Structure Theorem in Section 1. The following was also recently noticed independently by Koldobsky (but not published).

Proposition 3.14. *Let $K_1 \in \mathcal{I}^n_{k_1}$ and $K_2 \in \mathcal{I}^n_{k_2}$ for $k_1, k_2 \geqslant 1$ such that $l = k_1 + k_2 \leqslant n - 1$. Then the star-body L defined by $\rho_L^l = \rho_{K_1}^{k_1} \rho_{K_2}^{k_2}$ satisfies $L \in \mathcal{I}^n_l$.*

Proof. First, assume that $K_i \in \mathcal{I}^{n,\infty}_{k_i}$ for $i = 1, 2$, so that $(\|\cdot\|_K^{-k_i})^\wedge \in C^\infty_+(\mathbb{R}^n \setminus \{0\})$ and is homogeneous of degree $-n + k_i$. Since $l < n$ the convolution $(\|\cdot\|_K^{-k_1})^\wedge \star (\|\cdot\|_K^{-k_2})^\wedge$ as distributions is well defined (as explained above). Therefore:

$$\big(\|\cdot\|_L^{-l}\big)^\wedge = \big(\|\cdot\|_K^{-k_1}\|\cdot\|_K^{-k_2}\big)^\wedge = \big(\|\cdot\|_K^{-k_1}\big)^\wedge \star \big(\|\cdot\|_K^{-k_2}\big)^\wedge \geqslant 0,$$

as a function on $\mathbb{R}^n \setminus \{0\}$, which implies that $L \in \mathcal{I}^n_l$. The general case, when $K_i \in \mathcal{I}^n_{k_i}$ without any smoothness assumptions, follows from Remark 3.12 in the same manner as in the proof of Proposition 3.4. □

Applying Proposition 3.14 with $K_1 = K_2$, we have:

Corollary 3.15. $\mathcal{I}^n_{k_1} \cap \mathcal{I}^n_{k_2} \subset \mathcal{I}^n_{k_1+k_2}$ *for $k_1, k_2 \geqslant 1$ such that $k_1 + k_2 \leqslant n - 1$.*

By successively applying Corollary 3.15, we see that $\mathcal{I}^n_k \subset \mathcal{I}^n_l$ if k divides l. As for the class $\mathcal{BP}$, the question whether $\mathcal{I}^n_k \subset \mathcal{I}^n_l$ for general $1 \leqslant k < l \leqslant n - 1$ remains open. Nevertheless, we are able to show again the following "non-linear" embedding of $\mathcal{I}^n_k$ into $\mathcal{I}^n_l$, which is again an immediate corollary of Proposition 3.14 (using $K_2 = D_n \in \mathcal{I}^n_{l-k}$):

Corollary 3.16. *If $K \in \mathcal{I}^n_k$ then the star-body L defined by $\rho_L = \rho_K^{k/l}$ satisfies $L \in \mathcal{I}^n_l$ for $1 \leqslant k \leqslant l \leqslant n - 1$.*

We conclude this section with our last observation.

Proposition 3.17. *If $K \in \mathcal{I}^n_k$ then any m-dimensional central section L of K (for $m > k$) satisfies $L \in \mathcal{I}^m_k$.*

Proof. Let K be a star-body in $\mathbb{R}^n$, fix $k \in \{1, \ldots, n-2\}$, and let $H \in G(n, m)$ for $m > k$. In view of Theorem 2.5, we have to show that as distributions

$$\big(\|\cdot\|_K^{-k}\big)^\wedge \geqslant 0 \quad \text{implies} \quad \big(\|\cdot\|_K^{-k}\big|_H\big)^\wedge = \big(\|\cdot\|_{K\cap H}^{-k}\big)^\wedge \geqslant 0.$$

This becomes intuitively clear, after noticing that for a test function ϕ:

$$(\phi|_H)^\wedge(u) = \int\limits_{u+H^\perp} \hat{\phi}(y)\,dy.$$

Nevertheless, for a more general function $f = \|\cdot\|_K^{-k}$ such that $\hat{f} \geqslant 0$ as a distribution, we will need a somewhat different proof. Note that since $m > k$, f is locally integrable on any affine translate $z + H$, and that for any test function ϕ_H on H, $\int_H f(y+z)\phi(y)\,dy$ is continuous with respect to $z \in H^\perp$. Now let ϕ_H be any *non-negative* test function on H. For $\epsilon > 0$, denote by $\varphi_{H^\perp,\epsilon}$ the (positive) Gaussian function on $H^\perp$ such that $(\varphi_{H^\perp,\epsilon})^\wedge$ is the density function of a standard Gaussian variable on $H^\perp$ with covariance matrix $\epsilon I_{H^\perp}$. For $y \in H$ and $z \in H^\perp$, define $\phi_\epsilon(y+z) = \phi_H(y)\varphi_{H^\perp,\epsilon}(z)$. Clearly ϕ_ϵ is a test function on $\mathbb{R}^n$, $\phi_\epsilon \geqslant 0$, and $(\phi_\epsilon)^\wedge(y+z) = (\phi_H)^\wedge(y)(\varphi_{H^\perp,\epsilon})^\wedge(z)$. We therefore have:

$$\big\langle (f|_H)^\wedge, \phi_H \big\rangle = \big\langle f|_H, (\phi_H)^\wedge \big\rangle = \int\limits_H f(y)(\phi_H)^\wedge(y)\,dy \tag{3.7}$$

$$= \lim_{\epsilon\to 0} \int\limits_{H^\perp} (\varphi_{H^\perp,\epsilon})^\wedge(z) \int\limits_H f(y+z)(\phi_H)^\wedge(y)\,dy\,dz \tag{3.8}$$

$$= \lim_{\epsilon\to 0} \int\limits_{\mathbb{R}^n} f(x)(\phi_\epsilon)^\wedge(x)\,dx \tag{3.9}$$

$$= \lim_{\epsilon\to 0} \big\langle f, (\phi_\epsilon)^\wedge \big\rangle = \lim_{\epsilon\to 0} \langle \hat{f}, \phi_\epsilon \rangle \geqslant 0. \tag{3.10}$$

Since $\phi_H \geqslant 0$ was arbitrary, it follows that $(f|_H)^\wedge \geqslant 0$. □

4. The connection between Radon and Fourier transforms

We have seen that although the classes $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$ share the exact same structure and easily verify that $\mathcal{BP}_k^n \subset \mathcal{I}_k^n$, they are defined and handled using very different notions: Radon and Fourier transforms, respectively. The aim of this section is to establish a common ground that will enable to attack the question of whether $\mathcal{BP}_k^n = \mathcal{I}_k^n$ from a unified point of view. Since $\mathcal{BP}_k^n \subset \mathcal{I}_k^n$, it seems natural that this common ground will involve the language of Radon transforms, so we will have to translate the action of the Fourier transform to this language.

We will use the following notation. If $\mu \in \mathcal{M}(G(n, n-m))$, we denote by $\mu^\perp \in \mathcal{M}(G(n,m))$ the measure defined by $\mu^\perp(A) = \mu(A^\perp)$ for any Borel set $A \subset G(n,m)$, where $A^\perp = \{E^\perp \mid E \in A\}$. Note that the operation $\mu \to \mu^\perp$ is dual to the operator $I : C(G(n,m)) \to C(G(n, n-m))$ defined in Section 1, in the sense that $\langle \mu, I(f)\rangle_{G(n,n-m)} = \langle \mu^\perp, f\rangle_{G(n,m)}$. We recall that $I(f)(E) = f^\perp(E) = f(E^\perp)$ for any $E \in G(n, n-m)$. We therefore extend I to an operator $I : \mathcal{M}(G(n,m)) \to \mathcal{M}(G(n, n-m))$, defined as $I(\mu) = \mu^\perp$, and by abuse of notation we say that I is self-dual.

Theorem 2.7 in Section 2 was the first example relating the Radon and Fourier transforms. Using operator notations, this may be stated as:

$$R_{n-k} \circ E_{-k}^\wedge = c(n,k) I \circ R_k, \tag{4.1}$$

as operators from $C_e^\infty(S^{n-1})$ to $C_e^\infty(G(n,n-k))$. In view of the remark immediately after Theorem 2.7, a generalization of (4.1) is given by the spherical Parseval identity, which in the formulation of Remark 2.6, shows that $E_{-k}^\wedge$ is a self-adjoint operator on $C_e^\infty(S^{n-1})$:

$$\left(E_{-k}^\wedge\right)^* = E_{-k}^\wedge. \tag{4.2}$$

Passing to the dual in (4.1) and using (4.2), we immediately have:

$$E_{-k}^\wedge \circ R_{n-k}^* = c(n,k) R_k^* \circ I, \tag{4.3}$$

as operators on certain spaces. We formulate this more carefully in the next proposition.

Proposition 4.1. *Let $f \in C_e^\infty(S^{n-1})$, and assume that $f = R_{n-m}^*(d\mu)$ as measures in $\mathcal{M}(S^{n-1})$, for some measure $\mu \in \mathcal{M}(G(n,n-m))$. Then $E_{-m}^\wedge(f) = c(n,m) R_m^*(d\mu^\perp)$ as measures in $\mathcal{M}(S^{n-1})$, where $c(n,m)$ is the constant from Lemma* 2.4.

Proof. Let $g \in C_e^\infty(S^{n-1})$ be arbitrary. Then by the spherical Parseval identity and Theorem 2.7:

$$\begin{aligned}
&\int_{S^{n-1}} E_{-m}^\wedge(f)(\theta) g(\theta)\, d\sigma(\theta) \\
&= \int_{S^{n-1}} f(\theta) E_{-m}^\wedge(g)(\theta)\, d\sigma(\theta) = \int_{S^{n-1}} R_{n-m}^*(d\mu)(\theta) E_{-m}^\wedge(g)(\theta)\, d\sigma(\theta) \\
&= \int_{G(n,n-m)} R_{n-m}\big(E_{-m}^\wedge(g)\big)(F)\, d\mu(F) = c(n,m) \int_{G(n,n-m)} R_m(g)(F^\perp)\, d\mu(F) \\
&= c(n,m) \int_{G(n,m)} R_m(g)(F)\, d\mu^\perp(F) = c(n,m) \int_{S^{n-1}} R_m^*(d\mu^\perp)(\theta) g(\theta)\, d\sigma(\theta).
\end{aligned}$$

Since $C_e^\infty(S^{n-1})$ is dense in $C_e(S^{n-1})$ in the maximum norm, the proposition follows. □

In the context of star-bodies, the following is an immediate corollary of Proposition 4.1.

Corollary 4.2. *Let K be an infinitely smooth star-body in $\mathbb{R}^n$. Then for a measure $\mu \in \mathcal{M}(G(n,n-k))$:*

$$\|\cdot\|_K^{-k} = R_{n-k}^*(d\mu) \quad \textit{iff} \quad \left(\|\cdot\|_K^{-k}\right)^\wedge = c(n,k) R_k^*(d\mu^\perp),$$

where $c(n,k)$ is the constant from Lemma 2.4*, and the equalities are understood as equalities between measures in $\mathcal{M}(S^{n-1})$.*

Proof. The "only if" part follows immediately from Proposition 4.1 with $m = k$ and $f = \|\cdot\|_K^{-k}$. The "if" part follows by applying Proposition 4.1 with $m = n-k$ and $f = (\|\cdot\|_K^{-k})^\wedge$, and using the fact that $E_{-n+k}^\wedge(f) = (2\pi)^n \|\cdot\|_K^{-k}$ and that the constants $c(n,k)$ from Lemma 2.4 satisfy $c(n,k)c(n,n-k) = (2\pi)^n$. □

Proposition 4.1 has several interesting consequences. The first one is:

Theorem 4.3. *Let $n > 1$ and fix $1 \leqslant k \leqslant n-1$. Then*

$$\mathcal{BP}_k^n = \mathcal{I}_k^n \quad \text{iff} \quad \mathcal{BP}_{n-k}^n = \mathcal{I}_{n-k}^n.$$

Proof. Assume that $\mathcal{BP}_{n-k}^n = \mathcal{I}_{n-k}^n$, and let $K \in \mathcal{I}_{n-k}^{n,\infty}$. In view of Lemma 3.11, the fact that $\mathcal{BP}_k^n$ is closed under limit in the radial metric, and Corollary 2.2, it is enough to show that $K \in \mathcal{BP}_k^n$. Since $(\|\cdot\|_K^{-k})^\wedge \geqslant 0$ by Theorem 2.5, we may define the infinitely smooth star-body L as the body for which $\|\cdot\|_L^{-n+k} = (\|\cdot\|_K^{-k})^\wedge$. Therefore $(\|\cdot\|_L^{-n+k})^\wedge = (2\pi)^n \|\cdot\|_K^{-k} \geqslant 0$, hence $L \in \mathcal{I}_{n-k}^n$. It follows from our assumption that $L \in \mathcal{BP}_{n-k}^n$, so there exists a non-negative measure $\mu \in \mathcal{M}_+(G(n,k))$ so that $(\|\cdot\|_K^{-k})^\wedge = \|\cdot\|_L^{-n+k} = R_k^*(d\mu)$. By Corollary 4.2, this implies that $\|\cdot\|_K^{-k} = c(n,k) R_{n-k}^*(d\mu^\perp)$. Therefore $K \in \mathcal{BP}_k^n$, which concludes the proof. $\square$

Another immediate consequence of Proposition 4.1 is another elementary proof of the following corollary.

Corollary 4.4.

$$\mathcal{BP}_k^n \subset \mathcal{I}_k^n.$$

Proof. Let $K \in \mathcal{BP}_k^{n,\infty}$, so $\|\cdot\|_K^{-k} = R_{n-k}^*(d\mu)$ for some non-negative Borel measure $\mu \in \mathcal{M}_+(G(n,n-k))$. By Corollary 4.2 it follows that $(\|\cdot\|_K^{-k})^\wedge = c(n,k) R_k^*(d\mu^\perp)$, implying that $(\|\cdot\|_K^{-k})^\wedge \geqslant 0$, and hence $K \in \mathcal{I}_k^n$. By Lemma 3.1, and the fact that $\mathcal{I}_k^n$ is closed under limit in the radial metric, this concludes the proof. $\square$

Applying Proposition 4.1 to the function $f = 0$, once for $m = k$ and once for $m = n-k$, we also immediately deduce the following useful proposition.

Proposition 4.5.

$$\operatorname{Ker} R_{n-k}^* = \operatorname{Ker} R_k^* \circ I.$$

This is equivalent by a standard duality argument to the following proposition, which may be deduced directly from Theorem 2.7.

Proposition 4.6.

$$\overline{\operatorname{Im} R_{n-k}} = \overline{\operatorname{Im} I \circ R_k}.$$

We conclude this section by introducing a family of very natural operators acting on $C(G(n,k))$ to itself, and showing a few nice properties which they share. Denote by $V_k : C(G(n,k)) \to C(G(n,k))$ the operator defined as $V_k = I \circ R_{n-k} \circ R_k^*$.

Proposition 4.7. *V_k is self-adjoint.*

Proof. It is actually not hard to show this directly, just by using double-integration as in Section 3. Nevertheless, we prefer to use Proposition 4.1. Let $f, g \in C^\infty(G(n, n-k))$. Then by Proposition 4.1, the spherical Parseval identity and Proposition 4.1 again, we have:

$$\begin{aligned}
\langle V_{n-k}(f), g\rangle_{G(n,n-k)} &= \langle R^*_{n-k}(f), (I \circ R_k)^*(g)\rangle \\
&= c(n,k)^{-1}\langle R^*_{n-k}(f), (E^\wedge_{-k} \circ R^*_{n-k})(g)\rangle \\
&= c(n,k)^{-1}\langle (E^\wedge_{-k} \circ R^*_{n-k})(f), R^*_{n-k}(g)\rangle \\
&= \langle (I \circ R_k)^*(f), R^*_{n-k}(g)\rangle = \langle f, V_{n-k}(g)\rangle_{G(n,n-k)}.
\end{aligned}$$

Since $C^\infty(G(n, n-k))$ is dense in $C(G(n, n-k))$ in the maximum norm, and the operators R^*_{n-k} and R_k, and hence V_{n-k}, are continuous with respect to this norm, it follows that the same holds for any $f, g \in C(G(n, n-k))$. $\square$

Proposition 4.8.

$$V_{n-k} = I \circ V_k \circ I.$$

Proof. This time we give the proof in operator style notations. The formal details are filled in exactly the same manner as above. Using the definition of V_k, and the identities (4.3) and (4.1), we have:

$$\begin{aligned}
I \circ V_k \circ I = R_{n-k} \circ R^*_k \circ I &= c(n,k)^{-1} R_{n-k} \circ E^\wedge_{-k} \circ R^*_{n-k} \\
&= I \circ R_k \circ R^*_{n-k} = V_{n-k}. \qquad \square
\end{aligned}$$

It is known (e.g. [10]) that for $1 < k < n-1$, even if we restrict the operators R_m to infinitely smooth functions, $\operatorname{Ker} R^*_k \neq \{0\}$ and $\overline{\operatorname{Im} R_{n-k}} \neq C^\infty(G(n, n-k))$, and therefore V_k is neither injective nor surjective onto a dense set for those values of k. Since $\overline{\operatorname{Im} R^*_k} = C_e(S^{n-1})$ and $\operatorname{Ker} R_{n-k} = \{0\}$, it follows that $\operatorname{Ker} V_k = \operatorname{Ker} R^*_k$ and $\overline{\operatorname{Im} V_k} = \overline{\operatorname{Im} I \circ R_{n-k}} = \overline{\operatorname{Im} R_k}$ (by Proposition 4.6). A standard duality argument shows that $\overline{\operatorname{Im} R_k}$ is orthogonal to $\operatorname{Ker} R^*_k$ (as measures acting on continuous functions, and therefore as functions when R^*_k is restricted to $C(G(n,k))$), and therefore we may consider V_k as an operator from $\overline{\operatorname{Im} R_k}$ to $\overline{\operatorname{Im} R_k}$, which is injective and surjective onto a dense set. A natural question for integral geometrists would be to find a nice inversion formula for V_k. Note that by a standard double-integral argument, the operator $R^*_k \circ I \circ R_{n-k} : S^{n-1} \to S^{n-1}$ is exactly the usual spherical Radon transform R (for every k), and under the standard identification between $G(n, n-1)$, $G(n, 1)$ and S^{n-1}, so are V_1 and V_{n-1}.

5. Equivalent formulations of $\mathcal{BP}^n_k = \mathcal{I}^n_k$

In this section we use the results and techniques of the previous sections together with basic tools from functional analysis to derive equivalent formulations of the natural conjecture that $\mathcal{BP}^n_k = \mathcal{I}^n_k$. As mentioned in Section 1, the relevance of this conjecture to convex geometry stems from the generalized k-codimensional Busemann–Petty problem. It was shown in [36] that the answer to this problem is positive iff every convex body in $\mathbb{R}^n$ is in $\mathcal{BP}^n_k$, and this was shown to be false [4,21] for $k < n - 3$, but the cases of $k = n - 3$ and $k = n - 2$ remain open. The analogous question for $\mathcal{I}_k^n$ turned out to be easier using the analytic tools provided by the Fourier transform, and it was shown by Koldobsky in [20] that $\mathcal{I}_k^n$ contains all n-dimensional convex bodies iff $k \geqslant n - 3$. Hence, a positive answer to whether $\mathcal{BP}_k^n = \mathcal{I}_k^n$ would imply a positive answer to the generalized k-codimensional Busemann–Petty problem, for $k \geqslant n-3$. The equivalent formulations derived in this section indicate that the $\mathcal{BP}_k^n = \mathcal{I}_k^n$ question is connected and equivalent to very natural questions in integral geometry.

Before we start, we would like to give an intuitive equivalent formulation to $\mathcal{BP}_k^n = \mathcal{I}_k^n$. By Grinberg and Zhang's characterization (Theorem 2.1), $\mathcal{BP}_k^n$ is exactly the class of star-bodies generated from the Euclidean ball D_n by means of full-rank linear transformations, k-radial sums, and limit in the radial metric. Loosely speaking, we say that "modulo these operations," D_n is the only member of $\mathcal{BP}_k^n$. Since $\mathcal{I}_k^n$ is closed under these operations as well, we can ask whether "modulo these operations" D_n is the only star-body such $(\|\cdot\|_{D_n}^{-k})^\wedge \geqslant 0$. In terms of functions on the sphere, this is equivalent to asking whether "modulo these operations," the only function $f \in C_e^\infty(S^{n-1})$ such that $f \geqslant 0$ and $E_{-k}^\wedge(f) \geqslant 0$ is the constant function $f = 1$ (note that we may restrict our attention to infinitely smooth functions because of Lemma 3.11). This formulation transforms the problem to the language of Fourier transforms. As opposed to this, our other formulations in this section will use the language of the Radon transforms and integral geometry.

We will use the following notations. $R_m(C(S^{n-1}))_+$ will denote the non-negative functions in the image of R_m and $R_m(C_+(S^{n-1}))$ will denote the image of R_m acting on the cone $C_+(S^{n-1})$ (which is the same as its image acting on $C_{+,e}(S^{n-1})$). We denote $G = G(n, n-k)$ for short.

It is well known (e.g. [10,17,33]) that $R_{n-k} : C_e(S^{n-1}) \to C(G(n, n-k))$ is an injective operator, but it is not onto for $k < n - 1$, and $\overline{\operatorname{Im} R_{n-k}} \neq C(G(n, n-k))$ for $1 < k < n - 1$. We will restrict our discussion to this range of k. It follows by an elementary duality argument, that the image of the dual operator $R_{n-k}^* : \mathcal{M}(G(n, n-k)) \to \mathcal{M}_e(S^{n-1})$ is dense in $\mathcal{M}_e(S^{n-1})$ in the w^*-topology, but R_{n-k}^* is not injective and has a non-trivial kernel. It is known that the dense image in $\mathcal{M}_e(S^{n-1})$ contains $C_e^\infty(S^{n-1})$, and in fact an explicit inversion formula was obtained by Koldobsky in [21, Proposition 3] (which is not unique because of the kernel). It follows from Koldobsky's argument (or from the general results of [10]) the following lemma.

Lemma 5.1. *If $f \in C_e^\infty(S^{n-1})$ then there exists a $g \in C^\infty(G(n, n-k))$ such that $f = R_{n-k}^*(g)$.*

It will also be useful to note that

$$\operatorname{Ker} R_{n-k}^* = \bigl\{\mu \in \mathcal{M}\bigl(G(n, n-k)\bigr) \bigm| \langle \mu, f \rangle = 0 \ \forall f \in \operatorname{Im} R_{n-k}\bigr\}, \tag{5.1}$$

and to recall Propositions 4.5 and 4.6, which show that

$$\operatorname{Ker} R_{n-k}^* = \operatorname{Ker} R_k^* \circ I \quad \text{and} \quad \overline{\operatorname{Im} R_{n-k}} = \overline{\operatorname{Im} I \circ R_k}.$$

The latter immediately implies:

$$\overline{R_{n-k}\bigl(C_+(S^{n-1})\bigr)}, \overline{I \circ R_k\bigl(C_+(S^{n-1})\bigr)} \subset \overline{R_{n-k}\bigl(C(S^{n-1})\bigr)_+}. \tag{5.2}$$

It will be useful to consider the quotient space:

$$\mathcal{M}(n, n-k) = \mathcal{M}\bigl(G(n, n-k)\bigr) \big/ \operatorname{Ker} R_{n-k}^*,$$

which is the space of bounded linear functionals on the subspace $\overline{\operatorname{Im} R_{n-k}}$ of $C(G(n,n-k))$. By abuse of notation, we will also think of R^*_{n-k} as an operator from $\mathcal{M}(n,n-k)$ to $\mathcal{M}_e(S^{n-1})$, and although this does not change its image, it is now injective on $\mathcal{M}(n,n-k)$. The same is true for $R^*_k \circ I$, since $\operatorname{Ker} R^*_{n-k} = \operatorname{Ker} R^*_k \circ I$, and we may proceed to interpret $R^*_{n-k}(d\mu)$ and $R^*_k(d\mu^\perp)$ in the usual way for $\mu \in \mathcal{M}(n,n-k)$, since these values are the same for the entire co-set $\mu + \operatorname{Ker} R^*_{n-k}$. If R^*_{n-k} were onto $\mathcal{M}_e(S^{n-1})$, or even $C_e(S^{n-1})$, we could proceed by identifying between a star-body K and a signed Borel measure μ in $\mathcal{M}(n,n-k)$, by the correspondence $\|\cdot\|_K^{-k} = R^*_{n-k}(d\mu)$. Unfortunately, the general theory does not guarantee this, and in fact we believe that some star-bodies do not admit such a representation (although we have not been able to find a reference for this). But as remarked earlier, $C_e^\infty(S^{n-1})$ does lie in the image of R^*_{n-k}, and this is enough for our purposes.

Let us now review the definitions of $\mathcal{BP}^n_k$ and $\mathcal{I}^n_k$. Our original definition required that $K \in \mathcal{BP}^n_k$ iff $\rho_K^k = R^*_{n-k}(d\mu)$ for some non-negative measure $\mu \in \mathcal{M}_+(G(n,n-k))$. We claim that this is equivalent to requiring that $\mu \in \mathcal{M}_+(n,n-k)$, since by a version of the Hahn–Banach theorem [16, Lemma 4.3], any non-negative functional on $\overline{\operatorname{Im} R_{n-k}}$ may be extended to a non-negative functional on the entire $C(G(n,n-k))$, and the converse is trivially true. Defining $\mathcal{M}(\mathcal{BP}^n_k)$ as the set of non-negative functionals in $\mathcal{M}(n,n-k)$:

$$\mathcal{M}\big(\mathcal{BP}^n_k\big) = \mathcal{M}_+(n,n-k),$$

we see the following.

Lemma 5.2. *Let K be a star-body in $\mathbb{R}^n$. Then $K \in \mathcal{BP}^n_k$ iff $\rho_K^k = R^*_{n-k}(d\mu)$, for some $\mu \in \mathcal{M}(\mathcal{BP}^n_k)$.*

Let us also define $\mathcal{M}(\mathcal{I}^n_k)$ as

$$\mathcal{M}(\mathcal{I}^n_k) = \big\{\mu \in \mathcal{M}(n,n-k) \;\big|\; R^*_{n-k}(d\mu) \geqslant 0,\, R^*_k(d\mu^\perp) \geqslant 0\big\},$$

where "$\nu \geqslant 0$" means that ν is a non-negative measure in $\mathcal{M}_e(S^{n-1})$. Using co-set notations, let us also define

$$\mathcal{M}^\infty(n,n-k) = \big\{f + \operatorname{Ker} R^*_{n-k} \;\big|\; f \in C^\infty(G)\big\},$$

and denote

$$\mathcal{M}^\infty\big(\mathcal{I}^n_k\big) = \mathcal{M}\big(\mathcal{I}^n_k\big) \cap \mathcal{M}^\infty(n,n-k), \quad \text{and}$$
$$\mathcal{M}^\infty_+(n,n-k) = \mathcal{M}^\infty\big(\mathcal{BP}^n_k\big) = \mathcal{M}\big(\mathcal{BP}^n_k\big) \cap \mathcal{M}^\infty(n,n-k).$$

Unfortunately, we cannot give a completely analogous characterization to Lemma 5.2 for $\mathcal{I}^n_k$ and $\mathcal{M}(\mathcal{I}^n_k)$. However, we have the following.

Lemma 5.3. *Let K be an* infinitely smooth *star-body in $\mathbb{R}^n$. Then $K \in \mathcal{I}^n_k$ iff $\rho_K^k = R^*_{n-k}(d\mu)$, for some $\mu \in \mathcal{M}^\infty(\mathcal{I}^n_k)$.*

Proof. We will first prove the "only if" part. Assume that $K \in \mathcal{I}_k^{n,\infty}$. By Lemma 5.1, there exists a signed measure $\mu \in \mathcal{M}^\infty(n, n-k)$ so that $\|\cdot\|_K^{-k} = R_{n-k}^*(d\mu)$. By Corollary 4.2 of Proposition 4.1, it follows that $(\|\cdot\|_K^{-k})^\wedge = c(n,k) R_k^*(d\mu^\perp)$. Since $\|\cdot\|_K^{-k} \geqslant 0$ because K is a star-body and $(\|\cdot\|_K^{-k})^\wedge \geqslant 0$ because $K \in \mathcal{I}_k^n$, it follows that $R_{n-k}^*(d\mu) \geqslant 0$ and $R_k^*(d\mu^\perp) \geqslant 0$, proving that $\mu \in \mathcal{M}^\infty(\mathcal{I}_k^n)$. The "if" part follows from Corollary 4.2 in exactly the same manner, since $(\|\cdot\|_K^{-k})^\wedge = c(n,k) R_k^*(d\mu^\perp) \geqslant 0$ for a measure $\mu \in \mathcal{M}^\infty(\mathcal{I}_k^n)$ such that $\|\cdot\|_K^{-k} = R_{n-k}^*(d\mu)$. $\square$

Remark 5.4. It seems that any attempt to prove the "only if" part of the lemma for a general star-body $K \in \mathcal{I}_k^n$ by approximating it with $K_i \in \mathcal{I}_k^{n,\infty}$ will fail. The reason is that we have no way of controlling the norm of the (a-priori signed) measures $\mu_i \in \mathcal{M}(\mathcal{I}_k^n)$ for which $\rho_{K_i}^k = R_{n-k}^*(d\mu_i)$, and therefore it is not guaranteed that μ_i will converge to some measure (like in the usual argument which uses the w^*-compactness of the unit-ball of $\mathcal{M}(n, n-k)$). If it were known that the μ_i are non-negative (this would follow if $\mathcal{M}(\mathcal{BP}_k^n) = \mathcal{M}(\mathcal{I}_k^n)$), it would follow that $\|\mu_i\| = \|R_{n-k}^*(d\mu_i)\|$ (since $R_{n-k}^*(d\mu_i)$ is non-negative), and over the latter term we do have control. The "if" part of the lemma may be proved without any smoothness assumption by the standard approximation argument.

We now see that we have derived alternative definitions of $\mathcal{BP}_k^n$ and $\mathcal{I}_k^{n,\infty}$ using a common language of Radon transforms and without using the Fourier transform. Note that even if we could remove the restriction of infinite smoothness from Lemma 5.3, it would not be yet clear that $\mathcal{BP}_k^n = \mathcal{I}_k^n$ iff $\mathcal{M}(\mathcal{BP}_k^n) = \mathcal{M}(\mathcal{I}_k^n)$, since for a general $\mu \in \mathcal{M}(\mathcal{BP}_k^n)$ or $\mu \in \mathcal{M}(\mathcal{I}_k^n)$, $R_{n-k}^*(d\mu)$ may not be a measure with continuous density (and hence cannot equal ρ_K^k for a star-body K). We do, however, have:

Lemma 5.5.

$$\mathcal{M}(\mathcal{BP}_k^n) \subset \mathcal{M}(\mathcal{I}_k^n).$$

Proof. If $\mu \in \mathcal{M}_+(n, n-k)$ then trivially $R_{n-k}^*(d\mu) \geqslant 0$ and $R_k^*(d\mu^\perp) \geqslant 0$, hence $\mu \in \mathcal{M}(\mathcal{I}_k^n)$. Although the proof is trivial, note that underlying this statement are Propositions 4.5 and 4.6 which enabled us to restrict R_{n-k}^* and $R_k^* \circ I$ to $\mathcal{M}(n, n-k)$. $\square$

We may now formulate the main theorem of this section:

Theorem 5.6. *Let n and $1 \leqslant k \leqslant n-1$ be fixed. Then the following are equivalent*:

(1) $$\mathcal{BP}_k^n = \mathcal{I}_k^n.$$
(2) $$\mathcal{M}^\infty(\mathcal{BP}_k^n) = \mathcal{M}^\infty(\mathcal{I}_k^n).$$
(3) $$\mathcal{M}(\mathcal{BP}_k^n) = \mathcal{M}(\mathcal{I}_k^n).$$
(4) $$\overline{R_{n-k}\big(C(S^{n-1})\big)_+} = \overline{R_{n-k}\big(C_+(S^{n-1})\big) + I \circ R_k\big(C_+(S^{n-1})\big)}.$$
(5) *If $\mu + 1 \in \mathcal{M}(\mathcal{BP}_k^n)$ and $\mu \in \mathcal{M}(\mathcal{I}_k^n)$, then $\mu \in \mathcal{M}(\mathcal{BP}_k^n)$.*
(6) *There does not exist a measure $\mu \in \mathcal{M}_+^\infty(n, n-k)$ such that $R_{n-k}^*(d\mu) \geqslant 1$ and $R_k^*(d\mu^\perp) \geqslant 1$ (where "$\nu \geqslant 1$" means that $\nu - 1$ is a non-negative measure), and such that*

$$\inf\big\{\langle \mu, f\rangle \;\big|\; f \in R_{n-k}\big(C(S^{n-1})\big)_+ \text{ and } \langle 1, f\rangle = 1\big\} = 0. \tag{5.3}$$

We will show (2) $\Rightarrow$ (1), (1) $\Rightarrow$ (3), (3) $\Leftrightarrow$ (4), (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (2). Obviously, (3) $\Rightarrow$ (2) and (3) $\Rightarrow$ (5).

Proof of (2) $\Rightarrow$ (1). Let $K \in \mathcal{I}_k^{n,\infty}$. In view of Lemma 3.11, the fact that $\mathcal{BP}_k^n$ is closed under limit in the radial metric, and Corollary 2.2, it is enough to show that $K \in \mathcal{BP}_k^n$. By Lemma 5.3, $\rho_K^k = R_{n-k}^*(d\mu)$ for some $\mu \in \mathcal{M}^\infty(\mathcal{I}_k^n)$. By our assumption that $\mathcal{M}^\infty(\mathcal{BP}_k^n) = \mathcal{M}^\infty(\mathcal{I}_k^n)$ and by Lemma 5.2, it follows that $K \in \mathcal{BP}_k^n$ (in fact $K \in \mathcal{BP}_k^{n,\infty}$). $\square$

Proof of (1) $\Rightarrow$ (3). In view of Lemma 5.5, it is enough to prove $\mathcal{M}(\mathcal{I}_k^n) \subset \mathcal{M}(\mathcal{BP}_k^n)$. Let $\mu \in \mathcal{M}(\mathcal{I}_k^n)$, so $R_{n-k}^*(d\mu) \geqslant 0$ and $R_k^*(d\mu^\perp) \geqslant 0$. Let $\{u_i\} \subset C^\infty(O(n))$ be an approximate identity as in Lemma 2.3. Let K_i denote the infinitely smooth star-body defined by

$$\|\cdot\|_{K_i}^{-k} = u_i * R_{n-k}^*(\mu) \geqslant 0$$

(we used $R_{n-k}^*(\mu) \geqslant 0$ to verify that K_i is indeed a star-body). As in the proof of Lemma 3.1, it is easy to see that $\|\cdot\|_{K_i}^{-k} = R_{n-k}^*(u_i * \mu)$, so by Corollary 4.2 we have:

$$\left(\|\cdot\|_{K_i}^{-k}\right)^\wedge = c(n,k) R_k^*\left((u_i * \mu)^\perp\right) = R_k^*(u_i * \mu^\perp) = u_i * R_k^*(\mu^\perp) \geqslant 0.$$

Hence $K_i \in \mathcal{I}_k^n$, and by our assumption that $\mathcal{BP}_k^n = \mathcal{I}_k^n$, it follows that $K_i \in \mathcal{BP}_k^n$. By Lemma 5.2, this implies that $\|\cdot\|_{K_i}^{-k} = R_{n-k}^*(d\eta_i)$, where $\eta_i \in \mathcal{M}(BP_k^n)$. The injectivity of R_{n-k}^* on $\mathcal{M}(n, n-k)$ implies that $u_i * \mu = \eta_i \in \mathcal{M}(BP_k^n)$. Lemma 2.3 shows that $u_i * \mu$ tends to μ in the w^*-topology, and since $\mathcal{M}(BP_k^n)$ is obviously closed in this topology, it follows that $\mu \in \mathcal{M}(BP_k^n)$. $\square$

For the proof of (3) $\Leftrightarrow$ (4) and for later use, we will need to recall a few classical notions from functional analysis (e.g. [2]). A cone P in a Banach space X is a non-empty subset of X such that $x, y \in P$ implies $c_1 x + c_2 y \in P$ for every $c_1, c_2 \geqslant 0$. The dual cone $P^* \subset X^*$ is defined by $P^* = \{x^* \in X^* | \langle x^*, p\rangle \geqslant 0\ \forall p \in P\}$. Therefore P^* is always closed in the w^*-topology, and $P^* = (\overline{P})^*$. It is also easy to check that $P_1 \subset P_2$ implies $P_2^* \subset P_1^*$, $(P_1 + P_2)^* = P_1^* \cap P_2^*$ and $(P_1 \cap P_2)^* = P_1^* + P_2^*$. An immediate consequence of the Hahn–Banach theorem is that $\overline{P_1} = \overline{P_2}$ iff $P_1^* = P_2^*$.

Proof of (3) $\Leftrightarrow$ (4). All the sets appearing in (3) and (4) are clearly cones. It remains to show that the cones in both sides of (3) are exactly the dual cones to the ones in both sides of (4). The equivalence then follows by the Hahn–Banach theorem, as in the last statement of the previous paragraph.

By definition, $\mathcal{M}(\mathcal{BP}_k^n)$ is dual to $\overline{R_{n-k}(C(S^{n-1}))_+}$. The cones

$$\left\{\mu \in \mathcal{M}(n, n-k) \;\middle|\; R_{n-k}^*(d\mu) \geqslant 0\right\} \quad \text{and}$$
$$\left\{\mu \in \mathcal{M}(n, n-k) \;\middle|\; R_k^*(d\mu^\perp) \geqslant 0\right\}$$

are immediately seen to be dual to $R_{n-k}(C_+(S^{n-1}))$ and $I \circ R_k(C_+(S^{n-1}))$, respectively. Since $(P_1 + P_2)^* = P_1^* \cap P_2^*$, it follows that

$$\mathcal{M}\left(\mathcal{I}_k^n\right) = \left(\overline{R_{n-k}\left(C_+(S^{n-1})\right) + I \circ R_k\left(C_+(S^{n-1})\right)}\right)^*.$$

This concludes the proof. $\square$

Remark 5.7. By (5.2), we have:

$$\overline{R_{n-k}\big(C(S^{n-1})\big)_+} \supset \overline{R_{n-k}\big(C_+(S^{n-1})\big) + I \circ R_k\big(C_+(S^{n-1})\big)}.$$

By duality, we see again that

$$\mathcal{M}\big(\mathcal{BP}_k^n\big) \subset \mathcal{M}\big(\mathcal{I}_k^n\big).$$

Proof of (5) ⇒ (6). This follows immediately from the definitions. Assume that (6) is false, so that there exists a measure $\mu \in \mathcal{M}_+^\infty(n, n-k)$ such that $R_{n-k}^*(d\mu) \geqslant 1$ and $R_k^*(d\mu^\perp) \geqslant 1$ and such that (5.3) holds. Define $\mu' = \mu - 1$, and so $\mu' + 1 \in \mathcal{M}(\mathcal{BP}_k^n)$, $\mu' \in \mathcal{M}(\mathcal{I}_k^n)$, and (5.3) shows that μ' is not in $\mathcal{M}(\mathcal{BP}_k^n)$. Therefore μ' is a counterexample to (5). $\square$

Proof of (6) ⇒ (2). Assume that (2) is false, so $\mathcal{M}^\infty(\mathcal{BP}_k^n) \neq \mathcal{M}^\infty(\mathcal{I}_k^n)$. By Lemma 5.5, this means that there exists a measure $\mu' \in \mathcal{M}^\infty(\mathcal{I}_k^n) \setminus \mathcal{M}(\mathcal{BP}_k^n)$. Since $\mu' \in \mathcal{M}^\infty(n, n-k)$, we can write $\mu' = g + \operatorname{Ker} R_{n-k}^*$ with $g \in C^\infty(G(n, n-k))$. Assume that $\min(g) = -C$ where $C > 0$, otherwise we would have $\mu' \in \mathcal{M}(\mathcal{BP}_k^n)$.

Now consider the measure $\mu_\lambda = (1-\lambda)\mu' + \lambda \in \mathcal{M}^\infty(n, n-k)$ for $\lambda \in [0, 1]$. Since $\mathcal{M}(\mathcal{BP}_k^n)$ is convex, contains the measure 1, and is closed in the w^*-topology, it follows that there exists a $\lambda_0 \in (0, 1]$ so that $\mu_\lambda \in \mathcal{M}^\infty(\mathcal{BP}_k^n)$ iff $\lambda \in [\lambda_0, 1]$. But for $\lambda_1 = C/(1 + C)$ we already see that $\mu_{\lambda_1} \in \mathcal{M}(\mathcal{BP}_k^n)$, because $\mu_{\lambda_1} = g_{\lambda_1} + \operatorname{Ker} R_{n-k}^*$ and $g_{\lambda_1} = 1/(1 + C)g + 1 - 1/(1 + C) \in C_+^\infty(G(n, n-k))$. We conclude that $\lambda_0 \in (0, 1)$.

Now define $\mu = \mu_{\lambda_0}/\lambda_0 \in \mathcal{M}(\mathcal{BP}_k^n)$, and notice that $\mu - 1 = (1 - \lambda_0)/\lambda_0 \mu' \in \mathcal{M}^\infty(\mathcal{I}_k^n)$, implying that $R_{n-k}^*(d\mu) \geqslant 1$ and $R_k^*(d\mu^\perp) \geqslant 1$. It remains to show (5.3). Assume by negation that

$$\inf\big\{\langle \mu, f\rangle \;\big|\; f \in R_{n-k}\big(C(S^{n-1})\big)_+ \text{ and } \langle 1, f\rangle = 1\big\} = \delta > 0.$$

But then it is easy to check that for $\lambda_2 = \lambda_0(1 - \delta)/(1 - \delta\lambda_0) < \lambda_0$, $\langle \mu_{\lambda_2}, f\rangle \geqslant 0$ for all $f \in R_{n-k}(C(S^{n-1}))_+$, and hence for all $f \in \overline{R_{n-k}(C(S^{n-1}))_+}$. Therefore $\mu_{\lambda_2} \in \mathcal{M}^\infty(\mathcal{BP}_k^n)$, in contradiction to the definition of λ_0. Therefore (5.3) is shown, concluding the proof. $\square$

Remark 5.8. In formulation (6), it is equivalent to require that $\mu \in \mathcal{M}_+(n, n-k)$ and also $\mu \in \mathcal{M}(G(n, n-k))$ instead of $\mu \in \mathcal{M}_+^\infty(n, n-k)$. The equivalence of $\mu \in \mathcal{M}_+(n, n-k)$ follows since we have not used the fact that $\mu \in \mathcal{M}^\infty(n, n-k)$ in the proof (by negation) of (5) ⇒ (6). The equivalence of $\mu \in \mathcal{M}(G(n, n-k))$ follows by the previously mentioned version of the Hahn–Banach theorem (which was used to derive Lemma 5.2). This is the formulation which was used in Section 1.

We proceed to develop several more formulations of the $\mathcal{BP}_k^n = \mathcal{I}_k^n$ question. Unfortunately, we cannot show an equivalence with the original question, but rather a weak type of implication. We formulate a very natural conjecture, and show that together with a positive answer to the $\mathcal{BP}_k^n = \mathcal{I}_k^n$ question, the new formulations are implied.

Given an Borel set $Z \subset G(n, n-k)$, we define the restriction of a measure $\mu \in \mathcal{M}(G(n, n-k))$ to Z, denoted $\mu|_Z \in \mathcal{M}(G(n, n-k))$, as the measure satisfying $\mu|_Z(A) = \mu(A \cap Z)$ for any Borel set $A \subset G(n, n-k)$. We will say that μ is supported in a closed set Z, if $\mu|_{Z^C} = 0$, and define the support of μ, denoted $\mathrm{supp}(\mu)$, as the minimal closed set Z in which μ is supported (it is easy to check that this is well defined). It is also easy to check the following lemma.

Lemma 5.9. *If $f \in C(G(n, n-k))$, $\mu \in \mathcal{M}(G(n, n-k))$ and $\mathrm{supp}(\mu) \subset f^{-1}(0)$ then $\langle \mu, f \rangle = 0$. Conversely, if $f \in C_+(G(n, n-k))$, $\mu \in \mathcal{M}_+(G(n, n-k))$ and $\langle \mu, f \rangle = 0$, then $\mathrm{supp}(\mu) \subset f^{-1}(0)$.*

We also recall the definition of the covering property from Section 1. A set closed set $Z \subset G(n, n-k)$ is said to satisfy the *covering property* if

$$\bigcup_{E \in Z} E \cap S^{n-1} = S^{n-1} \quad \text{and} \quad \bigcup_{E \in Z} E^{\perp} \cap S^{n-1} = S^{n-1}. \tag{5.4}$$

Our starting point is formulation (6) in Theorem 5.6, which involves both a function f and a measure μ. Note that the requirement that if $f \in \overline{R_{n-k}(C(S^{n-1}))_+}$ and $\langle 1, f \rangle = 1$, then $\langle \mu, f \rangle$ is bounded away from zero, is stronger than demanding that $\langle \mu, f \rangle \neq 0$. The motivation for the following discussion stems from the impression that the conditions on μ, namely that $\mu \in \mathcal{M}_+(G(n, n-k))$ (following Remark 5.8), $R^*_{n-k}(d\mu) \geqslant 1$ and $R^*_k(d\mu^{\perp}) \geqslant 1$, may be equivalently specified by some condition on the support of μ. In that case, the condition that $\langle \mu, f \rangle \neq 0$ becomes a condition on the set $f^{-1}(0)$. Let us show the following necessary condition on the support of such a μ as above.

Lemma 5.10. *Let $\mu \in \mathcal{M}_+(G(n, n-k))$ so that $R^*_{n-k}(d\mu) \geqslant 1$ and $R^*_k(d\mu^{\perp}) \geqslant 1$. Then $\mathrm{supp}(\mu)$ satisfies the covering property.*

Proof. Denote by $Z = \mathrm{supp}(\mu)$ and $\widetilde{Z} = \bigcup_{E \in Z} E \cap S^{n-1}$. We will show that if $\mu \in \mathcal{M}_+(G(n, n-k))$ and $R^*_{n-k}(d\mu) \geqslant 1$ then $\widetilde{Z} = S^{n-1}$. The other "half" of the covering property follows similarly from $R^*_k(d\mu^{\perp}) \geqslant 1$.

Notice that for $E_1, E_2 \in G(n, n-k)$, the Hausdorff distance between $E_1 \cap S^{n-1}$ and $E_2 \cap S^{n-1}$ is equivalent to the distance between E_1 and E_2 in $G(n, n-k)$. It follows that since Z is closed, so is $\widetilde{Z}$. Now assume that $\widetilde{Z} \neq S^{n-1}$, so there exists a $\theta \in S^{n-1}$ and an $\epsilon > 0$, so that $\widetilde{B} = B_{S^{n-1}}(\theta, \epsilon) \cup B_{S^{n-1}}(-\theta, \epsilon) \subset \widetilde{Z}^C$. Let $f \in C_{e,+}(S^{n-1})$ be any non-zero function supported in $\widetilde{B}$. Since $\widetilde{B} \subset \widetilde{Z}^C$ it follows that $B = \mathrm{supp}(R_{n-k}(f)) \subset Z^C$, and therefore:

$$\langle R^*_{n-k}(\mu), f \rangle = \langle \mu, R_{n-k}(f) \rangle = 0.$$

But on the other hand, since $R^*_{n-k}(d\mu) \geqslant 1$ and $f \in C_{e,+}(S^{n-1})$ is non-zero:

$$\langle R^*_{n-k}(\mu), f \rangle \geqslant \langle 1, f \rangle > 0,$$

a contradiction. $\square$

We conjecture that the covering property is also a sufficient condition in the following sense.

Covering Property Conjecture. *For any $n > 0$, $1 \leqslant k \leqslant n-1$, if $Z \subset G(n, n-k)$ is a closed set satisfying $\bigcup_{E \in Z} E \cap S^{n-1} = S^{n-1}$, then there exists a measure $\mu \in \mathcal{M}_+(G(n, n-k))$ supported in Z, such that $R^*_{n-k}(d\mu) \geqslant 1$.*

Under this conjecture, we immediately have the following counterpart to Lemma 5.10.

Lemma 5.11. *Assume the Covering Property Conjecture, and let $Z \subset G(n, n-k)$ be a closed set satisfying the covering property. Then there exists a measure $\mu \in \mathcal{M}_+(G(n, n-k))$ supported in Z, such that $R^*_{n-k}(d\mu) \geqslant 1$ and $R^*_k(d\mu^\perp) \geqslant 1$.*

Proof. Apply the conjecture to the closed sets $Z \subset G(n, n-k)$ and $Z^\perp \subset G(n, k)$, and let $\mu_1 \in \mathcal{M}_+(G(n, n-k))$ and $\mu_2 \in \mathcal{M}_+(G(n, k))$ be the resulting measures. Then $\mu_1 + \mu_2^\perp$ is supported in Z and satisfies the requirements. □

Remark 5.12. A very natural way to approach the proof of the Covering Property Conjecture, is to assume that the closed set Z satisfying $\bigcup_{E\in Z} E \cap S^{n-1} = S^{n-1}$ is minimal with respect to set inclusion (indeed, by Zorn's lemma it is easy to verify that there exists such a minimal set). The natural candidate for a measure supported on Z is simply the Hausdorff measure H_Z on Z, and it remains to show that H_Z is a finite measure and that $R^*_{n-k}(dH_Z) \geqslant \epsilon$ for some $\epsilon > 0$, using the minimality of Z. In particular, one has to show that the Hausdorff dimension of Z is k. Although having some progress in this direction, we have not been able to give a complete proof. We also remark that it is easy to construct a *non-bounded* measure μ supported on Z for which $R^*_{n-k}(d\mu) \geqslant 1$, simply by using the counting measure on Z, i.e. $\mu(A) = |\{A \cap Z\}|$ for any Borel set $A \subset G(n, n-k)$ (where $|A|$ denotes the cardinality of A).

As opposed to Theorem 5.6, where R^*_{n-k} was treated as an operator on $\mathcal{M}(n, n-k)$, we now go back to the original definition of R^*_{n-k} as an operator acting on the entire $\mathcal{M}(G(n, n-k))$. We summarize this in the following lemma, abbreviating as usual $G = G(n, n-k)$:

Lemma 5.13.

$$\mathcal{M}(n, n-k) = \mathcal{M}(G)/\operatorname{Ker} R^*_{n-k}. \tag{1}$$

$$\mathcal{M}_+(n, n-k) = \{\mu + \operatorname{Ker} R^*_{n-k} \mid \mu \in \mathcal{M}_+(G)\}. \tag{2}$$

$$\{\mu \in \mathcal{M}(G) \mid \langle \mu, f\rangle \geqslant 0\ \forall f \in \overline{R_{n-k}(C(S^{n-1}))_+}\} = \mathcal{M}_+(G) + \operatorname{Ker} R^*_{n-k}. \tag{3}$$

Proof. (1) is simply the definition of $\mathcal{M}(n, n-k)$. (2) follows from (3), since $\mathcal{M}_+(n, n-k)$ is defined as the cone of non-negative linear functionals on $\overline{\operatorname{Im} R_{n-k}}$, and any linear functional on the subspace may be extended to the entire space, hence to $\mu \in \mathcal{M}(G)$. (3) was already implicitly used in the proof of Lemma 5.2, but we repeat the argument once more. The right-hand set is clearly a subset of the left-hand set, since $\operatorname{Ker} R^*_{n-k}$ is perpendicular to $\overline{\operatorname{Im} R_{n-k}}$ by (5.1). Conversely, any μ in the left-hand set is a non-negative linear functional on $\overline{\operatorname{Im} R_{n-k}}$, and by a version of the Hahn–Banach theorem (as in the proof of Lemma 5.2), may be extended to a $\mu' \in \mathcal{M}_+(G)$. Again by (5.1), the difference $\mu' - \mu$ must lie in $\operatorname{Ker} R^*_{n-k}$, concluding the proof. □

We now state several more formulations, which are shown to be equivalent each to the other. We then show that under the Covering Property Conjecture, a positive answer to the $\mathcal{BP}^n_k = \mathcal{I}^n_k$ question would imply these new statements. For a closed set $Z \subset G(n, n-k)$, we denote by $\mathcal{M}(Z)$ the set of all measures in $\mathcal{M}(G(n, n-k))$ supported in Z.

Theorem 5.14. *Let n and $1 \leqslant k \leqslant n-1$ be fixed, and let $Z \subset G(n, n-k)$ denote a closed subset. Then the following are equivalent*:

(1) *There does not exist a non-zero $f \in \overline{R_{n-k}(C(S^{n-1}))_+}$ such that $Z \subset f^{-1}(0)$.*

(2) $$\overline{R_{n-k}\big(C(S^{n-1})\big)_+} \cap \big\{f \in C(G) \bigm| f|_Z = 0\big\} = \{0\}.$$

(3) $$\mathcal{M}_+(G) + \operatorname{Ker} R^*_{n-k} + \mathcal{M}(Z) = \mathcal{M}(G).$$

(4) *There exists a measure $\mu \in \mathcal{M}(G)$ such that $R^*_{n-k}(d\mu) = 0$ and $\mu = \mu_1 + \mu_2$ where $\mu_i \in \mathcal{M}(G)$, $\mu_1 \geqslant 1$ and μ_2 is supported in Z.*

It is clear that (2) is just a convenient reformulation of (1). We will show that (2) $\Leftrightarrow$ (3) and (3) $\Leftrightarrow$ (4).

Proof of (2) $\Leftrightarrow$ (3). Again, we use the Hahn–Banach theorem which shows that for cones, $\overline{P_1} = \overline{P_2}$ iff $P_1^* = P_2^*$. The dual cone (in $\mathcal{M}(G)$) to $\overline{R_{n-k}(C(S^{n-1}))_+}$ is by definition:

$$\big\{\mu \in \mathcal{M}(G) \bigm| \langle \mu, f\rangle \geqslant 0 \ \forall f \in \overline{R_{n-k}(C(S^{n-1}))_+}\big\},$$

which by Lemma 5.13 is equal to $\mathcal{M}_+(G) + \operatorname{Ker} R^*_{n-k}$. The dual cone to $C_Z(G) = \{f \in C(G) \mid f|_Z = 0\}$ is obviously $\mathcal{M}(Z)$. Indeed, by definition, if $\mu \in \mathcal{M}(G)$ is not supported in Z, there exists a $f \in C_Z(G)$ such that $\langle \mu, f\rangle \neq 0$ (since Z is closed). Since also $-f \in C_Z(G)$, either $\langle \mu, f\rangle$ or $\langle \mu, -f\rangle$ is negative, and therefore μ cannot be in the dual cone to $C_Z(G)$. The dual cone to $\{0\}$ is of course $\mathcal{M}(G)$. Using $(P_1 \cap P_2)^* = P_1^* + P_2^*$, this concludes the proof. □

Proof of (3) $\Rightarrow$ (4). Apply (3) with the measure $-1 \in \mathcal{M}(G)$ on the right-hand side. Then there exist measures $\nu_1 \in \mathcal{M}_+(G)$, $\nu_2 \in \operatorname{Ker} R^*_{n-k}$ and $\nu_3 \in \mathcal{M}(Z)$, such that $\nu_1 + \nu_2 + \nu_3 = -1$. Denoting $\mu = -\nu_2$, $\mu_1 = \nu_1 + 1$ and $\mu_2 = \nu_3$, (4) follows immediately. □

Proof of (4) $\Rightarrow$ (3). $\mathcal{C}(G)$ is dense in $\mathcal{M}(G)$ in the w^*-topology, so it is enough to show that (4) implies $C(G) \subset \mathcal{M}_+(G) + \operatorname{Ker} R^*_{n-k} + \mathcal{M}(Z)$, as the cones on the right-hand side are closed in this topology. Let $g \in C(G)$, so there exists a constant $C \geqslant 0$ such that $g + C \geqslant 0$, and hence $g + C + \operatorname{Ker} R^*_{n-k} \in \mathcal{M}_+(n, n-k)$. By Lemma 5.13, this means that $g + C \in \mathcal{M}_+(G) + \operatorname{Ker} R^*_{n-k}$, and we see that it is enough to show that the measure $-C$ is in $\mathcal{M}_+(G) + \operatorname{Ker} R^*_{n-k} + \mathcal{M}(Z)$. Since all of the involved sets are cones, it is enough to show the claim for the measure -1. But this follows from formulation (4) in the same manner is in the previous proof. Indeed, let $\mu = \mu_1 + \mu_2$ as assured by (4), where $\mu \in \operatorname{Ker} R^*_{n-k}$, $\mu_1 - 1 \in \mathcal{M}_+(G)$ and $\mu_2 \in \mathcal{M}(Z)$. Then $-1 = (\mu_1 - 1) - \mu + \mu_2 \in \mathcal{M}_+(G) + \operatorname{Ker} R^*_{n-k} + \mathcal{M}(Z)$. This concludes the proof. □

Comparing formulations in Theorems 5.6(6) and 5.14(1) for a set Z satisfying the covering property, and using Lemmas 5.10 and 5.11, the following should now be clear.

Proposition 5.15. *Let n and $1 \leqslant k \leqslant n-1$ be fixed. Assuming the Covering Property Conjecture, if any of the formulations in Theorem* 5.6 *hold, then so do any of the formulations in Theorem* 5.14 *for* any *closed $Z \subset G(n, n-k)$* satisfying the covering property.

Proof. The statement follows immediately from the remark before the proposition, taking into account Remark 5.8 and Lemma 5.9. □

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Appendix A

In the appendix, we formulate and prove Proposition A.1, which is an extended version of the statement from Section 1 and of Corollary 3.3. We have left the proof of Proposition A.1 for the appendix, since the technique involved differs from those used in the rest of this note. Although the proposition is of elementary nature and fairly simple to prove, we have not been able to find a reference to it in the literature, so we give a self contained proof here. A similar formulation of the case $k_1, \dots, k_r = 1$ was given by Blaschke and Petkantschin (see [26,30] for an easy derivation), and used by Grinberg and Zhang in [16] to deduce that $\mathcal{BP}_1^n \subset \mathcal{BP}_l^n$ for all $1 \leqslant l \leqslant n-1$.

We assume some elementary knowledge of exterior products of differential forms on homogeneous spaces. A rigorous derivation may be found in [30], but we recommend the intuitive exposition in [26, Sections 2, 3]. We will also use the notations from Section 3.

We will use the following terminology. For a set of m vectors $\bar{v} = \{v_1, \dots, v_m\}$ in a Euclidean space V, denote by $\mathrm{Vol}_m(\bar{v}) = \det(\{\langle v_i, v_j\rangle\}_{i,j=1}^m)^{1/2}$, which is exactly the *$m$-dimensional volume of the parallelepiped spanned by* $\bar{v}$. If $m = \sum_{i=1}^r k_i$, let U_i be a k_i-dimensional subspace of V. Choose an arbitrary basis $\bar{u^i} = \{u_1^i, \dots, u_{k_i}^i\}$ of U_i such that $\mathrm{Vol}_{k_i}(\bar{u^i}) = 1$, and let $\bar{u} = \bigcup_{i=1}^r \bar{u^i}$. Then the *$m$-dimensional volume of the parallelepiped spanned by unit volume elements of $U_1, \dots, U_r$* is defined as $\mathrm{Vol}_m(\bar{u})$. It is easy to verify that this definition indeed does not depend on the basis $\bar{u^i}$ chosen for U_i, as long as $\mathrm{Vol}_{k_i}(\bar{u^i}) = 1$ (this will also be clear from the proof of Proposition A.1).

Proposition A.1. *Let $n > 1$ be fixed, let d be an integer between 0 and $n-1$, and let $D \in G(n,d)$. For $i = 1, \dots, r$, let $k_i \geqslant 1$ denote integers whose sum l satisfies $l \leqslant n-d$. For $a = 1, \dots, n-d$ denote by $G^a = G(n, n-a)$, and by μ_D^a the Haar probability measure on G_D^a. For $F \in G^l$ and $a = 1, \dots, l-1$, denote by μ_F^a the Haar probability measure on G_F^a. Denote by $\bar{E} = (E_1, \dots, E_r)$ an ordered set with $E_i \in G^{k_i}$. Then for any continuous function $f(\bar{E}) = f(E_1, \dots, E_r)$ on $G^{k_1} \times \dots \times G^{k_r}$*

$$\int\limits_{E_1 \in G_D^{k_1}} \dots \int\limits_{E_r \in G_D^{k_r}} f(\bar{E})\, d\mu_D^{k_1}(E_1) \dots d\mu_D^{k_r}(E_r)$$

$$= \int\limits_{F \in G_D^l} \int\limits_{E_1 \in G_F^{k_1}} \dots \int\limits_{E_r \in G_F^{k_r}} f(\bar{E}) \Delta(\bar{E})\, d\mu_F^{k_1}(E_1) \dots d\mu_F^{k_r}(E_r)\, d\mu_D^l(F),$$

where $\Delta(\bar{E}) = C_{n,\{k_i\},l,d}\Omega(\bar{E})^{n-d-l}$, $C_{n,\{k_i\},l,d}$ *is a constant depending only on* $n, \{k_i\}, l, d$, *and* $\Omega(\bar{E})$ *denotes the volume of the* l*-dimensional parallelepiped spanned by unit volume elements of* $E_1^\perp, \dots, E_r^\perp$.

Remark A.2. One way to compute the constant $C_{n,\{k_i\},l,d}$ is to use the function $f = 1$ in Proposition A.1. Perhaps a better way is to follow the proof, which gives

$$C_{n,\{k_i\},l,d} = \frac{|G(n-d, n-d-l)| \Pi_{i=1}^r |G(l, l-k_i)|}{\Pi_{i=1}^r |G(n-d, n-d-k_i)|},$$

where $|G(a,b)|$ denotes the volume of the Grassmann manifold $G(a,b)$, and is given by [26]

$$\big|G(a,b)\big| = \frac{|S^{a-1}| \cdots |S^{a-b}|}{|S^{b-1}| \cdots |S^0|}, \tag{A.1}$$

where $|S^m|$ denotes the volume of the Euclidean unit sphere S^m of dimension m (and $|S^0| = 2$).

Proof of Proposition A.1. We will show that the densities $d\mu_D^{k_1}(E_1) \dots d\mu_D^{k_r}(E_r)$ and $\Delta(\bar{E})\, d\mu_F^{k_1}(E_1) \dots d\mu_F^{k_r}(E_r)\, d\mu_D^l(F)$ with $F = \bigcap_{a=1}^r E_a$ coincide on a set of measure 1 with respect to both measures. It is easy to verify that the set consisting of all $(E_1, \dots, E_r)$ such that $\dim(\bigcap_{a=1}^r E_a) = n - l$ satisfies this requirement, and therefore F above is in $G(n, n-l)$, hence the second measure is well defined. Indeed, this set is exactly complementary to the set of all $(E_1, \dots, E_r)$ such that $\Omega(\bar{E}) = 0$, which defines a lower-dimensional analytic submanifold of $G^{k_1} \times \dots \times G^{k_r}$, hence having measure 0 with respect to the first (Haar) measure.

If $J \in G(a,c)$, it is well known [26] that the volume element of $G_J(a,b)$ for $b > c$ at $H \in G_J(a,b)$ is given by

$$dG_J(a,b)(H) = \bigwedge_{i=c+1}^{b} \bigwedge_{j=b+1}^{a} w_{i,j}, \tag{A.2}$$

where $w_{i,j} = \langle e_i, de_j \rangle$, and $\{e_1, \dots, e_a\}$ is any orthonormal basis of $\mathbb{R}^a$ such that $J = \operatorname{span}\{e_1, \dots, e_c\}$ and $H = \operatorname{span}\{e_1, \dots, e_b\}$. Indeed, it is easy to verify that this formula does not depend on the given orthonormal basis satisfying these conditions, by changing basis and applying a change of variables formula. With this normalization, the total volume of $G_J(a,b)$ is $|G(a-c, b-c)|$, as defined in (A.1) [26]. Since $d_1 \wedge d_2 = -d_2 \wedge d_1$, the volume element is signed, corresponding to the assumed orientation of the element. However, we will henceforth ignore the orientation and implicitly take the absolute value in all exterior products, except where it is mentioned otherwise. Note also that the skew-symmetry implies $d \wedge d = 0$.

Let $\{f_1, \dots, f_d\}$ be an orthonormal basis of D, and let $\{f_1, \dots, f_{n-l}\}$ be a completion to an orthonormal basis of F. For $a = 1, \dots, r$ let $\{e_{n-l+1}^a, \dots, e_{n-k_a}^a\}$ be an orthonormal basis of $F^\perp \cap E_a$, and let $\{e_{n-k_a+1}^a, \dots, e_n^a\}$ be an orthonormal basis of $E_a^\perp$. For every a we define $e_i^a = f_i$ for $i = 1, \dots, n-l$. Then

$$d\mu_D^{k_1}(E_1) \cdots d\mu_D^{k_r}(E_r) = C_{n,\{k_i\},l,d}^1 \bigwedge_{a=1}^{r} \bigwedge_{i=d+1}^{n-k_a} \bigwedge_{j=n-k_a+1}^{n} w_{i,j}^a,$$

where $w^a_{i,j} = \langle e^a_i, de^a_j\rangle$ and $C^1_{n,\{k_i\},l,d} = (\Pi^r_{i=1}|G(n-d,n-d-k_i)|)^{-1}$ accounts for the fact that the measure on the left is normalized to have total mass 1. Notice that by (A.2)

$$\bigwedge_{a=1}^{r}\bigwedge_{i=n-l+1}^{n-k_a}\bigwedge_{j=n-k_a+1}^{n} w^a_{i,j} = C^2_{\{k_i\},l}\, d\mu_F^{k_1}(E_1)\dots d\mu_F^{k_r}(E_r),$$

where $C^2_{\{k_i\},l} = \Pi^r_{i=1}|G(l,l-k_i)|$. It remains to show that

$$\bigwedge_{a=1}^{r}\bigwedge_{i=d+1}^{n-l}\bigwedge_{j=n-k_a+1}^{n} w^a_{i,j} = C'_{n,\{k_i\},l,d}\,\Delta(\bar{E})\, d\mu^l_D(F). \tag{A.3}$$

Now let $\{g_{n-l+1},\dots,g_n\}$ denote an orthonormal basis of $F^{\perp}$, and denote $\lambda^a_{j,v} = \langle e^a_j, g_v\rangle$ for $j,v = n-l+1,\dots,n$. Hence $e^a_j = \sum_{v=n-l+1}^n \lambda^a_{j,v} g_v$ and $de^a_j = \sum_{v=n-l+1}^n (d\lambda^a_{j,v} g_v + \lambda^a_{j,v} dg_v)$. Denoting $w_{j,v} = \langle f_j, dg_v\rangle$, we see that since $\langle f_i, g_v\rangle = 0$, then for $i = 1,\dots,n-l$ and $j = n-l+1,\dots,n$

$$w^a_{i,j} = \sum_{v=n-l+1}^{n} \lambda^a_{j,v} w_{i,v}. \tag{A.4}$$

As evident from (A.3), we will be interested in the values of $\lambda^a_{j,v}$ only in the range $j = n-k_a+1,\dots,n$. We therefore rearrange these values by defining a bijection $u : \bigcup_{a=1}^r\{(a,n-k_a+1),\dots,(a,n)\} \to \{1,\dots,l\}$, and denote $\Lambda_{u(a,j),v} = \lambda^a_{j,v}$. Plugging (A.4) into (A.3), we have:

$$\begin{aligned}\bigwedge_{a=1}^{r}\bigwedge_{i=d+1}^{n-l}\bigwedge_{j=n-k_a+1}^{n} w^a_{i,j} &= \bigwedge_{i=d+1}^{n-l}\bigwedge_{a=1}^{r}\bigwedge_{j=n-k_a+1}^{n}\sum_{v=n-l+1}^{n}\lambda^a_{j,v} w_{i,v} \\ &= \bigwedge_{i=d+1}^{n-l}\bigwedge_{u=1}^{l}\sum_{v=n-l+1}^{n}\Lambda_{u,v} w_{i,v} = \bigwedge_{i=d+1}^{n-l}\det(\Lambda) w_{i,n-l+1}\wedge\dots\wedge w_{i,n}.\end{aligned}$$

The last transition is standard and is explained by the skew-symmetry of the exterior product: all terms for which $w_{i,v_1}\wedge\dots\wedge w_{i,v_l}$ contains a recurring $v_i = v_j$ are 0, and we are only left with the case $v_i = \pi(i)$, where π is a permutation of $\{n-l+1,\dots,n\}$; these terms are equal to $(-1)^{\mathrm{sign}(\pi)} w_{i,n-l+1}\wedge\dots\wedge w_{i,n}$, producing the determinant of Λ. Continuing, since Λ does not depend on i and using (A.2), we see that

$$\bigwedge_{a=1}^{r}\bigwedge_{i=d+1}^{n-l}\bigwedge_{j=n-k_a+1}^{n} w^a_{i,j} = \det(\Lambda)^{n-l-d}\bigwedge_{i=d+1}^{n-l}\bigwedge_{j=n-l+1}^{n} w_{i,j} = \det(\Lambda)^{n-l-d} C^3_{n,l,d}\, d\mu^l_D(F),$$

where $C^3_{n,l,d} = |G(n-d,n-d-l)|$. To deduce (A.3), it remains to show that $\det(\Lambda) = \Omega(\bar{E})$.

Recall that $\lambda^a_{j,v} = \langle e^a_j, g_v\rangle$, and in the range $j = n-k_a+1,\dots,n$, these are exactly the coefficients of the orthonormal bases $\bar{e^a} = \{e^a_{n-k_a+1},\dots,e^a_n\}$ of $E_a^{\perp}$ with respect to the orthonormal basis $\bar{g} = \{g_{n-l+1},\dots,g_n\}$ of $F^{\perp}$. Using the orthogonality of $\bar{g}$, it is immediate that

$(\Lambda\Lambda^t)_{u(a_1,j_1),u(a_2,j_2)} = \langle e^{a_1}_{j_1}, e^{a_2}_{j_2}\rangle$, and therefore $\det(\Lambda) = \mathrm{Vol}_{F^\perp}(\bar{e})$ for $\bar{e} = \{\bar{e^1}, \ldots, \bar{e^r}\}$, which is exactly the definition of $\Omega(\bar{E})$. Incidentally, this also shows that $\mathrm{Vol}_{F^\perp}(\bar{e})$ is invariant to taking an arbitrary (not necessary orthonormal) basis $\bar{e^a}$ of $E_a^\perp$ with $\mathrm{Vol}_{E^\perp}(\bar{e^a}) = 1$, since this is easily checked for $\det(\Lambda)$. □

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