Noncommutative chaotic inflation and WMAP three year results

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Abstract

Noncommutative inflation is based upon the consideration of some effects of the space–time uncertainty principle motivated by ideas from string/M theory. The CMB anisotropies may carry a signature of this very early Universe correction from the space–time uncertainty principle and can be used to place constraints on the parameters of the noncommutative inflation model. In this Letter we analyze the noncommutative chaotic inflation model by means of the WMAP three year results. We show that the noncommutative chaotic inflation model can produce a large negative running of spectral index within a reasonable range of e-folding number N, provided that the value of p in the potential $V(\phi) \propto \phi^p$ is enhanced roughly to $p \sim 12$–18. Applying a likelihood analysis to the WMAP results, we find the best-fit values of parameters for the noncommutative chaotic inflation model. In addition, this model predicts a rather big gravitational wave which may be tested by the upcoming experiments.
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Einstein's general relativity will break down at very high energies in the early Universe when quantum effects are expected to be important. If inflation [1] happens at the very early time and if the period of inflation lasts sufficiently long, then the effects of quantum gravity should in principle leave an imprint on the primordial spectrum of perturbations, since the wavelengths of perturbations emerged from short distances in the early stages of inflation are stretched to the cosmic scales observable today by the rapid expansion during inflation. Though we lack a complete theory of quantum gravity presently, we can still look for the hint of these quantum gravity imprints through cosmological experiments such as WMAP and SDSS. Since string/M theory is a promising framework for quantum gravity, it is of interest to explore specific stringy corrections to the spectrum of fluctuations. Many authors have considered and discussed this problem. For the first discussion on the trace of the trans-Planckian effect in the primordial spectrum of perturbations see [2]; for a review containing a comprehensive list of references see [3]. In particular, it should be pointed out that the

possible effects of quantum gravity in the spectra of fluctuations in inflationary cosmology may be induced by the consequences of a basic stringy effect, namely the space–time uncertainty relation [4]

$$\Delta t \, \Delta x_{\text{phys}} \geqslant L_s^2, \tag{1}$$

where t, x_{phys} are the physical space–time coordinates and L_s is the string scale. It was shown in [5] that this space–time uncertainty principle may yield inflation from pure radiation. A more modest approach was pioneered in [6], where the consequences were studied by imposing Eq. (1) into the action for cosmological perturbations on an inflationary background. On a general ground, if inflation is indeed affected by physics at a scale close to string scale or a related scale, it can be expected that space–time uncertainty must leave traces in the CMB power spectrum [6–8]. The primordial power spectrum based upon the noncommutative inflation model [6] has been given in [9–12], and the explicit calculation indeed shows that the effects can be observed.

The standard concordance ΛCDM model, which can arise from an inflationary background cosmology in which the quasi-exponential expansion of space is driven by a scalar field, still

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provides a fairly good fit to the recent three-year WMAP [13] and earlier observations. Assuming that the primordial fluctuations are adiabatic with a power law spectrum, the WMAP three-year data require a spectral index that is significantly less than the Harrison–Zel'dovich–Peebles scale-invariant spectrum ($n_s = 1$, $r = 0$). This suggests, for power law inflationary models, a detectable level of gravity waves. However, many inflationary models can only predict a much smaller gravity wave amplitude. Moreover, the WMAP three-year data can place significant constraints on inflationary models. For example, the chaotic inflationary model with the inflaton potential, $V(\phi) \propto \phi^p$, has been tested in light of the current data, with the result that the data prefer the $m^2\phi^2$ model over the $\lambda\phi^4$ model, assuming a power-law primordial power spectrum. When allowing for a running spectral index, the data will favor a large negative running index and a large tensor amplitude (characterized by r, the ratio of the tensor to scalar power spectrum). However, it is rather hard to produce a large absolute value of the running spectral index within the framework of the usual slow-roll inflationary models [14]. Hence, the confirmation of this suggestive trend is very important for our understanding of the early Universe physics, especially of the quantum gravity relevant physics.

It has been mentioned that the CMB anisotropies may carry a signature of the very early Universe corrections from the space–time uncertainty principle motivated by string theory, and can be used to place constraints on the parameters appearing in the noncommutative inflation model. A recent paper [15] shows that the noncommutative inflation model can nicely produce a large running spectral index. In [15], some concrete noncommutative inflation models such as the chaotic inflation and power-law inflation have also been re-examined. It is well known that the usual chaotic inflation model can only predict the negligible amount of running index, $dn_s/d \ln k \approx -10^{-3}$. Though the noncommutative chaotic inflation model can realize a considerable running index, for the $m^2\phi^2$ model and the $\lambda\phi^4$ model, the authors of [15] find that a low number of e-folds, say $N \sim 10$, is required. Needless to say, this is fairly unnatural since such low values of N may not be sufficient to resolve the naturalness problems such as the flatness and the horizon puzzles. A reasonable range of N has been derived in [16], that is $47 < N < 61$ (or $N = 54 \pm 7$). In this Letter, we will show that the noncommutative chaotic inflation model can provide a large negative running spectral index within the reasonable range of N, provided that the value of p in the potential $V(\phi) \propto \phi^p$ is enhanced, say $p \sim 15$. In addition, we perform a likelihood analysis to find the best-fit values of the parameters of the noncommutative chaotic inflation model to the WMAP results of the spectral index and its running. Thus we are able to constrain the noncommutative model so as to place limits on the space–time noncommutativity in the string scale.

The space–time noncommutative effects can be encoded in a new product, star product, replacing the usual algebra product. The evolution of a homogeneous and isotropic background will not change and the standard cosmological equations based upon the Friedmann–Robertson–Walker (FRW) metric remain the same:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{2}$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_{\mathrm{Pl}}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{3}$$

where M_{Pl} is the reduced Planck mass. If $\dot{\phi}^2 \ll V(\phi)$ and $\ddot{\phi} \ll 3H\dot{\phi}$, the scalar field shall slowly roll down its potential. Define some slow-roll parameters,

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{M_{\mathrm{Pl}}^2}{2}\left(\frac{V'}{V}\right)^2, \tag{4}$$

$$\eta = \epsilon - \frac{\ddot{H}}{2H\dot{H}} = M_{\mathrm{Pl}}^2\frac{V''}{V}, \tag{5}$$

$$\xi^2 = 7\epsilon\eta - 5\epsilon^2 - 2\eta^2 + \zeta^2 = M_{\mathrm{Pl}}^4\frac{V'V'''}{V^2}, \tag{6}$$

where $\zeta^2 \equiv \dddot{H}/(2H^2\dot{H})$, then the slow-roll condition can be expressed as $\epsilon, \eta \ll 1$.

The stringy space–time uncertainty relation leads to changes in the action for the metric fluctuations. The action for scalar metric fluctuations can be reduced to the action of a real scalar field u with a specific time-dependent mass which depends on the background cosmology. Thus the modified action of the perturbation led by the stringy space–time uncertainty relation can be expressed as [6]

$$S = V_T \int\limits_{k<k_0} d\tilde{\eta}\, d^3k\, \frac{1}{2} z_k^2(\tilde{\eta})\left(u'_{-k}u'_k - k^2 u_{-k}u_k\right), \tag{7}$$

where V_T is the total spatial coordinate volume, a prime denotes the derivative with respect to the modified conformal time $\tilde{\eta}$, k is the comoving wave number, and

$$z_k^2(\tilde{\eta}) = z^2(\beta_k^+\beta_k^-)^{1/2}, \quad z = \frac{a\dot{\phi}}{H}, \tag{8}$$

$$\frac{d\tilde{\eta}}{d\tau} = \left(\frac{\beta_k^-}{\beta_k^+}\right)^{1/2},$$

$$\beta_k^\pm = \frac{1}{2}\left(a^{\pm 2}(\tau + kL_s^2) + a^{\pm 2}(\tau - kL_s^2)\right), \tag{9}$$

here L_s is the string length scale, $k_0 = (\beta_k^+/\beta_k^-)^{1/4}L_s$ and τ denotes a new time variable in terms of which the stringy uncertainty principle takes the simple form $\Delta\tau\Delta x \geqslant L_s^2$, using comoving coordinates x. The case of general relativity corresponds to $L_s = 0$.

In [12], the power spectrum in the noncommutative space–time of the curvature perturbation, $\mathcal{R} \equiv \Phi + H\delta\phi/\dot{\phi}$, where Φ is the scalar metric perturbation in longitudinal gauge, has been calculated by evaluating the time when fluctuations are generated,

$$\mathcal{P}_{\mathcal{R}} \simeq \frac{1}{2\epsilon}\frac{1}{M_{\mathrm{Pl}}^2}\left(\frac{H}{2\pi}\right)^2 (1+\mu)^{-4-6\epsilon+2\eta}$$

$$= \frac{V/M_{\mathrm{Pl}}^4}{24\pi^2\epsilon}(1+\mu)^{-4-6\epsilon+2\eta}, \tag{10}$$

where $\mu = H^2k^2/(a^2M_s^4)$ is the noncommutative parameter, H and V takes the values when the fluctuation mode k crosses the

Hubble radius, and $M_s = L_s^{-1}$ is the string mass scale. Then the spectral index of the power spectrum of the scalar fluctuations and its running can be given,

$$n_s - 1 \equiv s = \frac{d \ln \mathcal{P}_{\mathcal{R}}}{d \ln k} = -6\epsilon + 2\eta + 16\epsilon\mu, \qquad (11)$$

$$\frac{dn_s}{d \ln k} \equiv \alpha_s = -24\epsilon^2 + 16\epsilon\eta - 2\xi^2 - 32\epsilon\eta\mu. \qquad (12)$$

When $L_s \to 0$ or $M_s \to +\infty$, the noncommutative parameter $\mu = H^2 k^2 / (a^2 M_s^4) \to 0$, Eqs. (11) and (12) reproduce the results of general relativity in the commutative case. The tensor–scalar ratio is also given,

$$r = 16\epsilon. \qquad (13)$$

On the other hand, the best fit results of the three-year WMAP data only for the ΛCDM model with both a running spectral index and tensor modes are [13]

$$n_s = 1.21^{+0.13}_{-0.16}, \qquad dn_s/d \ln k = -0.102^{+0.050}_{-0.043}, \qquad (14)$$

and

$$r \leqslant 1.5 \quad \text{at 95\% CL.} \qquad (15)$$

The fit of the WMAP data shows that a large negative running spectral index is favored by the data, and if allowing for a running index, then models with large tensor components are consistent with the data.

The chaotic inflation has the potential of the form $V(\phi) = \lambda M_{\text{Pl}}^{4-p} \phi^p$ with p a positive integer and λ dimensionless, which provides a way of descending from the Planck scale with chaotic initial field conditions [17]. The slow-roll parameters are related to the number of e-folds N of inflation between the epoch when the horizon scale modes left the horizon and the end of inflation by

$$\epsilon = \frac{p}{4N}, \qquad \eta = \frac{p-1}{2N}, \qquad \xi^2 = \frac{(p-1)(p-2)}{4N^2}. \qquad (16)$$

In the noncommutative case, the spectral index and its running as well as the tensor–scalar ratio can be expressed as

$$n_s = 1 + \left(\mu - \frac{1}{8} - \frac{1}{4p}\right)r, \qquad (17)$$

$$\alpha_s = -\frac{p+2}{32p^2}r^2\left(1 + \frac{8p(p-1)}{p+2}\mu\right), \qquad (18)$$

$$r = \frac{4p}{N}. \qquad (19)$$

It can be seen that the running of the spectral index is always negative for the chaotic inflation model. Moreover, the power spectrum is always a red spectrum in the usual commutative case. But it will become blue if $\mu > (p+2)/8p$ in the noncommutative chaotic inflation. For the chaotic inflation in the usual commutative case, the amount of the running of the spectral index is negligible, $\alpha_s \sim -10^{-3}$. Thus in [13], this model in the quadratic case $m^2\phi^2$ and in the quartic case $\lambda\phi^4$ have been tested by using three-year WMAP data assuming a power-law primordial power spectrum, with the result that the data rule out rather firmly $p \geqslant 4$. However, the chaotic inflation in the noncommutative case will be capable of producing a large enough
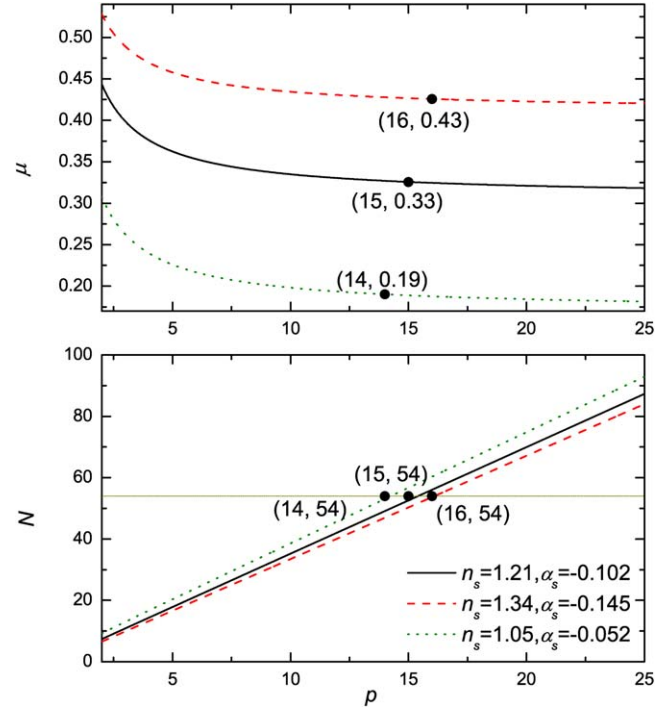

Fig. 1. The constraint relationship curves for $\mu(p)$ and $N(p)$. Points locate the coordinates corresponding to $N = 54$. Note that we restrict p to be an integer.

running of the spectral index provided that choosing a noncommutative parameter μ properly, as shown in [15]. The authors of [15] only analyzed the simplest cases of noncommutative chaotic inflation model, i.e. the cases of $p = 2$ and $p = 4$, and found that in order to realize a large running of the index for matching the WMAP data roughly, a rather low e-folding number N is required, say $N \sim 10$. Such a low value of N, however, seems fairly unnatural as confessed in [15]. Actually, reasonable value of N should be around 50, as demonstrated in [16], $N = 54 \pm 7$. We will show that in order to reconcile this conflict within the noncommutative framework the model parameter p should be enhanced to $p \sim 12$–18.

Utilizing the noncommutative chaotic inflation results (11) and (12) to fit the fixed values of spectral index and its running, for instance the WMAP central values $n_s = 1.21$ and $\alpha_s = -0.102$, gives a constraint relationship between the noncommutative parameter μ and the potential parameter p, namely $\mu(p)$, and then obtains the constraint relationship of $N(p)$. This case is plotted in Fig. 1, as indicated by the black solid lines. The plot explicitly shows that the value of N increases with the value of p, and $N(p)$ is nearly linear. We also see that $\mu(p)$ becomes nearly flat when p goes beyond 10. The relation between the tensor–scalar ratio r and the noncommutative parameter μ in this case can also be obtained, as shown in Fig. 2, the black solid line. We see that the noncommutative chaotic inflation model can predict a rather big gravitational wave, $r \sim 1$, which is within the range of the WMAP and can be confirmed by the upcoming experiments eventually. If the e-folds N is taken to be a reasonable value, say 54, then we get $p = 15$, $\mu = 0.33$ and $r = 1.14$, as shown in Figs. 1 and 2. Note that here p is restricted to be an integer and is not needed
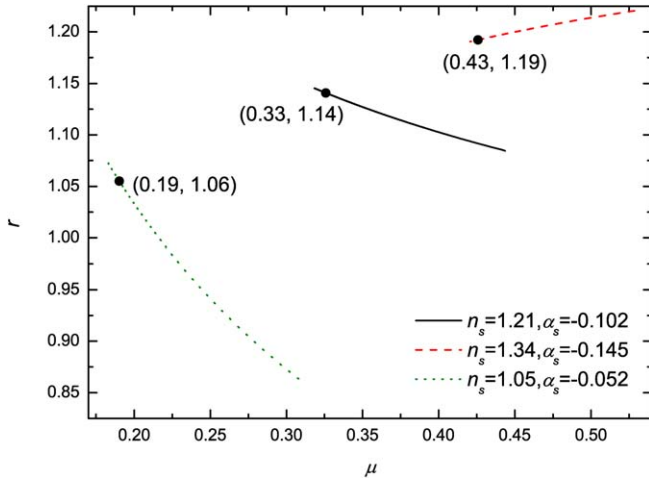
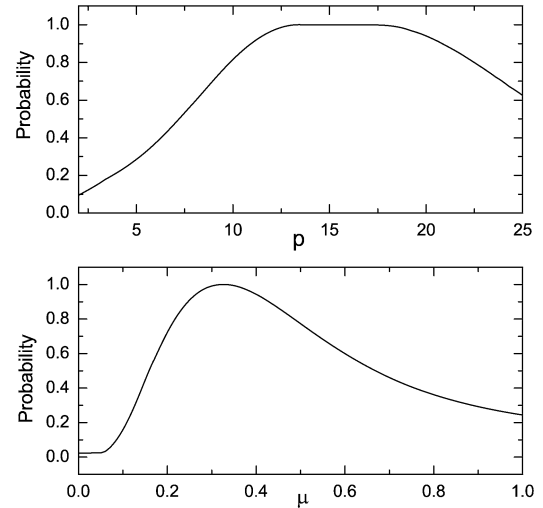Fig. 2. The relationship between μ and r. Points locate the coordinates corresponding to $N = 54$.

Fig. 3. Probability distributions of the likelihood analysis for the parameters p and μ, assuming that $47 < N < 61$. The best fit happens at $p = 13.5$, $\mu = 0.33$ and $N = 47.56$.

to be an even number. Furthermore, we also consider the uncertainties of n_s and α_s in (14) simply. We take two cases into account, $n_s = 1.34$ and $\alpha_s = -0.145$ as well as $n_s = 1.05$ and $\alpha_s = -0.052$, and show the results in Figs. 1 and 2, represented by the red dashed lines and green dotted lines, respectively.[1] According to $N = 54$, we have, for these two cases respectively, $p = 16$, $\mu = 0.43$ and $r = 1.19$ as well as $p = 14$, $\mu = 0.19$ and $r = 1.06$. Hence, fitting the WMAP three year results and at the same time considering a reasonable e-folds N, we conclude that the noncommutative chaotic inflation model should have a potential parameter $p \sim 14$–16. What is more, the noncommutative chaotic inflation predicts a rather big gravitational wave, $r \sim 1$.

Finally, we perform a likelihood analysis to find the best-fit values of the model parameters, including p, μ and N, to the WMAP three-year results. According to the WMAP results of the spectral index and its running (14), the likelihood values of model parameters can be derived by evaluating the χ^2-distribution. First, we assume a rather large range for the e-folds N, $14 < N < 75$. The likelihood analysis shows that the best fit happens at $p = 20.86$, $\mu = 0.32$ and $N = 72.87$. Then, we assume a more reasonable range for the e-folds, $47 < N < 61$. The best fit for this case happens at $p = 13.5$, $\mu = 0.33$ and $N = 47.56$. The probability distributions of the likelihood values of p and μ are shown in Fig. 3. In a word, the WMAP three year results require a large p for the noncommutative chaotic inflation.

In summary, we analyze the noncommutative chaotic inflation model by means of the WMAP three year results. Noncommutative inflation is a modification of standard general relativity inflation which takes into account some effects of the space–time uncertainty principle motivated by ideas from string theory. This effect of quantum gravity of the very early Universe may in principle lead to some corrections to the primordial power spectrum of perturbations and thus leave an

imprint in the CMB anisotropy measurements. Indeed, the noncommutative inflation model can account for the WMAP three year results gracefully. In this Letter, we show that the noncommutative chaotic inflation model can provide a large negative running of the spectral index within a reasonable range of e-folding number N, provided that the value of p in the potential $V(\phi) \propto \phi^p$ is enhanced roughly to $p \sim 12$–18. In addition, we find that the noncommutative chaotic inflation model predicts a rather big gravitational wave which may be tested by the upcoming experiments.

Acknowledgements

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