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A differential model for the deformation of the Plancherel growth process

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Abstract

In the present paper we construct and solve a differential model for the q-analog of the Plancherel growth process. The construction is based on a deformation of the Markov–Krein correspondence between continual diagrams and probability distributions.

Keywords: Growth of Young diagrams; The Markov–Krein correspondence

1. Introduction

The Plancherel growth of Young diagrams has been the subject of an intensive research for many years (see, for example, the book by Kerov [13], and also the expository article by Vershik [19] for a recent review). An important result in the field is the asymptotics of the shape of the Young diagram (called the limit shape) in the course of the Plancherel growth process (Logan and Shepp [16], Vershik and Kerov [20]). In [12] Kerov constructed a dynamical model for the Plancherel growth process, and showed that the limit shape is a fixed point of the Burgers equation: it attracts asymptotically all solutions of the Burgers equation of a certain class. The construction of the dynamical model is based on the correspondence between continual diagrams and probability distributions, called by Kerov the Markov–Krein correspondence.

The Plancherel measure admits a natural deformation as it follows from the representation theory of the Iwahori–Hecke algebras. The deformed Plancherel measure defines a stochastic

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process which is a natural q-analog of the Plancherel growth. The main goal of the present paper is to construct and to solve a differential model for this process. The method is in a deformation of the Markov–Krein correspondence, which results in a deformation of the differential equations responsible for the dynamic of continual diagrams. It is shown in this paper that the deformed Burgers equation has a fixed point, and it is proved that the fixed point attracts asymptotically all relevant solutions.

1.1. Background and remarks on the related works

1.1.1. The Plancherel growth process

Let $\mathbb{Y}$ denote the Young graph, and let $\mathbb{Y}_n$ be the level of $\mathbb{Y}$ consisting of the Young diagrams with n boxes. Thus $\mathbb{Y} = \bigcup_{n=0}^{\infty} \mathbb{Y}_n$. By definition, the Plancherel growth process is the Markov chain on $\mathbb{Y}$ whose initial state is the empty diagram, and whose transition probabilities $p(\lambda, \Lambda)$ are given by

$$p(\lambda, \Lambda) = \frac{1}{|\Lambda|} \frac{\dim \Lambda}{\dim \lambda},$$

if λ is obtained from Λ by removing one box, and by $p(\lambda, \Lambda) = 0$ otherwise. Here $\dim \lambda$ denotes the number of standard Young diagrams of shape λ, and $|\lambda|$ denotes the number of boxes in λ. It can be shown (see, for example, Kerov [13]) that the distribution M_n of the state $\lambda \in \mathbb{Y}_n$ coincides with the Plancherel measure on $\mathbb{Y}_n$, $M_n(\lambda) = \frac{\dim^2 \lambda}{|\lambda|!}$. If the Young diagrams on each level of $\mathbb{Y}$ are distributed according to the Plancherel measure then the Plancherel growth process is defined. It is known (Vershik and Kerov [20], Logan and Shepp [16]) that in the course of the Plancherel growth processes almost all Young diagrams with the normalized area become uniformly close to a common universal curve. In a natural coordinate system this curve is given by

$$\Omega(s) = \begin{cases} \frac{2}{\pi}\left(s \arcsin \frac{s}{2} + \sqrt{4 - s^2}\right), & \text{if } |s| \leqslant 2, \\ |s|, & \text{if } |s| \geqslant 2. \end{cases} \tag{1}$$

1.1.2. A formula for the transition probabilities

Given $\lambda \in \mathbb{Y}$ define a piecewise linear function $\lambda(s)$ with slopes ± 1 and local minima and maxima at two interlacing sequences of integer points

$$x_1 < y_1 < x_2 < \cdots < x_m < y_m < x_{m+1},$$

where the x_i's are the local minima, and the y_i's are the local maxima of $\lambda(s)$, see Fig. 1.

Write $\mu_k(\lambda)$ instead of $p(\lambda, \Lambda)$ if the box that distinguishes Λ from λ is attached to the minimum of the function $\lambda(s)$ with the coordinate x_k. Then $\{\mu_k(\lambda)\}_{k=1}^{m+1}$ are precisely the coefficients of the partial fraction expansion

$$\sum_{k=1}^{m+1} \frac{\mu_k(\lambda)}{x - x_k} = \frac{\prod_{i=1}^{m}(x - y_i)}{\prod_{i=1}^{m+1}(x - x_i)}, \tag{2}$$

see Kerov [9,10]. Formula (2) determines the one-to-one correspondence between the set of Young diagrams, and the set of discrete probability distributions.

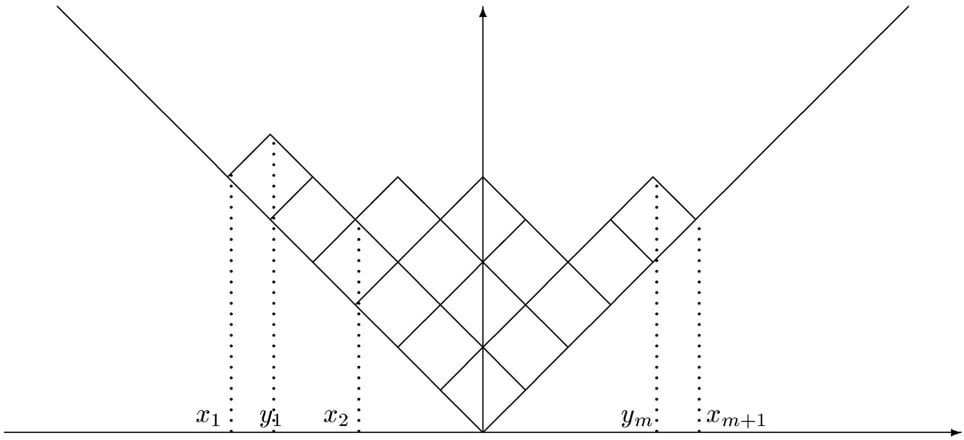

Fig. 1. Young diagrams as interlacing sequences.

1.1.3. The Markov–Krein correspondence

Kerov showed in [9,13] that the correspondence between Young diagrams and discrete probability distributions defined by Eq. (2) can be extended by continuity. This extension was called by Kerov the Markov–Krein correspondence. More precisely, there is a bijective correspondence $\mu \longleftrightarrow w$ between the set of probability measures on $\mathbb{R}$ with compact support, and the set of continual diagrams (the definition of continual diagrams can be found in Section 4). It is characterized by the relation

$$R_\mu(x) = R_w(x),$$

where

$$R_\mu(x) := \int \frac{\mu(ds)}{x-s}, \qquad R_w(x) := \frac{1}{x} \exp\left[-\int \frac{\sigma'(s)\,ds}{s-x}\right],$$

$x \in \mathbb{C}/I$, $I \subset \mathbb{R}$ stands for a sufficiently large interval, and $\sigma(s) = \frac{1}{2}(w(s) - |s|)$. The function $R_w(x)$ is called the R-function of the diagram w, and the function $R_\mu(x)$ is called the R-function of the measure μ. If equation $R_\mu(x) = R_w(x)$ is satisfied the measure μ is referred to as the transition distribution of the diagram w.

Let w be a continual diagram, and define the function $F(s)$ by the formula

$$F(s) = \frac{1}{2}\big(1 + w'(s)\big).$$

$F(s)$ can be regarded as the distribution function of a signed measure τ, and τ is referred to as the Rayleigh measure of the continual diagram w. The Markov–Krein correspondence turns into the relationship between a probability distribution μ, and a bounded signed measure τ on the real line satisfying the identity

$$\int \frac{\mu(ds)}{z-s} = \exp \int \ln \frac{1}{z-s} \tau(ds). \tag{3}$$

To see the relation with transition probabilities of the Plancherel growth process assume that μ is the discrete distribution with weights $\mu_k(\lambda)$ at the points x_k, where $\{x_k\}_{k=1}^{m+1}$ are the local minima of the function $\lambda(s)$, see Fig. 1. Let τ be the signed measure with the weights $+1$ at the points $\{x_i\}_{i=1}^{m+1}$, and the weights -1 at the points $\{y_i\}_{i=1}^{m}$. Then identity (3) specializes to (2).

Besides its relevance to the Plancherel growth process the Markov–Krein correspondence plays a role in diverse topics in analysis including

(1) the connection between additive and multiplicative integral representations of analytic functions of negative imaginary type;
(2) the Markov moment problem;
(3) distributions of mean values of Dirichlet random measures;
(4) the theory of spectral shift function in the scattering theory,

see the expository paper by Kerov [11], where a variety of applications of the Markov–Krein correspondence are described. Note also that more general versions of the Markov–Krein correspondence were used in Kerov and Tsilevich [15] in connection with the Dirichlet measures, and in Vershik, Yor, and Tsilevich [23].

1.1.4. Differential model for the Plancherel growth of Young diagrams

Kerov showed in [12] that the limiting diagram $\Omega(s)$ is a fixed point of the Burgers equation (Eq. (6) below), i.e. $\Omega(s)$ attracts asymptotically all solutions of this equation. The Burgers equation naturally arises in the framework of the differential model for the growth of Young diagrams constructed in [12]. The main assumptions behind this model are:

(1) The history of a growth of a continual diagram w is described by a curve $w(.,t)$, $t_0 < t < \infty$ in the space of continual diagrams. The diagrams $w(.,t)$ are assumed to increase (with respect to the inclusion of subgraphs) with t.
(2) The diagram $w(s,t)$ is required to grow in the direction of its transition distribution μ_t, which means (see [12])

$$\mu_t(ds) = \frac{\partial\sigma(s,t)}{\partial t}\,ds.$$

The equation above is called the basic dynamic equation. The bijection between the continual diagrams and the probability measures leads to different equivalent forms of the basic dynamic equation:

$$\int \frac{1}{x-s}\frac{\partial\sigma(s,t)}{\partial t}\,ds = \frac{1}{x}\exp\left[-\int \frac{1}{x-s}\frac{\partial\sigma(s,t)}{\partial s}\,ds\right], \tag{4}$$

$$\frac{d}{dt}p_n(t) = (n+1)(n+2)h_n(t), \quad n = 0, 1, \ldots, \tag{5}$$

$$\frac{\partial R(x;t)}{\partial t} + R(x,t)\frac{\partial R(x,t)}{\partial x} = 0, \tag{6}$$

where

$$R(x,t)=\int\frac{\mu_{w(.,t)}(ds)}{x-s}=\int\frac{\partial}{\partial t}\sigma(s,t)\frac{ds}{x-s},$$

$p_n(t)$ are the moments of a diagram $w(s,t)$, and $h_n(t)$ be the moments of its transition distribution $\mu_{w(.,t)}(ds)$. The main result in [12] is the following

Theorem 1.1.1. *Assume that the function $\sigma(s,t)=(w(s,t)-|s|)/2$ satisfies Eq.* (4) *(which is equivalent to the basic dynamic equation via the Krein correspondence, and to the Burgers equation, Eq.* (6)*). Then*

$$\lim_{t\to\infty}\frac{1}{\sqrt{t}}w(s\sqrt{t},t)=\Omega(s)$$

uniformly in $s\in\mathbb{R}$, where $\Omega(s)$ is given by Eq. (1)*.*

Theorem 1.1.1 means that all solutions of Eq. (4) have the common asymptotics as $t\to\infty$. Therefore the differential model of the growth described above is a continuous time deterministic process with the same asymptotic behavior as the (random) Plancherel growth process.

Eq. (6) is a quasi-linear differential equation which describes the free motion of a one-dimensional medium of noninteracting particles [12]. In terms of Eq. (6) the curve $\Omega(s)$ corresponds to the automodel solution

$$R(x,t)=\frac{1}{\sqrt{t}}r\left(\frac{x}{\sqrt{t}}\right),$$

where $r(x)$ satisfies the nonlinear differential equation

$$2rr'-xr'-r=0. \tag{7}$$

The only solution of this equation vanishing at $x\to\infty$ is

$$r(x)=\frac{1}{2}\bigl(x-\sqrt{x^2-4}\bigr).$$

This is precisely the R-function of the diagram $\Omega(s)$.

1.2. Statement of main results

We start with a natural q-deformation of the Plancherel measure $M_q^{(n)}$ (Eq. (22)) which is originated from the representation theory of the Iwahori–Hecke algebras (Sections 2.1, 2.2). It is shown in Section 2.3 how $M_q^{(n)}$ is related with nonuniform random permutations. $M_q^{(n)}$ defines a q-analog of the Plancherel growth process, which is a Markov chain on the Young graph. The transition probabilities of this Markov chain can be described as follows. Suppose that the Young diagrams λ and Λ are distinguished by one box attached to the minimum of the function $\lambda(s)$

with the coordinate x_k, see Fig. 1. Then the probability of the transition from λ to Λ is denoted by $\mu_k(\lambda; q)$. $\mu_k(\lambda; q)$ satisfies the equation

$$\sum_{k=1}^{m+1} \frac{\mu_k(\lambda; q)}{1-q^{x-x_k}} = \frac{\prod_{k=1}^{m}(1-q^{x-y_k})}{\prod_{k=1}^{m+1}(1-q^{x-x_k})} \tag{8}$$

for sufficiently large values of the parameter x. Note that as q approaches 1 Eq. (8) is reduced to Eq. (2) for the transition probabilities in the Plancherel growth process. Therefore (8) defines transition probabilities for a q-analog of the Plancherel growth process.

The main goal of the present paper is to construct a differential model for this growth process. For this purpose we introduce the q-deformation of the Krein correspondence between continual diagrams and probability measures with compact supports.

1.2.1. The q-deformation of the Markov–Krein correspondence

Let $0 < q \leqslant 1$, and assume that a real variable x takes values outside an interval $[a, b]$. Denote by $\mathcal{D}[a, b]$ the set of continual diagrams with the property $w(s) = |s - s_0|$ for $s \notin [a, b]$. In addition, denote by $\mathcal{M}[a, b]$ the space of probability measures on the interval $[a, b]$. For $0 < q < 1$ the q-deformation of the R-function of a continual diagram $w \in \mathcal{D}[a, b]$ is defined by the expression

$$R_w(x; q) = \frac{1-q}{1-q^x}\exp\left[-\ln q^{-1}\int_a^b \frac{d\sigma(s)}{1-q^{x-s}}\right] = \frac{1-q}{1-q^x}\exp\left[-\frac{1}{2}\ln q^{-1}\int_a^b \frac{d(w(s)-|s|)}{1-q^{x-s}}\right],$$

and the q-deformation of the R-function of a measure $\mu \in \mathcal{M}[a, b]$ is defined by the expression

$$R_\mu(x; q) = (1-q)\int_a^b \frac{\mu(ds)}{1-q^{x-s}}.$$

For $q = 1$ the q-deformation of the R-function of a diagram $w \in \mathcal{D}[a, b]$ is defined to be $R_w(x)$, and the q-deformation of the R-function of a measure μ is defined to be $R_\mu(x)$.

Theorem 1.2.1. *Let q be a fixed parameter which takes values in the interval* $(0, 1]$. *The relation $R_{\mu_q}(x; q) = R_{w(.;q)}(x; q)$ defines the one-to-one correspondence $w_q \longleftrightarrow \mu_q$ between continual diagrams from $\mathcal{D}[a, b]$, and the probability measures from $\mathcal{M}[a, b]$.*

If equation $R_{\mu_q}(x; q) = R_{w(.;q)}(x; q)$ is satisfied, then μ_q is referred to as the q-transition measure of the diagram $w(.; q)$. An equivalent form of Theorem 1.2.1 is

Theorem 1.2.2. *There is a relationship between a probability measure μ_q on $[a, b]$ and a Rayleigh measure τ_q on $[a, b]$ defined by the identity*

$$\int_a^b \frac{\mu_q(ds)}{1-q^{x-s}} = \exp\left[\int_a^b \ln\left(\frac{1}{1-q^{x-s}}\right)\tau_q(ds)\right]. \tag{9}$$

The probability measure μ_q and the Rayleigh measure τ_q determine each other uniquely.

Eq. (9) can be considered as the q-deformation of the Markov–Krein correspondence (3). Let us emphasize that in the equality $R_{\mu_q}(x;q)=R_{w(.;q)}(x;q)$ both the diagram, $w(.;q)$, and the measure, μ_q, generally depend on the parameter q. Assume that the interval $[a,b]$ is chosen to be large enough, and that μ_q is the discrete distribution with weights $\mu_k(\lambda;q)$ at the points x_k, where $\{x_k\}_{k=1}^{m+1}$ are the local minima of the function $\lambda(s)$, see Fig. 1. Let τ_q be the signed measure with the weights $+1$ at the points $\{x_i\}_{i=1}^{m+1}$, and the weights -1 at the points $\{y_i\}_{i=1}^{m}$. Then Eq. (9) specializes to (8). In this work we apply (9) to derive the differential model for the q-analog of the Plancherel growth process. It is of interest to investigate the role of (9) in other topics of analysis, but we leave this issue for the future research.

1.2.2. The differential model for the q-analog of the Plancherel growth process

We start from the same assumptions as in the Kerov growth model, see Section 1.1.4. Thus the history of the growth of a continual diagram $w(.;q)$ is described by a curve $w(.,t;q)$, $t_0<t<\infty$, the diagram $w(.,t;q)$ is assumed to increase, and to grow in the direction of its q-transition distribution $\mu_{t,q}$. Introduce the functions $\{h_n[\mu_{t,q};q]\}_{n=1}^{\infty}$

$$h_n[\mu_{t,q};q]=\int\limits_a^b q^{-ns}\mu_{t,q}(ds),$$

and the functions $\{p_n[w(.,t;q);q]\}_{n=1}^{\infty}$

$$p_n\big[w(.,t;q);q\big]=-n\ln q^{-1}\int\limits_a^b q^{-ns}\frac{\partial\sigma(s,t;q)}{\partial s}\,ds+1.$$

The theorem below gives the analog of the dynamic equations (4)–(6):

Theorem 1.2.3. *The following dynamic equations are equivalent*

$$\int\limits_a^b \big(1-q^{x-s}\big)^{-1}\frac{\partial\sigma(s,t;q)}{\partial t}\,ds=\big(1-q^{x}\big)^{-1}\exp\Bigg[-\ln q^{-1}\int\limits_a^b \big(1-q^{x-s}\big)^{-1}\frac{\partial\sigma(s,t;q)}{\partial s}\,ds\Bigg]; \tag{10}$$

$$\frac{\partial}{\partial t}p_n\big[w(.,t;q);q\big]=n^2\ln^2 q^{-1}\sum_{|\lambda|=n}\prod_{k=1}^{m(\lambda)}\frac{p_k^{r_k}[w(.,t;q);q]}{k^{r_k}r_k!}, \tag{11}$$

where $n=1,2,\dots$, $\lambda=(1^{r_1},2^{r_2},\dots,m^{r_m})$, and $m=m(\lambda)$;

$$\frac{\partial R_{w(.,t;q)}(x;q)}{\partial x}+\frac{1-q}{\ln q^{-1}}R^{-1}_{w(.,t;q)}(x;q)\frac{\partial R_{w(.,t;q)}(x;q)}{\partial t}=0. \tag{12}$$

Partial differential equation (12) can be understood as a q-analog of the Burgers equation (6). The main difference between (12) and (6) is that $R_{w(.,t;q)}(x;q)$ is a function of three variables: x, t, and q. We remark that the crucial observation behind Theorem 1.2.3 is that the functions

$\{h_n[\mu_{t,q};q]\}_{n=1}^{\infty}$ and $\{p_n[w(.,t;q);q]\}_{n=1}^{\infty}$ are related to each other as the generators of the algebra Λ of the symmetric functions, $\{\boldsymbol{h}_n\}_{n=1}^{\infty}$, and $\{\boldsymbol{p}_n\}_{n=1}^{\infty}$.

1.2.3. The description of the limiting diagram

Theorem 1.2.4. *Let q be a fixed parameter which takes values from the open interval* $(0,1)$. *Assume that $w(s,t;q)$ is a solution of the equivalent dynamic equations* (10)–(12).

Claim 1. *There exists a limiting continual diagram $\Omega(s;q)$ such that*

$$\lim_{t\to\infty}\frac{1}{\sqrt{t}}w\big(s\sqrt{t},t;q^{\frac{1}{\sqrt{t}}}\big)=\Omega(s;q)$$

uniformly in $s\in\mathbb{R}$ and $q\in(0,1)$.

Claim 2. *The limiting diagram is uniquely determined by the function $R_{\Omega(.;q)}(x;q)$ defined by*

$$R_{\Omega(.;q)}(x;q)=\big(1-q^{x}\big)^{-1}\exp\left[-\frac{1}{2}\ln q^{-1}\int\limits_a^b\frac{d(\Omega(s;q)-|s|)}{1-q^{x-s}}\,ds\right],$$

and this function, $R_{\Omega(.;q)}(x;q)$, is the solution of the equation

$$r\big(1-q^{x-\frac{\ln q^{-1}}{1-q}r}\big)=1-q. \tag{13}$$

Claim 3. *Let $\tau^{\Omega(.;q)}$ be the Rayleigh measure of the limiting diagram $\Omega(s;q)$. For $0<q<1$ the moments $\{p_n[\Omega(.;q);q]\}_{n=1}^{\infty}$ of $\tau^{\Omega(.;q)}$ defined by*

$$p_n\big[\Omega(.;q);q\big]=\int\limits_a^b q^{-ns}\tau_q^{\Omega(.;q)}(ds)$$

can be expressed in terms of the solution $\{y_n\}_{n=1}^{\infty}$ of the system of differential equations

$$\frac{dy_n(\varsigma)}{d\varsigma}=n^2\left\{\sum_{|\lambda|=n}\prod_{k=1}^{m(\lambda)}\frac{y_k^{r_k}(\varsigma)}{k^{r_k}r_k!}\right\},\quad n=1,2,\ldots$$

(where $n=1,2,\ldots,$ $\lambda=(1^{r_1},2^{r_2},\ldots,m^{r_m})$, and $m=m(\lambda)$) are subjected to the initial conditions $y_n(0)=1$, $n=1,2,\ldots$. Namely,

$$p_n\big[\Omega(.;q);q\big][q]=y_n\big[\ln^2 q\big],\quad n=1,2,\ldots.$$

The theorem does not provide an explicit form for the limiting curve $\Omega(s;q)$, but it determines the function $R_{\Omega(s;q)}(x;q)$, and the moments of the curve $\Omega(s;q)$ explicitly. In terms of Eq. (12) the curve $\Omega(s;q)$ corresponds to the q-auto-model solution

$$R_{w(.,t;q)}(x;q)=\frac{1-q}{1-q^{\sqrt{t}}}r\left(\frac{x}{\sqrt{t}};q^{\sqrt{t}}\right),$$

where $r(x;q)$ satisfies the partial differential equation

$$2r\frac{\partial}{\partial x}r-\frac{1-q}{\ln q^{-1}}x\frac{\partial}{\partial x}r-qr-q(1-q)\frac{\partial}{\partial q}r=0. \tag{14}$$

Note that if we take $q=1$ in (14), then (14) is reduced to (7). Eq. (14) can be reduced to a quasi-linear partial differential equation which can be solved by the method of characteristics. The result is that the solution of (14) is uniquely determined by Eq. (13). Clearly, the solution of (13) can be understood as the q-deformation of the R-function corresponding to the diagram $\Omega(s)$ defined by Eq. (1). Indeed, if q approaches 1 in Eq. (13), then this equation turns into $r(x-r)=1$. The only solution of this equation vanishing at $x\to+\infty$ coincides with the R-function of the diagram $\Omega(s)$.

2. A deformation of the Plancherel measure

2.1. Iwahori–Hecke algebras

This section recalls few facts on the Iwahori–Hecke algebras associated with the finite Coxeter groups. The general references on the Hecke algebras are Curtis and Reiner [4, §67–68], Carter [3, §10.8–10.11]. We follow the presentation in the paper by Diaconis and Ram [5], which contains the necessary representation theoretic background (Sections 3 and 7).

Let W be a finite Coxeter group generated by simple reflections $s_1,\dots,s_n$. A choice of reflection generators gives rise to a length function l on a Coxeter group. l is defined as the minimum number of the reflection generators required to express a group element. Thus the length function $l(w)$ is the smallest k such that $w=s_{i_1}s_{i_2}\dots s_{i_k}$. The length function has the following properties: $l(id)=0$, $l(s_i)=1$, and $l(s_iw)=l(w)\pm 1$ for each $w\in W$, $1\leqslant i\leqslant n$. Let q be a parameter which takes values in the interval (0, 1).

Definition 2.1.1. The *Iwahori–Hecke algebra H* corresponding to W is the vector space with the basis $\{T_w \mid w\in W\}$ and the multiplication given by

$$T_iT_w=\begin{cases}T_{s_iw}, & \text{if } l(s_iw)=l(w)+1,\\ (q-1)T_w+qT_{s_iw}, & \text{if } l(s_iw)=l(w)-1,\end{cases} \tag{15}$$

where $T_i=T_{s_i}$ for $1\leqslant i<n$.

The irreducible representations of the Iwahori–Hecke algebra H are in one-to-one correspondence with the irreducible representations of the Coxeter group W. Let $\hat{W}$ be an index set for the irreducible representations of W, and for each $\lambda\in\hat{W}$ let χ_W^{λ} be the corresponding irreducible character of W. If χ_H^λ is the character of the irreducible representation of H indexed by $\lambda \in \hat{W}$ then

$$\chi_H^\lambda(T_w)\big|_{q=1} = \chi_W^\lambda(w)$$

for all $w \in W$. In particular, the irreducible representations of W and H indexed by the same λ, $\lambda \in \hat{W}$, have the same dimensions. Define a trace $\vec{t}: H \to \mathbb{C}$ on H by

$$\vec{t}(T_w) = \begin{cases} P_W(q), & \text{if } w = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $P_W(q) = \sum_{w \in W} q^{l(w)}$ is the Poincaré polynomial of the group W. The *generic degrees* are the constants t_λ defined by

$$\vec{t} = \sum_{\lambda \in \hat{W}} t_\lambda \chi_H^\lambda. \tag{16}$$

Let $S(n)$ be the symmetric group. $S(n)$ is generated by the simple transpositions $s_i = (i, i+1)$, $1 \leqslant i \leqslant n-1$. The irreducible representation of $S(n)$, and of the corresponding Iwahori–Hecke algebra H are indexed by Young diagrams with n boxes. Let $l(\lambda)$ be the number of nonzero rows in the Young diagram λ (the length of the Young diagram), and let $|\lambda|$ be the number of boxes of λ. We number the rows and columns as for matrices, and denote by λ_i and λ'_j the length of the ith row and of the jth respectively. The hook length $h(u)$ of a box u in a position (i, j) of λ is

$$h(u) = \lambda_i - i + \lambda'_j - j + 1.$$

Let $b(\lambda) = \sum_{i=1}^{l(\lambda)} (i-1)\lambda_i$, and introduce notations $[k] = 1 - q^k$, $[k]! = [1][2]\ldots[k]$. With these notations the generic degrees t_λ are given by

$$t_\lambda = \frac{q^{b(\lambda)}[n]!}{\prod_{u \in \lambda}[h(u)]}. \tag{17}$$

For the Poincaré polynomial of $S(n)$ there is an explicit formula

$$P_{S(n)}(q) = \prod_{i=1}^{n-1} \frac{q^{i+1} - 1}{q - 1} = \frac{[n]!}{(1-q)^n}. \tag{18}$$

2.2. *q-Deformation of the Plancherel measures*

Consider Eq. (16) in the case of $W = S(n)$. When w is the unit element of $S(n)$ Eq. (16) takes the form

$$P_{S(n)}(q) = \sum_{|\lambda| = n} t_\lambda \dim \lambda. \tag{19}$$

where $\dim\lambda$ is the dimension of the irreducible representation of $S(n)$ parameterized by λ, $|\lambda|=n$. Alternatively, $\dim\lambda$ can be understood as the number of the standard Young tableaux of the shape λ. A convenient explicit formula for $\dim\lambda$ is

$$\dim\lambda = \frac{n!}{\prod_{u\in\lambda} h(u)}, \tag{20}$$

see, for example, Fulton and Harris [7, §4.1]. Inserting expressions for the generic degrees t_λ (Eq. (17)), and for the Poincaré polynomial $P_{S(n)}$ of $S(n)$ (Eq. (18)) into formula (19) we obtain the identity

$$\sum_{|\lambda|=n} \frac{q^{b(\lambda)}\dim\lambda}{\prod_{u\in\lambda}[h(u)]} = \frac{1}{(1-q)^n}. \tag{21}$$

Denote by $\mathbb{Y}_n$ the set of Young diagrams λ with n boxes, and introduce the following function of λ on $\mathbb{Y}_n$

$$M_q^{(n)}(\lambda) = (1-q)^n \dim\lambda \frac{q^{b(\lambda)}}{\prod_{u\in\lambda}[h(u)]}. \tag{22}$$

Then $\sum_{|\lambda|=n} M_q^{(n)}(\lambda)=1$, and each $M_q^{(n)}$ is a probability distribution on the set $\mathbb{Y}_n$ of Young diagrams with n boxes. We have

$$M_{q=1}^{(n)}(\lambda) = M_{\mathrm{Plancherel}}^{(n)}(\lambda), \quad \text{where } M_{\mathrm{Plancherel}}^{(n)}(\lambda) = \frac{(\dim\lambda)^2}{n!}$$

is the Plancherel measure. Thus $M_q^{(n)}$ can be understood as a q-deformation of the Plancherel measure.

2.3. Relation with nonuniform random permutations

It is a well-known fact that the Plancherel measure is a push forward of the uniform distribution on the symmetric group. In this section we show that $M_q^{(n)}$ can be understood as a push forward of a nonuniform distribution on the symmetric group $S(n)$.

We assume that $S(n)$ is realized as the group of permutations of the set $\{1,2,\dots,n\}$. Let σ be a permutation from $S(n)$. We say that i, $i\in\{1,2,\dots,n\}$, is a descent if $\sigma(i)>\sigma(i+1)$. Denote by $D(\sigma)$ the set of all descents of σ, and define the major index of σ, $\mathrm{maj}(\sigma)$, by the formula

$$\mathrm{maj}(\sigma) = \sum_{i\in D(\sigma)} i.$$

Introduce a probability distribution on $S(n)$ by setting

$$\mathbb{P}(\sigma) = \frac{q^{\mathrm{maj}(\sigma)}}{\sum_{\sigma\in S(n)} q^{\mathrm{maj}(\sigma)}}. \tag{23}$$

If the value of the parameter q approaches to 1 then $\mathbb{P}$ approaches to the uniform distribution on the symmetric group $S(n)$, and if the value of the parameter q approaches to 0 then $\mathbb{P}$ approaches to the distribution concentrated at the unit element $\sigma = e$ of the group $S(n)$.

Let T be a standard Young tableau with entries $1, 2, 3, \ldots, n$. Define a descent of T to be an integer i such that $i + 1$ appears in a row of T lower than i, and define the descent set $D(T)$ to be the set of all descents of T. For instance, the standard Young tableau

$$\begin{array}{|c|c|c|c|}
\hline
1 & 3 & 4 & 6 \\
\hline
2 & 5 & 8 & 10 \\
\hline
9 & 7 & \multicolumn{2}{c}{} \\
\cline{1-2}
11 & 12 & \multicolumn{2}{c}{} \\
\cline{1-2}
13 & \multicolumn{3}{c}{} \\
\cline{1-1}
\end{array}$$

has the descent set $\{1, 4, 6, 8, 10, 12\}$. For any standard Young tableau T define the major index $\mathrm{maj}(T)$ by

$$\mathrm{maj}(T) = \sum_{i \in D(T)} i.$$

Proposition 2.3.1. *For any Young diagram λ we have*

$$\sum_T q^{\mathrm{maj}(T)} = \frac{q^{b(\lambda)}[n]!}{\prod_{u \in \lambda}[h(u)]},$$

where T ranges over all standard Young tableaux of shape λ.

Proof. The proof is given in Stanley [18, Chapter 7, pp. 374–376]. $\square$

Proposition 2.3.2. *Let $\sigma \in S(n)$, and assume that σ corresponds to the pair (P, Q) of standard Young tableaux of the same shape via the Robinson–Schensted–Knuth algorithm. Then $D(P) = D(\sigma^{-1})$, and $D(Q) = D(\sigma)$, where D denotes the descent set.*

Proof. See Stanley [18, Chapter 7, p. 382]. $\square$

Proposition 2.3.3. *The probability distribution $M_q^{(n)}$ defined by Eq.* (22) *is a push forward of the nonuniform distribution $\mathbb{P}$ on $S(n)$ (defined by Eq.* (23)*) via the Robinson–Schensted–Knuth correspondence.*

Proof. Assume that $(P(\sigma), Q(\sigma))$ is the pair of standard Young tableaux which is in one-to-one correspondence with σ via the Robinson–Schensted–Knuth algorithm, σ is an element of $S(n)$. Suppose that the probability of σ is $\mathbb{P}(\sigma)$, where $\mathbb{P}(\sigma)$ is given explicitly by Eq. (23). Then $\mathbb{P}(\sigma)$ equals the probability to find the pair $(P(\sigma), Q(\sigma))$ among all possible pairs of Young diagrams with n boxes, and of the same shape. Denote this probability by $\mathbb{P}\{(P(\sigma), Q(\sigma))\}$. Since $D(\sigma) = D(Q(\sigma))$ we have $\mathrm{maj}(\sigma) = \mathrm{maj}(Q)$, and $\mathbb{P}\{(P(\sigma), Q(\sigma))\}$ takes the form

$$\mathbb{P}\big\{\big(P(\sigma), Q(\sigma)\big)\big\} = \frac{q^{\mathrm{maj}(Q)}}{\sum_{\mathrm{Sp}(P)=\mathrm{Sp}(Q)} q^{\mathrm{maj}(Q)}},$$

where the sum is over all standard Young tableaux with n boxes, and of the same shape. Let us compute the probability of the event that the tableaux in the pair $(P(\sigma), Q(\sigma))$ are of the same shape λ, $|\lambda| = n$. This probability is

$$\sum_{\mathrm{Sp}(P)=\mathrm{Sp}(Q)=\lambda} \mathbb{P}\big\{\big(P(\sigma), Q(\sigma)\big)\big\} = \frac{\dim\lambda \sum_{Q:\mathrm{Sp}(Q)=\lambda} q^{\mathrm{maj}(Q)}}{\sum_{|\lambda|=n}(\dim\lambda \sum_{Q:\mathrm{Sp}(Q)=\lambda} q^{\mathrm{maj}(Q)})}, \tag{24}$$

where $\dim\lambda$ is the number of the standard Young tableaux of the shape λ. The expression in the right-hand side of (24) can be rewritten further using Proposition 2.3.1 and formula (21). The result is

$$\sum_{\mathrm{Sp}(P)=\mathrm{Sp}(Q)=\lambda} \mathbb{P}\big\{\big(P(\sigma), Q(\sigma)\big)\big\} = (1-q)^n \dim\lambda \frac{q^{b(\lambda)}}{\prod_{u\in\lambda}[h(u)]} = M_q^{(n)}(\lambda).$$

Therefore, $M_q^{(n)}(\lambda)$ is exactly the probability of the event that the tableaux in the pair $(P(\sigma), Q(\sigma))$ are of the same shape λ, where the pair $(P(\sigma), Q(\sigma))$ corresponds to permutation σ via the Robinson–Schensted–Knuth algorithm, and σ is a random permutation from $S(n)$ with respect to the probability distribution $\mathbb{P}$ defined by (23). We conclude that $M_q^{(n)}$ is push forward of the nonuniform distribution $\mathbb{P}$ on $S(n)$. □

Remark 2.3.4. (1) Several q-analogs of the Plancherel measure were studied in a paper by Fulman [6] in connection with increasing and decreasing subsequences in nonuniform random permutations. However the measures considered in Ref. [6] are different from $M_q^{(n)}$.

(2) The measure $M_q^{(n)}$ is a particular case of knot ergodic central measures, see the book by Kerov [13, Section 3, §4]. The description of knot measures in the content of the representation theory of the infinite-dimensional Hecke algebra $H_\infty(q)$ can be found in the paper by Vershik and Kerov [22].

3. Transition probabilities on the Young graph

3.1. The Young graph

For two Young diagrams λ and μ write $\mu \nearrow \lambda$ (equivalently, $\lambda \searrow \mu$) if $\mu \subset \lambda$ and $|\mu| = |\lambda| - 1$, i.e. μ is obtained from λ by removing one box. Let $\mathbb{Y}$ denote the lattice of Young diagrams ordered by inclusion. We consider $\mathbb{Y}$ as a graph whose vertices are arbitrary Young diagrams μ and the edges are couples (μ, λ) such that $\lambda \searrow \mu$. We shall call $\mathbb{Y}$ the Young graph, and shall denote the level consisting of the Young diagrams with n boxes by $\mathbb{Y}_n$. In this content a standard Young tableau can be understood as a directed path

$$\emptyset \nearrow \lambda^{(1)} \nearrow \cdots \nearrow \lambda^{(n)} = \lambda$$

exiting from the initial vertex $\lambda=\emptyset$ of the Young graph. The dimension of a Young diagram λ is the number $\dim\lambda$ defined recursively as follows: $\dim\emptyset=0$ for the empty diagram $\lambda=\emptyset$, and

$$\dim\Lambda=\sum_{\lambda:\ \lambda\nearrow\Lambda}\dim\lambda. \tag{25}$$

It is clear from the definition above that $\dim\lambda$ is the number of standard Young tableaux of the shape λ. Also note that $\dim\lambda$ coincides with the dimension of the corresponding representation of the symmetric group, and Eq. (25) follows from the Young branching rule for the characters of the finite symmetric group $S(n)$, $n=1,2,\dots$.

Remark 3.1.1. The Young graph is a particular case of multiplicative graphs. Other examples of multiplicative graphs are the Jack graph, the Kingman graph, the Schur graph, see the paper by Borodin and Olshanski [2] for details and further references.

3.2. Harmonic functions on the Young graph

A (real) valued function $\varphi(\lambda)$ is called a harmonic function on the Young graph if it satisfies the condition

$$\varphi(\lambda)=\sum_{\Lambda:\ \Lambda\searrow\lambda}\varphi(\Lambda) \tag{26}$$

for any $\lambda\in\mathbb{Y}$. For the representation-theoretic meaning of the harmonic functions on the Young graph see Refs. [2,14,21]. We are interested in nonnegative harmonic functions φ normalized at the empty diagram: $\varphi(\emptyset)=1$. As in Ref. [2] we denote the set of such functions by $\mathcal{H}_1^+(\mathbb{Y})$.

Proposition 3.2.1. *Let $\varphi\in\mathcal{H}_1^+(\mathbb{Y})$, and let $M(\lambda)$ be a function on the vertices of $\mathbb{Y}$ defined by $M(\lambda)=\dim\lambda\varphi(\lambda)$. Denote by $M^{(n)}$ the restriction of the function $M(\lambda)$ to the nth level $\mathbb{Y}_n$, $n=0,1,2,\dots$. Then $\sum_{|\lambda|=n}M^{(n)}(\lambda)=1$, i.e. $M^{(n)}$ is a probability distribution on $\mathbb{Y}_n$.*

Proof. The proof is by induction. Since $\varphi(\emptyset)=1$, $\dim(\emptyset)=1$ the claim is obviously valid for the level $\mathbb{Y}_0$. Assume that the claim holds for the level $\mathbb{Y}_{n+1}$, i.e.

$$\sum_{|\Lambda|=n+1}M^{(n+1)}(\Lambda)=\sum_{|\Lambda|=n+1}\varphi(\Lambda)\dim\Lambda=1.$$

Insert the expression for $\dim\Lambda$ (Eq. (25)) into the formula written above, and obtain

$$1=\sum_{|\Lambda|=n+1}\varphi(\Lambda)\left(\sum_{\lambda:\ \lambda\nearrow\Lambda}\dim\lambda\right)=\sum_{|\Lambda|=n+1}\varphi(\Lambda)\sum_{|\lambda|=n}\dim\lambda.$$

It is possible to rewrite the right side further as follows

$$\sum_{|\lambda|=n}\dim\lambda\sum_{\Lambda:\ \Lambda\searrow\lambda}\varphi(\Lambda).$$

But the second sum above is precisely $\varphi(\lambda)$, see Eq. (26), and the multiplication of the second sum on $\dim\lambda$ is $M^{(n)}(\lambda)$. Thus the formula $\sum_{|\lambda|=n} M^{(n)}(\lambda)=1$ is obtained, and the claim of the proposition follows. □

3.3. Transition and co-transition probabilities

Definition 3.3.1. For two vertices λ and Λ of the Young graph $\mathbb{Y}$ such that $\lambda\in\mathbb{Y}_n$ and $\Lambda\in\mathbb{Y}_{n+1}$ set

$$q(\lambda,\Lambda)=\begin{cases}\frac{\dim\lambda}{\dim\Lambda}, & \lambda\nearrow\Lambda,\\ 0, & \text{otherwise}.\end{cases} \tag{27}$$

Then $\sum_{\lambda\nearrow\Lambda} q(\lambda,\Lambda)=1$ (where the summation is over diagrams $\lambda\in\mathbb{Y}_n$ such that $\lambda\nearrow\Lambda$), and we will refer to the numbers $q(\lambda,\Lambda)$ as to the *co-transition probabilities* on the Young graph $\mathbb{Y}$.

Definition 3.3.2. Assume that $\varphi(\lambda)$ is a strictly positive valued harmonic function, $\varphi\in\mathcal{H}_1^+(\mathbb{Y})$, and set

$$p(\lambda,\Lambda)=\begin{cases}\frac{\varphi(\Lambda)}{\varphi(\lambda)}, & \Lambda\searrow\lambda,\\ 0, & \text{otherwise}.\end{cases} \tag{28}$$

Then $\sum_{\Lambda\searrow\lambda} p(\lambda,\Lambda)=1$ (where the summation is over diagrams $\Lambda\in\mathbb{Y}_{n+1}$ such that $\Lambda\searrow\lambda$), and we refer to the numbers $p(\lambda,\Lambda)$ as the *transition probabilities* on the Young graph $\mathbb{Y}$.

Proposition 3.2.1 implies that the transition probabilities define M and $M^{(n)}$ uniquely.

3.4. Transition probabilities for $M_q^{(n)}$

A possible way to introduce transition probabilities on $\mathbb{Y}$ is to use the Pieri rule for the Schur symmetric functions

$$p_1\cdot s_\lambda=\sum_{\Lambda\searrow\lambda} s_\Lambda, \tag{29}$$

see Macdonald [17, Section 1, §5]. Let $\alpha=\{\alpha_i\}_{i=1}^\infty$, $\beta=\{\beta_i\}_{i=1}^\infty$ be pairs of nonincreasing sequences of nonnegative numbers satisfying the condition

$$\sum_{i=1}^\infty\alpha_i+\sum_{i=1}^\infty\beta_i\leqslant 1. \tag{30}$$

Define the *extended Schur functions* $s_\alpha(\alpha,\beta)$ by the Frobenius formula

$$s_\lambda(\alpha,\beta)=\sum_{|\rho|=n}\frac{1}{z_\rho}\chi_\rho^\lambda p_\rho(\alpha,\beta),\quad |\lambda|=n, \tag{31}$$

where

$$p_\rho(\alpha,\beta)=p_{\rho_1}(\alpha,\beta)\cdot p_{\rho_2}(\alpha,\beta)\cdots,$$

and the power sums $p_k(\alpha,\beta)$ are given by

$$p_k(\alpha,\beta)=\begin{cases}1, & k=1,\\ \sum_{i=1}^{\infty}\alpha_i^k+(-1)^{k+1}\sum_{i=1}^{\infty}\beta_i^k, & k\geqslant 2.\end{cases} \tag{32}$$

With this realization of the algebra Λ of the symmetric functions, condition (29) implies that the ratios

$$p(\lambda,\Lambda)=\begin{cases}\frac{s_\Lambda(\alpha,\beta)}{s_\lambda(\alpha,\beta)}, & \Lambda\searrow\lambda,\\ 0, & \text{otherwise},\end{cases} \tag{33}$$

can be understood as transition probabilities.

Let $\alpha=\{(1-q)q^k\}_{k=0}^{\infty}$, $\beta=0$. In this case

$$s_\lambda(\alpha,\beta)=(1-q)^{|\lambda|}q^{b(\lambda)}\prod_{b\in\lambda}\big[h(b)\big]^{-1}, \tag{34}$$

where $[k]=(1-q^k)$, $b(\lambda)=\sum_{i=1}^{l(\lambda)}(i-1)\lambda_i$. Moreover, $s_\lambda(\alpha,\beta)$ can be understood as harmonic functions on the Young graph, as it follows from the Pieri rule (29), and from Eq. (32). Harmonic functions determine uniquely transition probabilities and distributions on the levels of the Young graph, see Sections 3.2, 3.3. In particular, $s_\lambda(\alpha,\beta)$ defined by Eq. (34), and the transition probabilities defined by Eq. (33) lead to the q-deformation of the Plancherel measure $M_q^{(n)}$, defined by Eq. (22). In this context, the q-deformation of the Plancherel measure, $M_q^{(n)}$, is a Markov probability measure on the Young graph $\mathbb{Y}$, with transition probabilities defined by (33) and (34).

Remark 3.4.1. Other choices of the parameters α,β result in probability distributions different from $M_q^{(n)}$, see Kerov [13, Section 3.4.2, Examples 1–5].

4. Continual diagrams and q-transition distributions

4.1. Continual diagrams

Continual diagrams were introduced by Kerov in Refs. [9–12], and used further in Refs. [1,8]. Here we recall the definition and some properties of the continual diagrams.

Definition 4.1.1. A *continual diagram* is a function $w(s)$ on $\mathbb{R}$ such that

(i) $|w(s_1)-w(s_2)|\leqslant|s_1-s_2|$ for any $s_1,s_2\in\mathbb{R}$ (the Lipschitz condition).
(ii) There exists a point $s_0\in\mathbb{R}$, called the center of w, such that $w(s)=|s-s_0|$ when $|s|$ is large enough.

The set of all continual diagrams is denoted by $\mathcal{D}$, and the subset of continual diagrams with the center 0 is denoted by $\mathcal{D}^0$.

To any $w \in \mathcal{D}$ assign a function

$$\sigma(s) = \frac{1}{2}\big(w(s) - |s|\big). \tag{35}$$

This function is called the charge of the continual diagram w, $w \in \mathcal{D}$.

Proposition 4.1.2.

(a) $\sigma'(s)$ *exists almost everywhere and satisfies*

$$\big|\sigma'(s)\big| \leqslant 1.$$

(b) $w(s)$ *is uniquely determined by the second derivative* $\sigma''(s)$.
(c) $\sigma'(s)$ *is compactly supported, and*

$$\sigma'(s) = \begin{cases} (w'(s)+1)/2 \geqslant 0, & \text{for } s < 0, \\ (w'(s)-1)/2 \leqslant 0, & \text{for } s > 0. \end{cases}$$

Proof. The first property follows from the Lipschitz condition (i) in Definition 4.1.1. The condition (ii) of Definition 4.1.1 implies the second and the third properties of the function $\sigma(s)$. □

Definition 4.1.3. A continuous piecewise linear function $w : \mathbb{R} \to \mathbb{R}$ is called a rectangular diagram if $w'(s) = \pm 1$ and there exists a constant s_0 such that $w(s) = |s - s_0|$ for sufficiently large $|s|$.

A rectangular diagram is completely determined by the coordinates of its minima $\{x_k\}_{k=1}^{m+1}$ and those of its maxima $\{y_k\}_{k=1}^{m}$. The sequences $\{x_k\}_{k=1}^{m+1}$ and $\{y_k\}_{k=1}^{m}$ interlace

$$x_1 < y_1 < x_2 < \cdots < x_m < y_m < x_{m+1}.$$

Conversely, any pair of interlacing sequences uniquely determines a rectangular diagram. The set of rectangular diagrams (or, equivalently, the set of interlacing sequences) will be denoted by $\mathcal{D}_R$.

Example 4.1.4 *(Young diagrams).* Given $\lambda \in \mathbb{Y}$ define a piecewise linear function $\lambda(s)$ with slopes ± 1 and local minima and maxima at two interlacing sequences of integer points

$$x_1 < y_1 < x_2 < \cdots < x_m < y_m < x_{m+1},$$

where the x_i's are the local minima, and the y_i's are the local maxima of $\lambda(s)$, see Fig. 1. The correspondence $\lambda \to \lambda(s)$ gives an embedding

$$\mathbb{Y} \to \mathcal{D}^0,$$

i.e. the set $\mathbb{Y}$ of Young diagrams is embedded into the subspace $\mathcal{D}^0$ of continual diagrams with zero center.

Example 4.1.5 *(Orthogonal polynomials).* Let $\{P_m(x)\}_{m=0}^{\infty}$ be a sequence of orthogonal polynomials defined with respect to a probability measure μ. The roots of two consecutive polynomials $P_{m+1}(x)$, $P_m(x)$ interlace, so the roots of $P_{m+1}(x)$ can be understood as minima, and the roots of $P_m(x)$ can be understood as maxima of a rectangular diagram.

4.2. q-Deformations of R-functions

Fix an interval $[a,b]$, where a is strictly negative, and b is strictly positive. Denote by $\mathcal{D}[a,b]$ the set of continual diagrams with the property $w(s)=|s-s_0|$ for $s\notin[a,b]$. The space $\mathcal{D}[a,b]$ is endowed with the uniform convergence topology. Denote by $\mathcal{D}_R[a,b]$ the subspace of rectangular diagrams in $\mathcal{D}[a,b]$. Note that the subspace of rectangular diagrams, $\mathcal{D}_R[a,b]$, is dense in $\mathcal{D}[a,b]$. In addition, denote by $\mathcal{M}[a,b]$ the space of probability measures on the interval $[a,b]$.

Definition 4.2.1. (1) *An R-function of a diagram* $w\in\mathcal{D}[a,b]$ is a function $R_w(x)$ holomorphic outside the interval $[a,b]$, and defined by

$$R_w(x)=\frac{1}{x}\exp\left[-\int_a^b\frac{d\sigma(s)}{s-x}\right]=\frac{1}{x}\exp\left[-\frac{1}{2}\int_a^b\frac{d(w(s)-|s|)}{s-x}\right].$$

(2) *An R-function of a measure* $\mu\in\mathcal{M}[a,b]$ is a function $R_\mu(x)$ holomorphic outside the interval $[a,b]$, and defined by

$$R_\mu(x)=\int_a^b\frac{\mu(ds)}{x-s}.$$

Definition 4.2.1 is due to Kerov, see Ref. [9, Section 2.2]. Now let us introduce natural *q-deformations* of the functions $R_w(x)$ and $R_\mu(x)$.

Definition 4.2.2. Let $0<q\leqslant 1$, and assume that a real variable x takes values outside the interval $[a,b]$. For $0<q<1$ *the q-deformation of the R-function of a diagram* $w\in\mathcal{D}[a,b]$ is defined by the expression

$$R_w(x;q)=\frac{1-q}{1-q^x}\exp\left[-\ln q^{-1}\int_a^b\frac{d\sigma(s)}{1-q^{x-s}}\right]=\frac{1-q}{1-q^x}\exp\left[-\frac{1}{2}\ln q^{-1}\int_a^b\frac{d(w(s)-|s|)}{1-q^{x-s}}\right],$$

and *the q-deformation of the R-function of a measure* $\mu\in\mathcal{M}[a,b]$ is defined by the expression

$$R_\mu(x;q)=(1-q)\int_a^b\frac{\mu(ds)}{1-q^{x-s}}.$$

For $q=1$ the q-deformation of the R-function of a diagram $w\in\mathcal{D}[a,b]$ is defined to be $R_w(x)$, and the q-deformation of the R-function of a measure $\mu\in\mathcal{M}[a,b]$ is defined to be $R_\mu(x)$.

4.3. q-Transition measures

Definition 4.3.1. Fix $0 < q \leqslant 1$. We call μ_q, $\mu_q \in \mathcal{M}[a,b]$, *a q-transition measure of a continual diagram* $w(.;q)$, $w(.;q) \in \mathcal{D}[a,b]$, if the functions $R_{\mu_q}(x;q)$ and $R_{w(.;q)}(x;q)$ coincide.

According to Definition 4.3.1, if $0 < q < 1$, and μ_q is the q-transition measure of the diagram $w(.;q)$, $w(.;q) \in \mathcal{D}[a,b]$, then

$$\int_a^b \frac{\mu_q(ds)}{1-q^{x-s}} = \frac{1}{1-q^x}\exp\left[-\frac{1}{2}\ln q^{-1}\int_a^b \frac{d(w(s;q)-|s|)}{1-q^{x-s}}\right], \tag{36}$$

and if $q=1$ then the transition measure $\mu := \mu_{q=1}$ of a diagram $w(.) := w(.;q=1)$, and the diagram $w(.)$ are related by the identity

$$\int_a^b \frac{\mu(ds)}{x-s} = \frac{1}{x}\exp\left[-\frac{1}{2}\int_a^b \frac{d(w(s)-|s|)}{s-x}\right]. \tag{37}$$

Proposition 4.3.2. *For a rectangular diagram w with the minima $\{x_k\}_{k=1}^{m+1}$ and the maxima $\{y_k\}_{k=1}^{m}$ the q-transition measure is the probability measure supported by the finite set $\{x_1,\dots,x_{m+1}\}$ whose weights $\{\mu_k(w;q)\}_{k=1}^{m+1}$ are given explicitly by the formula*

$$\mu_k(w;q) = \prod_{i=1}^{k-1}\frac{1-q^{x_k-y_i}}{1-q^{x_k-x_i}}\prod_{i=k+1}^{m+1}\frac{1-q^{x_k-y_{i-1}}}{1-q^{x_k-x_i}}. \tag{38}$$

Proof. Let w be a rectangular diagram taken from $\mathcal{D}_R[a,b]$ with the minima $\{x_k\}_{k=1}^{m+1}$ and the maxima $\{y_k\}_{k=1}^{m}$. Then the function $R_w(x;q)$ takes the form

$$R_w(x;q) = (1-q)\frac{\prod_{j=1}^{m}(1-q^{x-y_j})}{\prod_{j=1}^{m+1}(1-q^{x-x_j})}.$$

Indeed, if w is a rectangular diagram then the second derivative of the function $\sigma(s) = \frac{1}{2}(w(s)-|s|)$ is given by

$$\sigma''(s) = \sum_{k=1}^{m+1}\delta(s-x_k) - \sum_{k=1}^{m}\delta(s-y_k) - \delta(s),$$

and we can write

$$\frac{\prod_{k=1}^{m}(1-q^{x-y_k})}{\prod_{k=1}^{m+1}(1-q^{x-x_k})} = \exp\left[-\int_a^b \ln\bigl(1-q^{x-s}\bigr)\sigma''(s)\,ds - \ln\bigl(1-q^x\bigr)\right]$$
$$= \frac{1}{1-q^x}\exp\left[-\int_a^b \ln\bigl(1-q^{x-s}\bigr)\sigma''(s)\,ds\right]. \tag{39}$$

The integration by parts shows that the right-hand side of Eq. (39) coincides with the function $R_w(x;q)$ divided by $(1-q)$. The left-hand side of (39) can be rewritten as

$$\frac{\prod_{k=1}^{m}(1-q^{x-y_k})}{\prod_{k=1}^{m+1}(1-q^{x-x_k})} = \sum_{k=1}^{m+1}\frac{\mu_k(w;q)}{1-q^{x-x_k}}, \tag{40}$$

where $\mu_k(w;q)>0$ for $k=1,\dots,m+1$; $\sum_{k=1}^{m+1}\mu_k(w;q)=1$; and the weights $\{\mu_k(w;q)\}_{k=1}^{m+1}$ are given by (38). Therefore, the equation $R_\mu(x;q)=R_{w(.)}(x;q)$ implies in the case of rectangular diagram that μ is supported by $\{x_k\}_{k=1}^{m+1}$, and the weights $\{\mu_k(w;q)\}_{k=1}^{m+1}$ of μ are given by (38). $\square$

Recall that the q-deformation $M_q^{(n)}$ of the Plancherel measure defined by Eq. (22) can be understood as a Markov probability measure on the Young graph $\mathbb{Y}$, with the transition probabilities $p(\lambda,\Lambda)$ defined by Eqs. (33) and (34). Consider the Young diagram λ as a rectangular diagram, see Fig. 1. Denote by $\{x_k\}_{k=1}^{m+1}$ the minima of λ, and by $\{y_k\}_{k=1}^{m}$ the maxima of λ. Let us write $\mu_k(\lambda;q)$ instead of $p(\lambda,\Lambda)$ if the square that distinguishes Λ from λ is attached to the minimum x_k of λ.

Proposition 4.3.3. *The transition probabilities $\mu_k(\lambda;q)$ of the q-deformation $M_q^{(n)}$ of the Plancherel measure are given by formula* (38), *where in the left-hand side w must be replaced by λ. Thus, $\mu_k(\lambda;q)$ are the weights of the q-transition measure of the diagram λ in the sense of Definition* 4.3.1.

Proof. See Kerov [13, Section 3.4.3]. $\square$

Formula (38) defines a bijection between the set $\mathcal{M}^0[a,b]$ of probability measures on $[a,b]$ with finite support, and the set of rectangular diagrams $\mathcal{D}_R[a,b]$. This bijection can be extended by continuity to a homeomorphism of $\mathcal{D}[a,b]$ to $\mathcal{M}[a,b]$.

4.4. q-Moments of continual diagrams

Define the functions $p_1[w(.);q]$, $p_2[w(.);q],\dots$ on the space of diagrams $\mathcal{D}[a,b]$ by setting

$$p_n\bigl[w(.);q\bigr] = 1-\frac{n}{2}\ln q^{-1}\int_a^b q^{-ns}\,d\bigl(w(s)-|s|\bigr), \tag{41}$$

and define the functions $h_1[\mu; q], h_2[\mu; q], \dots$ (where μ is a probability measure from $\mathcal{M}[a, b]$) by setting

$$h_n[\mu; q] = \int_a^b q^{-ns}\mu(ds). \tag{42}$$

We will refer to $h_1[\mu; q], h_2[\mu; q], \dots$ as to *q-moments* of the probability measure μ.

Proposition 4.4.1. *Fix a real parameter q from the open interval* $(0, 1)$. *Assume that a diagram $w(.; q) \in \mathcal{D}[a, b]$ and a probability measure $\mu_q \in \mathcal{M}[a, b]$ are chosen in such a way that the relation* (36) *is satisfied for all x outside the interval* $[a, b]$. *Then the two sequences*

$$\left\{1 - \frac{n}{2}\ln q^{-1}\int_a^b q^{-ns}\, d\big(w(s; q) - |s|\big)\right\}_{n=1,2,\dots},$$

and

$$\left\{\int_a^b q^{-ns}\mu_q(ds)\right\}_{n=1,2,\dots}$$

are related to each other in the same way as the systems of generators of the algebra Λ of the symmetric functions, $\{\boldsymbol{p}_n\}_{n=1,2,\dots}$ and $\{\boldsymbol{h}_n\}_{n=1,2,\dots}$. In other words, relation (36) *is equivalent to*

$$1 + \sum_{n=1}^{\infty} h_n[\mu_q; q]q^{nx} = \exp\left[\sum_{n=1}^{\infty} p_n\big[w(.; q); q\big]q^{nx}\right]. \tag{43}$$

Proof. Rewrite the right-hand side of relation (36) as

$$\begin{aligned}
&\frac{1}{1 - q^x}\exp\left[-\ln q^{-1}\int_a^b \big(1 - q^{x-s}\big)^{-1}\sigma'(s)\, ds\right] \\
&= \exp\left[\sum_{n=0}^{\infty}\left[-\ln q^{-1}\int_a^b q^{-ns}\sigma'(s)\, ds\right]q^{nx} + \sum_{n=1}^{\infty}\frac{q^{nx}}{n}\right] \\
&= \exp\left[\sum_{n=1}^{\infty}\left(1 - n\ln q^{-1}\int_a^b q^{-ns}\sigma'(s)\, ds\right)\frac{q^{nx}}{n}\right] \\
&= \exp\left[\sum_{n=1}^{\infty} p_n\big[w(.; q); q\big]\frac{q^{nx}}{n}\right],
\end{aligned}$$

where $\sigma(s)$ denotes the charge of the diagram w_q. (The fact that $\int_a^b \sigma'(s)\,ds = 0$ was used to get the last equation.) Thus the right-hand side of (36) coincides with that of (43). The left-hand side of (36) can be rewritten as

$$1 + \sum_{n=1}^{\infty} \int_a^b q^{-ns} \mu_q(ds) q^{nx},$$

which is $1 + \sum_{n=1}^{\infty} h_n[\mu_q; q] q^{nx}$. The proposition is proved. $\square$

Corollary 4.4.2. *If μ_q is the q-transition measure of the diagram $w(,;q)$, and the parameter q takes values in the open interval* $(0, 1)$ *then Proposition* 4.4.1 *implies the relation*

$$\begin{aligned} R_{\mu_q}(x; q) &= (1-q)^{-1}\left(1 + \sum_{n=1}^{\infty} h_n[\mu_q; q] q^{nx}\right) \\ &= (1-q)^{-1} \exp\left[\sum_{n=1}^{\infty} p_n\big[w(.;q); q\big] q^{nx}\right] = R_{w(.;q)}(x; q). \end{aligned}$$

4.5. q-Deformation of the Markov–Krein correspondence

Let $w \in \mathcal{D}[a, b]$, and define the function $F(s)$ by the formula

$$F(s) = \frac{1}{2}\big(1 + w'(s)\big).$$

It is clear from Definition 4.1.1 of continual diagrams that $F(s)$ can be regarded as the distribution function of a signed measure τ. We will refer to the measure τ as to the Rayleigh measure. Simple calculations show that the functions $p_n[w(.); q]$ defined by Eq. (41) can be rewritten as

$$p_n\big[w(.); q\big] = p_n[\tau; q] = \int_a^b q^{-ns} \tau(ds). \tag{44}$$

Therefore the functions $p_n[w(.); q]$ can be regarded as the *q-moments of the Rayleigh measure τ*.

Theorem 4.5.1. *There is a relationship between a probability measure μ_q on $[a, b]$, and a Rayleigh measure τ_q on $[a, b]$ defined by the identity*

$$\int_a^b \frac{\mu_q(ds)}{1 - q^{x-s}} = \exp\left[\int_a^b \ln\left(\frac{1}{1 - q^{x-s}}\right) \tau_q(ds)\right]. \tag{45}$$

The probability measure μ_q and the Rayleigh measure τ_q determine each other uniquely via Eq. (45).

Proof. Eq. (45) can be obtained from Eq. (36) with the integration by parts. To prove the fact that τ_q and μ_q determine each other uniquely recall that the moments

$$h_n = h_n[\mu] = \int_a^b s^n \mu(ds)$$

determine the finite measure μ uniquely. (This fact is known as the uniqueness of a solution for the Hausdorff Moment Problem.) Using the obvious change of variables we can deduce that $h_n[\mu_q; q]$ defined by Eq. (42) determine the probability measure μ_q uniquely. The moments $p_n[\tau; q]$ defined by Eq. (44) also determine the Rayleigh measure τ_q uniquely. Furthermore, we have proved (see Proposition 4.4.1) that Eq. (36) is equivalent to the fact that the moments $h_n[\mu_q; q]$ and $p_n[\tau_q; q]$ are related to each other as the corresponding systems of generators of the algebra Λ of symmetric functions. This implies that the moments $h_n[\mu_q; q]$ and $p_n[\tau_q; q]$ determine each other uniquely. The same arguments as in the proof of Theorem 2.3 in Kerov [9, Section 2.5], can be applied to complete the proof. □

Theorem 4.5.2. *Let q be a fixed parameter which is taken from the interval* $(0; 1]$. *Then the relation $R_{\mu_q}(x; q) = R_{w(.;q)}(x; q)$ defines the one-to-one correspondence between continual diagrams from $\mathcal{D}[a, b]$, and the probability measures from $\mathcal{M}[a, b]$.*

Proof. The statement of the theorem in the case $q = 1$ is proved in Kerov [9,11]. For $q \in (0, 1)$ the statement of the theorem follows immediately from Theorem 4.5.1, and from the known fact that a diagram can be uniquely recovered from its Rayleigh measure (see, for example, Kerov [11]). □

5. Continual tableaux

5.1. Definition of continual tableaux

Definition 5.1.1. The region $\mathcal{D}_w[a, b] = \{(s, v)\colon |s| \leqslant v < w(s)\}$ is called the subgraph of a continual diagram w, $w \in \mathcal{D}[a, b]$.

Definition 5.1.2. Let $w_1, w_2 \in \mathcal{D}[a, b]$. We say that $w_1 \prec w_2$ if the subgraph of w_1 is a subset of the subgraph of w_2, i.e. $\mathcal{D}_{w_1}[a, b] \subset \mathcal{D}_{w_2}[a, b]$.

Definition 5.1.3. Let t be a parameter which takes values in some interval $[t_0, \infty)$. A continual tableau is a family of continual diagrams from $\mathcal{D}[a, b]$, $w(., t)$, which increases in t (with respect to the ordering introduced in Definition 5.1.2).

The function $\sigma(s, t) = \frac{1}{2}(w(s, t) - |s|)$ will be referred to as the charge of a tableau $w(., t)$.

5.2. q-Moments of continual tableaux

Proposition 5.2.1. *Given a real number q from the open interval $(0,1)$ assume that a tableau $w(.,t;q)$ and a family $\mu_{t,q}$ of probability measures from $\mathcal{M}[a,b]$ are related to each other by the formula*

$$\int\limits_a^b \frac{\mu_{t,q}(ds)}{1-q^{x-s}} = \frac{1}{1-q^x}\exp\left[-\ln q^{-1}\int\limits_a^b \frac{\partial\sigma(s,t;q)}{\partial s}\left(1-q^{x-s}\right)^{-1} ds\right] \tag{46}$$

for all x outside the interval $[a,b]$. Then two sequences

$$\left\{1-n\ln q^{-1}\int\limits_a^b q^{-ns}\frac{\partial\sigma(s,t;q)}{\partial s}\right\}_{n=1,2,\ldots},$$

and

$$\left\{\int\limits_a^b q^{-ns}\mu_{t,q}(ds)\right\}_{n=1,2,\ldots},$$

are related to each other in the same way as the systems of generators of the algebra Λ of the symmetric functions, $\{\boldsymbol{p}_n\}_{n=1,2,\ldots}$ and $\{\boldsymbol{h}_n\}_{n=1,2,\ldots}$. In other words, relation (46) is equivalent to

$$1+\sum_{n=1}^{\infty} h_n[\mu_{t,q};q]q^{nx} = \exp\left[\sum_{n=1}^{\infty} p_n\left[w(.,t;q);q\right]q^{nx}\right], \tag{47}$$

where the functions $\{h_n[\mu_{t,q};q]\}_{n=1}^{\infty}$ are defined by

$$h_n[\mu_{t,q};q] = \int\limits_a^b q^{-ns}\mu_{t,q}(ds), \tag{48}$$

and the functions $\{p_n[w(.,t;q);q]\}_{n=1}^{\infty}$ are defined by

$$p_n\left[w(.,t;q);q\right] = -n\ln q^{-1}\int\limits_a^b q^{-ns}\frac{\partial\sigma(s,t;q)}{\partial s}\,ds+1. \tag{49}$$

Proof. The proof of this proposition is a step by step repetition of the proof of Proposition 4.4.1. □

Remark 5.2.2. The equivalent form of Eq. (46) is

$$R_{\mu_{t,q}}(x;q) = R_{w(.,t;q)}(x;q),$$

where the functions $R_{\mu_{t,q}}(x;q)$ and $R_{w(.,t;q)}(x;q)$ are defined by

$$R_{\mu_{t,q}}(x;q) = (1-q)\int_a^b \frac{\mu_{t,q}(ds)}{1-q^{x-s}}, \tag{50}$$

$$R_{w(.,t;q)}(x;q) = \frac{1-q}{1-q^x}\exp\left[-\ln q^{-1}\int_a^b \frac{\partial\sigma(s,t;q)}{\partial s}\left(1-q^{x-s}\right)^{-1} ds\right] \tag{51}$$

for all x outside the interval $[a,b]$, and for all q taking values from the open interval (0, 1).

5.3. Dynamic equations

Theorem 5.3.1. *The following dynamic equations are equivalent*

$$\begin{aligned}&\int_a^b \left(1-q^{x-s}\right)^{-1}\frac{\partial\sigma(s,t;q)}{\partial t}\,ds\\ &\quad = \left(1-q^x\right)^{-1}\exp\left[-\ln q^{-1}\int_a^b \left(1-q^{x-s}\right)^{-1}\frac{\partial\sigma(s,t;q)}{\partial s}\,ds\right];\end{aligned} \tag{52}$$

$$\frac{\partial}{\partial t}p_n\left[w(.,t;q);q\right] = n^2\ln^2 q^{-1}\sum_{|\lambda|=n}\prod_{k=1}^{m(\lambda)}\frac{p_k^{r_k}[w(.,t;q);q]}{k^{r_k}r_k!}, \tag{53}$$

where $n=1,2,\ldots,$ $\lambda=(1^{r_1},2^{r_2},\ldots,m^{r_m})$, *and* $m=m(\lambda)$;

$$\frac{\partial R_{w(.,t;q)}(x;q)}{\partial x} + \frac{1-q}{\ln q^{-1}}R^{-1}_{w(.,t;q)}(x;q)\frac{\partial R_{w(.,t;q)}(x;q)}{\partial t} = 0. \tag{54}$$

Proof. Let us show that the first equation in the statement of the theorem, Eq. (52), implies Eq. (53). Let $w(s,t;q)$ be a tableau satisfying (52), and let $\sigma(s,t;q)$ be the charge of $w(.,t;q)$. Set

$$\mu_{t,q}(ds) = \frac{\partial\sigma(s,t;q)}{\partial t}\,ds. \tag{55}$$

It is not hard to see that $\mu_{t,q}$ defined by Eq. (55) is a family of probability measures from $\mathcal{M}[a,b]$. If the charge $\sigma(s,t;q)$ of $w(s,t;q)$ satisfies Eq. (52), then for every admissible t the measure $\mu_{t,q}$ is the q-transition measure of the diagram $w(s,t;q)$, see Definition 4.2.1. The moments $h_n[\mu_{t,q};q]$ of $\mu_{t,q}$ can be expressed as

$$
\begin{aligned}
h_n[\mu_{t,q};q] &= \int_a^b q^{-ns}\frac{\partial\sigma(s,t;q)}{\partial t}\,ds \\
&= \frac{\partial}{\partial t}\left(\int_a^b q^{-ns}\sigma(s,t;q)\,ds\right) \\
&= \frac{1}{n^2\ln^2 q^{-1}}\frac{\partial}{\partial t}p_n\big[w(.,t;q);q\big],
\end{aligned}
\tag{56}
$$

where we have used the integration by parts to get the last equation in (56). If $\mu_{t,q}$ is defined by Eq. (55), then the first equation in the statement of the theorem coincides with Eq. (46), and we can apply Proposition 5.2.1. Namely, Proposition 5.2.1 says that the moments $\{h_n[\mu_{q,t};q]\}_{n=1,2,\dots}$ and $\{p_n[w(.,t;q);q]\}_{n=1,2,\dots}$ are related to each other in the same way as the systems of the generators of the algebra Λ of the symmetric functions, $\{\boldsymbol{p}_n\}_{n=1,2,\dots}$ and $\{\boldsymbol{h}_n\}_{n=1,2,\dots}$. Therefore the following relation holds

$$
h_n[\mu_{t,q};q] = \sum_{|\lambda|=n}\prod_{k=1}^{m(\lambda)}\frac{p_k^{r_k}[w(.,t;q);q]}{k^{r_k}r_k!},
\tag{57}
$$

where $n = 1,2,\dots,$ $\lambda = (1^{r_1},2^{r_2},\dots,m^{r_m})$, and $m = m(\lambda)$, see Macdonald [17, I, §2]. From (56) and (57) we obtain Eq. (53).

Let us show that the second equation in the statement of the theorem implies Eq. (54). To this end define

$$
S(x,t;q) = \ln\left[\frac{R_{w(.,t;q)}(x;q)}{1-q}\right],
$$

where $R_{w(.,t;q)}(x;q)$ is given explicitly by Eq. (51). $S(x,t;q)$ can also be represented as

$$
S(x,t;q) = \sum_{n=1}^{\infty}\frac{p_n[w(.,t;q);q]q^{nx}}{n}.
$$

Differentiation of $S(x,t;q)$ with respect to the variable t gives

$$
\frac{\partial S(x,t;q)}{\partial t} = R^{-1}_{w(.,t;q)}(x;q)\frac{\partial R_{w(.,t;q)}(x;q)}{\partial t} = \sum_{n=1}^{\infty}\frac{q^{nx}}{n}\frac{\partial}{\partial t}p_n\big[w(.,t;q);q\big].
\tag{58}
$$

Observe that the first equation in the statement of the theorem is equivalent to $R_{\mu_{t,q}}(x;q) = R_{w_{.,t;q}}(x;q)$ where $\mu_{t,q}$ is defined by Eq. (55). Note also that the function $R_{\mu_{t,q}}(x;q)$ can be expanded in terms of the moments $h_n[\mu_{t,q};q]$ as follows

$$
R_{\mu_{t,q}}(x;q) = (1-q)\sum_{n=0}^{\infty}q^{nx}h_n[\mu_{t,q};q].
$$

Therefore the derivative of $R_{w(.,t;q)}(x;q)$ with respect to the variable x can be written as

$$\begin{aligned}\frac{\partial R_{w(.,t;q)}(x;q)}{\partial x} &= (1-q)\sum_{n=1}^{\infty} h_n[\mu_{t,q};q](n\ln q)q^{nx}\\ &= -\frac{(1-q)}{\ln q^{-1}}\sum_{n=1}^{\infty}\frac{\partial}{\partial t}\big(p_n\big[w(.,t;q);q\big]\big)\frac{q^{nx}}{n},\end{aligned} \tag{59}$$

where we have used (56). The comparison of (58) and (59) gives the third equation in the statement of the theorem, Eq. (54). □

6. q-Auto-model solutions

6.1. Definition of q-auto-models

Definition 6.1.1. Let $w(s,q)$ be a continuous function of two variables, s and q, where s takes values in the interval $[a,b]$, and q takes values in the open interval $(0,1)$. Assume further that for a given q the function $w(s;q)$ (considered as a function of the variable s) is an element of $\mathcal{D}[a,b]$, and that the subgraph $\mathcal{D}_w[a,b]$ of $w(s;q)$ is of unit area. A continual tableau $w(s,t;q)$ defined in terms of $w(s;q)$ by equation

$$w(s,t;q) = \sqrt{t}\,w\left(\frac{s}{\sqrt{t}};q^{\sqrt{t}}\right), \quad t>0, \tag{60}$$

is called a *q-auto-model.*

6.2. A definition of the q-deformation of the limiting diagram

Definition 6.2.1. Let $R_{\Omega(.;q)}(x;q)$ be the q-deformation of the R-function of a continual diagram $\Omega(.;q)$, see Definition 4.2.2. If $R_{\Omega(.;q)}$ satisfies the equation

$$R_{\Omega(.;q)}\big(1-q^{x-\frac{\ln q^{-1}}{1-q}R_{\Omega(.;q)}}\big) = 1-q, \tag{61}$$

then $\Omega(.;q)$ is referred to as the q-deformation of the limiting diagram $\Omega(s)$ defined by Eq. (1).

Remark 6.2.2. If q in Eq. (61) approaches 1, then (61) is reduced to equation $R_\Omega(x-R_\Omega)=1$. The solution of this equation vanishing at $x\to+\infty$ is the R-function of the diagram $\Omega(s)$ defined by Eq. (1).

6.3. The q-auto-model solution as the q-deformation of the limiting diagram

Theorem 6.3.1. *Let $w(s;q)$ be an arbitrary diagram of unit area, and $w(s,t;q)=\sqrt{t}\,w(\frac{s}{\sqrt{t}};q^{\sqrt{t}})$ be the corresponding q-auto-model. If the charge $\sigma(s,t;q)$ of $w(s,t;q)$ satisfies Eq.* (52)*, then $w(s;q)=\Omega(s;q)$.*

Proof. It is easy to check that the moments $p_1[w(.,t;q);q]$, $p_2[w(.,t;q);q],\dots$ of the q-auto-model $w(s,t;q)$ coincide with the moments $p_1[w(.;q^{\sqrt{t}});q^{\sqrt{t}}]$, $p_2[w(.;q^{\sqrt{t}});q^{\sqrt{t}}],\dots$ of the diagram $w(\frac{s}{\sqrt{t}},q^{\sqrt{t}})$:

$$p_n\big[w(.,t;q);q\big]=p_n\big[w\big(.;q^{\sqrt{t}}\big);q^{\sqrt{t}}\big].$$

Indeed, we have

$$\begin{aligned}
p_n\big[w(.,t;q);q\big]&=-n\ln q^{-1}\int_a^b q^{-ns}\frac{\partial}{\partial s}\sigma(s,t;q)\,ds+1\\
&=-n\ln q^{-1}\int_a^b q^{-ns}\frac{\partial}{\partial s}\left[\frac{1}{2}\big(w(s,t;q)-|s|\big)\right]ds+1\\
&=-n\ln q^{-1}\int_a^b q^{-ns}\frac{\partial}{\partial s}\left[\frac{1}{2}\left(\sqrt{t}\cdot w\left(\frac{s}{\sqrt{t}};q^{\sqrt{t}}\right)-\sqrt{t}\cdot\left|\frac{s}{\sqrt{t}}\right|\right)\right]ds+1\\
&=-n\ln\big[q^{\sqrt{t}}\big]^{-1}\int_{a/\sqrt{t}}^{b/\sqrt{t}}\big[q^{\sqrt{t}}\big]^{-nu}\frac{\partial}{\partial u}\left[\frac{1}{2}\big(w\big(u;q^{\sqrt{t}}\big)-|u|\big)\right]du+1\\
&=p_n\big[w\big(.;q^{\sqrt{t}}\big);q^{\sqrt{t}}\big].
\end{aligned}$$

This enables us to express the function $R_{w(.,t;q)}(x;q)$ which corresponds to the q-auto-model $w(.,t;q)$ in terms of the function $R_{w(.;q^{\sqrt{t}})}(x;q^{\sqrt{t}})$ which corresponds to the diagram $w(u,q^{\sqrt{t}})$:

$$\begin{aligned}
R_{w(.,t;q)}(x;q)&=(1-q)\exp\left[\sum_{n=1}^{\infty}\frac{p_n[w(.,t;q);q]q^{nx}}{n}\right]\\
&=(1-q)\exp\left[\sum_{n=1}^{\infty}\frac{p_n[w(.;q^{\sqrt{t}});q^{\sqrt{t}}]q^{n(\frac{x}{\sqrt{t}})\sqrt{t}}}{n}\right]\\
&=\frac{1-q}{1-q^{\sqrt{t}}}R_{w(.;q^{\sqrt{t}})}\left(\frac{x}{\sqrt{t}};q^{\sqrt{t}}\right).
\end{aligned}\tag{62}$$

Introduce new variables

$$u=\frac{x}{\sqrt{t}},\qquad Q=q^{\sqrt{t}}.$$

By Eq. (62) we have

$$R_{w(.,t;q)}(x;q)=\frac{1-q}{1-Q}R(u;Q),\quad \text{where } R(u;Q)=R_{w(.;q^{\sqrt{t}})}\left(\frac{x}{\sqrt{t}};q^{\sqrt{t}}\right).$$

The differential equation for the function $R_{w(.,t;q)}(x;q)$ (the third equation in Theorem 5.3.1) leads to the following partial differential equation for the function $R(u;Q)$:

$$\frac{\partial}{\partial u}\left[R^2(u;Q)\right]-\frac{1-Q}{\ln Q^{-1}}u\frac{\partial R(u;Q)}{\partial u}-QR(u;Q)-Q(1-Q)\frac{\partial R(u;Q)}{\partial Q}=0.$$

Set $R(u;Q)=\frac{1-Q}{\ln Q^{-1}}\mathrm{r}(u;Q)$, and introduce a real parameter ϱ, $\varrho>0$, by the relation $Q=\exp(-\varrho)$. Then $\mathrm{r}(u;\varrho)$ satisfies the following partial quasi-linear differential equation

$$2\mathrm{r}\frac{\partial}{\partial u}\mathrm{r}-u\frac{\partial}{\partial u}\mathrm{r}+\varrho\frac{\partial}{\partial\varrho}\mathrm{r}=\mathrm{r}. \tag{63}$$

Eq. (63) is a quasi-linear partial differential equation in two variables, and can be solved by the method of characteristics. Namely, for the partial differential equation (63) the characteristic equations are:

$$\frac{du}{ds}=2\mathrm{r}-u,\qquad \frac{d\varrho}{ds}=\varrho,\qquad \frac{d\mathrm{r}}{ds}=\mathrm{r}. \tag{64}$$

Eqs. (64) can be rearranged into two ordinary differential equations:

$$\frac{d\mathrm{r}}{\mathrm{r}}=\frac{d\varrho}{\varrho}, \tag{65}$$

$$\frac{du}{2\mathrm{r}-u}=\frac{d\varrho}{\varrho}. \tag{66}$$

The integration of the first equation above gives $\mathrm{r}=c_1\varrho$. Inserting this into (66) we obtain:

$$\frac{du}{2c_1\varrho-u}=\frac{d\varrho}{\varrho},\quad\text{or}\quad\frac{du}{d\varrho}=2c_1-\frac{u}{\varrho}.$$

Integrating the last equation we find

$$u=c_1\varrho+\frac{c_2}{\varrho}.$$

Our first integrals, therefore, are $c_1=f(u,\varrho,\mathrm{r})=\frac{\mathrm{r}}{\varrho}$ and $c_2=g(u,\varrho,\mathrm{r})=\varrho(u-\mathrm{r})$. The general solution is found by setting $f=F(g)$, which leads to the relation

$$\frac{\mathrm{r}}{\varrho}=F\bigl(\varrho(u-\mathrm{r})\bigr), \tag{67}$$

where F is an arbitrary function. Observe that

$$\mathrm{r}(u;\varrho)=\frac{\varrho}{1-\exp(-\varrho)}R_{w(.;e^{-\varrho})}\bigl(u;e^{-\varrho}\bigr)=\frac{\varrho}{1-\exp(-\varrho)}R_{\mu_{e^{-\varrho}}}\bigl(u;e^{-\varrho}\bigr),$$

where $\mu_{e^{-\varrho}}$ is the q-transition measure of the diagram $w(.; e^{-\varrho})$. It follows that

$$\mathrm{r}(u; \varrho) = \varrho \int_a^b \frac{\mu_{e^{-\varrho}}(ds)}{1 - e^{-\varrho(u-s)}},$$

and from this equation we conclude that $\mathrm{r}(u, \varrho)$ approaches to $\varrho(1 - e^{-\varrho u})^{-1}$ as $u \to \infty$. This enables us to determine the function F in (67) explicitly:

$$F(x) = \frac{1}{1 - \exp(-x)}.$$

Consequently, the function $\mathrm{r}(u, \varrho)$ satisfies the equation

$$\mathrm{r} = \varrho\bigl(1 - e^{-\varrho(u-\mathrm{r})}\bigr)^{-1}.$$

If we rewrite this equation in terms of $R_{w(.,e^{-\varrho})}(u; e^{-\varrho})$, and replace ϱ by $\ln q^{-1}$, and u by x, we obtain

$$R_{w(.;q)}\bigl(1 - q^{x - \frac{\ln q^{-1}}{1-q} R_{w(.;q)}}\bigr) = 1 - q.$$

Therefore the functions $R_{w(.;q)}(x; q)$ and $R_{\Omega(.;q)}(x; q)$ coincide for all admissible values of x and q. This implies $w(s; q) = \Omega(s; q)$. □

7. The asymptotics of the general solution

7.1. The large t asymptotics of the functions $p_n[w(., t; q)]$ and $h_n[w(., t; q)]$

Assume that the charge $\sigma(s, t; q)$ of a diagram $w(s, t; q)$ satisfies Eq. (52) of Theorem 5.3.1. Then the q-transition measure $\mu_{t,q}$ of the diagram $w(s, t; q)$ is

$$\mu_{t,q}(ds) = \frac{\partial \sigma(s, t; q)}{\partial t}\, ds.$$

Let $\{h_n[\mu_{t,q}; q]\}_{n=1}^{\infty}$ be the q-moments of $\mu_{t,q}$ (see Eq. (48)), and for every $n = 1, 2, \ldots$ set $h_n[w(., t; q); q] := h_n[\mu_{t,q}; q]$. Recall that the functions $p_n[w(., t; q); q]$ are defined by Eq. (49).

Lemma 7.1.1. *There exist functions $\{\check{p}_n[q]\}_{n=1}^{\infty}$ and $\{\check{h}_n[q]\}_{n=1}^{\infty}$, which are independent on t, such that*

$$p_n\bigl[w\bigl(., t; q^{\frac{1}{\sqrt{t}}}\bigr); q^{\frac{1}{\sqrt{t}}}\bigr] = \check{p}_n[q] + o\bigl(t^{-1/2}\bigr), \tag{68}$$

$$h_n\bigl[w\bigl(., t; q^{\frac{1}{\sqrt{t}}}\bigr); q^{\frac{1}{\sqrt{t}}}\bigr] = \check{h}_n[q] + o\bigl(t^{-1/2}\bigr), \tag{69}$$

as $t \to \infty$.

Proof. Comparing the right-hand sides of Eqs. (56) and (57) we obtain the following system of differential equations

$$\frac{\partial}{\partial t} p_n\big[w(.,t;q);q\big] = n^2 \ln^2 q^{-1} \left\{ \sum_{|\lambda|=n} \prod_{k=1}^{m(\lambda)} \frac{p_k^{r_k}[w(.,t;q);q]}{k^{r_k} r_k!} \right\}, \tag{70}$$

where $n = 1, 2, \ldots,$ $\lambda = (1^{r_1}, 2^{r_2}, \ldots, m^{r_m})$, and $m = m(\lambda)$. Setting

$$\varsigma := t \ln^2 q^{-1}, \quad \text{and} \quad y_n(\varsigma) := p_n\big[w(.,t;q);q\big],$$

we obtain differential equations for functions $\{y_n(\varsigma)\}_{n=1}^{\infty}$

$$\frac{dy_n(\varsigma)}{d\varsigma} = n^2 \left\{ \sum_{|\lambda|=n} \prod_{k=1}^{m(\lambda)} \frac{y_k^{r_k}(\varsigma)}{k^{r_k} r_k!} \right\}, \quad n = 1, 2, \ldots. \tag{71}$$

The first equations of the system above are

$$\begin{gathered}
\dot{y}_1 = y_1, \\
\dot{y}_2 = 2y_1^2 + 2y_2, \\
\dot{y}_3 = \frac{3}{2} y_1^3 + \frac{9}{2} y_2 y_1 + 3y_3, \\
\dot{y}_4 = \frac{2}{3} y_1^4 + 4y_2 y_1^2 + \frac{16}{3} y_3 y_1 + 2y_2^2 + 4y_4, \\
\dot{y}_5 = \frac{5}{24} y_1^5 + \frac{25}{12} y_2 y_1^3 + \frac{25}{6} y_3 y_1^2 + \frac{25}{8} y_1 y_2^2 + \frac{25}{4} y_4 y_1 + \frac{25}{6} y_2 y_3 + 5y_5, \\
\vdots
\end{gathered}$$

Successively solving these equations we find

$$\begin{gathered}
y_1(\varsigma) = y_1(0) e^{\varsigma}, \\
y_2(\varsigma) = \big[y_2(0) + 2y_1^2(0)\varsigma\big] e^{2\varsigma}, \\
y_3(\varsigma) = \left[y_3(0) + \frac{3}{2} y_1(0)\big[3y_2(0) + y_1^2(0)\big]\varsigma + \frac{9}{2} y_1^3(0)\varsigma^2 \right] e^{3\varsigma}, \\
y_4(\varsigma) = \left[y_4(0) + \left[\frac{2}{3} y_1^4(0) + 4y_1^2(0) y_2(0) + \frac{16}{3} y_1(0) y_3(0) + 2y_2^2(0) \right] \varsigma, \right. \\
\left. + \big[16 y_1^2(0) y_2(0) + 8 y_1^4(0)\big]\varsigma^2 + \frac{32}{3} y_1^4(0)\varsigma^3 \right] e^{4\varsigma}, \\
\vdots
\end{gathered}$$

Generally, $y_n(\varsigma)$ is a polynomial in ς of degree $n-1$ multiplied by $e^{n\varsigma}$, and the coefficients of this polynomial are homogeneous.

Returning to the functions $\{p_n[w(.,t;q);q]\}_{n=1,2,\ldots}$ we obtain

$$
\begin{aligned}
&p_1\big[w(.,t;q);q\big] = p_1\big[w(.,t=0;q);q\big]e^{t\ln^2 q^{-1}},\\
&p_2\big[w(.,t;q);q\big]\\
&\quad = \big[p_2\big[w(.,t=0;q);q\big] + 2p_1^2\big[w(.,t=0;q);q\big]\big(t\ln^2 q^{-1}\big)\big]e^{2t\ln^2 q^{-1}} p_3\big[w(.,t;q);q\big]\\
&\quad = \bigg[p_3\big[w(.,t=0;q);q\big] + \frac{3}{2}p_1\big[w(.,t=0;q);q\big]\big[3p_2\big[w(.,t=0;q);q\big]\\
&\qquad + p_1^2\big[w(.,t=0;q);q\big]\big]\big(t\ln^2 q^{-1}\big) + \frac{9}{2}p_1^3\big[w(.,t=0;q);q\big]\big(t\ln^2 q^{-1}\big)^2\bigg]e^{3t\ln^2 q^{-1}},\\
&p_4\big[w(.,t;q);q\big]\\
&\quad = [p_4\big[w(.,t=0;q);q\big] + \bigg[\frac{2}{3}p_1^4\big[w(.,t=0;q);q\big]\\
&\qquad + 4p_1^2\big[w(.,t=0;q);q\big]p_2\big[w(.,t=0;q);q\big]\\
&\qquad + \frac{16}{3}p_1\big[w(.,t=0;q);q\big]p_3\big[w(.,t=0;q);q\big] + 2p_2^2\big[w(.,t=0;q);q\big]\bigg]\big(t\ln^2 q^{-1}\big)\\
&\qquad + \big[16p_1^2\big[w(.,t=0;q);q\big]p_2\big[w(.,t=0;q);q\big] + 8p_1^4\big[w(.,t=0;q);q\big]\big]\big(t\ln^2 q^{-1}\big)^2\\
&\qquad + \frac{32}{3}p_1^4\big[w(.,t=0;q);q\big]\big(t\ln^2 q^{-1}\big)^3]e^{4t\ln^2 q^{-1}},\\
&\qquad \vdots
\end{aligned}
$$

Now it is clear that if

$$
p_n\big[w\big(.,t=0;q^{\frac{1}{\sqrt{t}}}\big);q^{\frac{1}{\sqrt{t}}}\big] = 1 + o\left(\frac{1}{\sqrt{t}}\right) \tag{72}
$$

as $t\to\infty$, then (68) holds. But (72) follows immediately from (49). Since the relation between functions $\{p_n[w(.,t;q)]\}_{n=1}^{\infty}$ and $\{h_n[w(.,t;q)]\}_{n=1}^{\infty}$ is homogeneous, Eq. (69) holds as well. □

Corollary 7.1.2. *Let $\{y_n(\varsigma)\}_{n=1,2,\ldots}$ be the solution of the system of differential equations given by Eq.* (71) *which satisfies the initial conditions $y_n(0)=1$, $n=1,2,\ldots$. Then the limiting values $\check{p}_n[q]$ in Eq.* (68) *are given by*

$$
\check{p}_n[q] = y_n\big[\ln^2 q\big], \quad n=1,2,\ldots,
$$

and the limiting values $\check{h}_n[q]$ in Eq. (69) *are given by*

$$
\check{h}_n[q] = \left\{\sum_{|\lambda|=n}\prod_{k=1}^{m(\lambda)}\frac{y_k^{r_k}[\ln^2 q]}{k^{r_k}r_k!}\right\},
$$

where $n=1,2,\ldots$, $\lambda=(1^{r_1},2^{r_2},\ldots,m^{r_m})$, and $m=m(\lambda)$.

Proof. In order to obtain the values of $\check{p}_n[q]$ we need to compute the limits $\lim_{t\to\infty} p_n[w(.,t;q^{\frac{1}{\sqrt{t}}});q^{\frac{1}{\sqrt{t}}}]$. Since $\lim_{t\to\infty} p_n[w(.,t=0;q^{\frac{1}{\sqrt{t}}});q^{\frac{1}{\sqrt{t}}}]=1$, $n=1,2,\ldots,$ it is not hard to conclude from the proof of the lemma above that $\lim_{t\to\infty} p_n[w(.,t;q^{\frac{1}{\sqrt{t}}});q^{\frac{1}{\sqrt{t}}}]=y_n(\ln^2 q)$, $n=1,2,\ldots,$ where $\{y_n(\varsigma)\}_{n=1,2,\ldots}$ is the solution of the system of differential equations (71) which satisfies the initial conditions $y_n(0)=1$, $n=1,2,\ldots$. $\square$

7.2. The common asymptotics of solutions

Theorem 7.2.1. *Assume that the charge $\sigma(s,t;q)$ of a tableau $w(s,t;q)$ satisfies* (52). *Then*

$$\lim_{t\to\infty}\frac{1}{\sqrt{t}}w\bigl(s\sqrt{t},t;q^{\frac{1}{\sqrt{t}}}\bigr)=\Omega(s;q)$$

uniformly in s and q.

Proof. Define the normalized tableau

$$W(s,t;q)=\frac{1}{\sqrt{t}}w\bigl(s\sqrt{t},t;q^{\frac{1}{\sqrt{t}}}\bigr),\quad t>0. \tag{73}$$

If the tableau $w(s,t;q)$ is a family of continual diagrams from $\mathcal{D}[a,b]$ then the normalized tableau $W(s,t;q)$ is the family of continual diagrams from $\mathcal{D}[a\sqrt{t},b\sqrt{t}]$. The functions $p_n[W(s,t;q);q]$ can be expressed as

$$p_n\bigl[W(.,t;q);q\bigr]=-n\ln q^{-1}\int\limits_{a\sqrt{t}}^{b\sqrt{t}} q^{-ns}\frac{\partial\Xi(s,t;q)}{\partial s}\,ds+1,$$

where $\Xi(s,t;q)$ is the charge of the normalized diagram $W(s,t;q)$. Let us express $\Xi(s,t;q)$ in terms of the charge $\sigma(s,t;q)$ of the initial diagram $w(s,t;q)$:

$$\begin{aligned}\Xi(s,t;q)&=\frac{1}{2}\bigl(W(s,t;q)-|s|\bigr)\\&=\frac{1}{2}\left(\frac{1}{\sqrt{t}}w\bigl(s\sqrt{t},t;q^{\frac{1}{\sqrt{t}}}\bigr)-\frac{1}{\sqrt{t}}|s\sqrt{t}|\right)\\&=\frac{1}{\sqrt{t}}\sigma\bigl(s\sqrt{t},t;q^{\frac{1}{\sqrt{t}}}\bigr).\end{aligned}$$

Inserting this into the integral for $p_n[W(.,t;q);q]$, and changing the variables of the integration we obtain

$$p_n\bigl[W(.,t;q);q\bigr]=p_n\bigl[w\bigl(.,t;q^{\frac{1}{\sqrt{t}}}\bigr);q^{\frac{1}{\sqrt{t}}}\bigr]. \tag{74}$$

By Lemma 7.1.1 this implies the large t asymptotic relation

$$p_n\bigl[W(.,t;q);q\bigr]=\check{p}_n+o\bigl(t^{-1/2}\bigr), \tag{75}$$

where $\check{p}_n$ are independent on t. Let the functions $h_n[W(.,t;q);q]$ be defined in terms of the functions $p_n[W(.,t;q);q]$ by the formula

$$h_n\big[W(.,t;q);q\big]=\sum_{|\lambda|=n}\prod_{k=1}^{m(\lambda)}\frac{p_k^{r_k}[W(.,t;q);q]}{k^{r_k}r_k!},\tag{76}$$

where $n=1,2,\ldots$, $\lambda=(1^{r_1},2^{r_2},\ldots,m^{r_m})$, and $m=m(\lambda)$. From Eq. (76) we obtain the large t asymptotic relation for the functions $h_n[W(.,t;q);q]$

$$h_n\big[W(.,t;q);q\big]=\check{h}_n+o\big(t^{-1/2}\big),\tag{77}$$

where $\check{h}_n$ are independent on t. The sequences $\{p_n[W(.,t;q);q]\}_{n=1,2,\ldots}$ and $\{h_n[W(.,t;q);q]\}_{n=1,2,\ldots}$ are related with each other as the sequences of the corresponding generators of the algebra Λ of symmetric functions, $\{\boldsymbol{p}_n\}_{n=1,2,\ldots}$ and $\{\boldsymbol{h}_n\}_{n=1,2,\ldots}$. Therefore the function $R_{W(.,t;q)}(x;q)$ can be represented in two ways:

$$R_{W(.,t;q)}(x;q)=(1-q)\exp\left[\sum_{n=1}^{\infty}\frac{p_n[W(.,t;q);q]q^{nx}}{n}\right],$$

and

$$R_{W(.,t;q)}(x;q)=(1-q)\left(1+\sum_{n=1}^{\infty}h_n\big[W(.,t;q);q\big]q^{nx}\right).$$

From the equation just written above, and from asymptotic relation (77) we conclude that

$$\lim_{t\to\infty}R_{W(.,t;q)}(x;q)=\check{R}(x;q),\quad\text{and}\quad\lim_{t\to\infty}\left[t\frac{\partial R_{W(.,t;q)}(x;q)}{\partial t}\right]=0,$$

where $\check{R}(x;q)$ is defined in terms of $\check{h}_n$ by

$$\check{R}(x;q)=(1-q)\left(1+\sum_{n=1}^{\infty}\check{h}_nq^{nx}\right).\tag{78}$$

Eq. (74) also implies the relation

$$R_{w(.,t;q)}(x;q)=\frac{1-q}{1-q^{\sqrt{t}}}R_{W(.,t;q^{\sqrt{t}})}\left(\frac{x}{\sqrt{t}};q^{\sqrt{t}}\right).\tag{79}$$

Indeed,

$$\begin{aligned}
R_{W(.,t;q)}(x;q) &= (1-q)\exp\left[\sum_{n=1}^{\infty}\frac{p_n[W(.,t;q);q]q^{nx}}{n}\right]\\
&= (1-q)\exp\left[\sum_{n=1}^{\infty}\frac{p_n[w(.,t;q^{\frac{1}{\sqrt{t}}});q^{\frac{1}{\sqrt{t}}}](q^{\frac{1}{\sqrt{t}}})^{n\sqrt{t}x}}{n}\right]\\
&= \frac{1-q}{1-q^{\sqrt{t}}}\bigl[R_{w(.,t;q^{\frac{1}{\sqrt{t}}})}\bigl(\sqrt{t}x;q^{\frac{1}{\sqrt{t}}}\bigr)\bigr],
\end{aligned}$$

which is clearly equivalent to Eq. (79). The third equation in Theorem 5.3.1 (which is a partial differential equation for $R_{w(.,t;q)}(x;q)$) and a change of variables result in the partial differential equation

$$\begin{aligned}
&\frac{\partial}{\partial u}\bigl[R^2(u,t;Q)\bigr]-\frac{1-Q}{\ln Q^{-1}}u\frac{\partial R(u,t;Q)}{\partial u}-QR(u,t;Q)-Q(1-Q)\frac{\partial R(u,t;Q)}{\partial Q}\\
&\quad=2t\frac{1-Q}{\ln Q^{-1}}\frac{\partial R(u,t;Q)}{\partial t},
\end{aligned}\tag{80}$$

where $R(u,t;Q):=R_{W(.,t;Q)}(u;Q)$. Let us write the function $R(u,t;Q)$ in the form

$$R(u,t;Q)=\check{R}(u;Q)+\overline{R(u,t;Q)},\tag{81}$$

where $\check{R}(u;Q)$ is defined in terms of $\check{h}_n$ by (78). Then

$$\lim_{t\to\infty}\overline{R(u,t;Q)}=0,\quad\text{and}\quad\lim_{t\to\infty}\left[t\frac{\partial\overline{R(u,t;Q)}}{\partial t}\right]=0.$$

Substituting (81) into partial differential equation (80) and taking the large t limit we obtain

$$\frac{\partial}{\partial u}\bigl[\check{R}^2(u;Q)\bigr]-\frac{1-Q}{\ln Q^{-1}}u\frac{\partial\check{R}(u;Q)}{\partial u}-Q\check{R}(u;Q)-Q(1-Q)\frac{\partial\check{R}(u;Q)}{\partial Q}=0.$$

The partial differential equation just written above is precisely that which have appeared previously in the proof of Theorem 6.3.1. The proof of Theorem 6.3.1 shows that the solution $\check{R}(u;Q)$ must satisfy the same equation as the function $R_{\Omega(.;q)}(x;q)$ (Eq. (61)). Therefore $\check{R}(u;Q)$ coincides with $R_{\Omega(.;q)}(x;q)$, and the limiting moments $\check{p}_n$, $\check{h}_n$ coincide with the corresponding moments of the diagram $\Omega(s;q)$. Thus the normalized diagram $W(s,t;q)$ converges uniformly to $\Omega(s;q)$ as $t\to\infty$. □

8. Growth of rectangular diagrams

The aim of this section is to relate the growth of the diagrams in the q-analog of the Plancherel process, and Eq. (54) more directly.

8.1. The definition of the growth

Let $\{x_k\}_{k=1}^{m+1}$ and $\{y_k\}_{k=1}^{m}$ be the points of minima and maxima of a rectangular diagram w correspondingly, see Fig. 1. Consider a one-parameter deformation w_t of w by attaching a tiny square of area $\mu_k(w;q)t$ above each minimum x_k. Such a deformation is referred to as *the growth of a rectangular diagram.* The interlacing sequences associated with the deformed diagram w_t are

$$x_t = \{x_1, y_1, \dots, x_m, y_m, x_{m+1}\},$$

and

$$y_t = \{x_1 - \sqrt{\mu_1 t}, x_1 + \sqrt{\mu_1 t}, x_2 - \sqrt{\mu_2 t}, x_2 + \sqrt{\mu_2 t}, \dots, x_{m+1} - \sqrt{\mu_{m+1} t}, x_{m+1} + \sqrt{\mu_{m+1} t}\}.$$

Thus x_t and y_t defined above are the sequences of the minima and of the maxima of the deformed diagram w_t.

8.2. The differential equation for the infinitesimal growth

Proposition 8.2.1. *If a rectangular diagram w_t grows according to the transition probabilities $\mu_k(w;q)$ defined by Eq.* (38)*, then infinitesimally (for small t) the function $R_{w_t}(x;q)$ evolves according to differential equation* (54).

Proof. Let μ_q be the q-transition measure of w. The q-deformation of the R-function of the diagram w, $R_w(x;q)$, and the q-deformation of the R-function of the q-transition measure μ_q of w, $R_{\mu_q}(x;q)$, are given by

$$R_w(x;q) = (1-q)\frac{\prod_{i=1}^{m}(1-q^{x-y_i})}{\prod_{i=1}^{m+1}(1-q^{x-x_i})}, \quad \text{and} \quad R_{\mu_q}(x;q) = (1-q)\sum_{k=1}^{m+1}\frac{\mu_k(w;q)}{1-q^{x-x_k}}.$$

We have

$$R_w(x;q) = R_{\mu_q}(x;q).$$

The q-deformation of the R-function of the deformed diagram w_t, $R_{w_t}(x;q)$, and the q-deformation of the R-function of the q-transition measure $\mu_{q,t}$ of w_t, $R_{\mu_{q,t}}(x;q)$, are given by

$$R_{\mu_{q;t}}(x;q) = (1-q)\sum_{j=1}^{2(m+1)}\frac{\nu_j(w_t;q)}{1-q^{x-x_t^{(j)}}}, \tag{82}$$

$$R_{w_t}(x;q) = (1-q)\frac{\prod_{i=1}^{m+1}(1-q^{x-x_i})\prod_{i=1}^{m}(1-q^{x-y_i})}{\prod_{i=1}^{m+1}(1-q^{x-x_i-\sqrt{\mu_i t}})(1-q^{x-x_i+\sqrt{\mu_i t}})}, \tag{83}$$

where $x_t^{(j)}$, $j = 1, \dots, 2(m+1)$, are the elements of the set x_t. The deformation preserves the equality between the q-deformation of the R-function of the diagram, and the q-deformation of the R-function of the corresponding q-transition measure, i.e.

$$R_{\mu_{q;t}}(x;q) = R_{w_t}(x;q).$$

Since $R_{\mu_{q;t}}(x;q)$ at $t = 0$ must coincide with $R_{\mu_q}(x;q)$ the following relations between transition probabilities must be true

$$\nu_1 + \nu_2 = \mu_1, \quad \nu_3 + \nu_4 = \mu_2, \quad \dots, \quad \nu_{2m+1} + \nu_{2m+2} = \mu_{m+1}. \tag{84}$$

The functions $R_{w_t}(x;q)$ and $R_w(x;q)$ are related to each other by the expression

$$R_{w_t}(x;q) = R_w(x;q) \prod_{k=1}^{m+1} \left(\frac{1 - q^{x - x_k - \sqrt{\mu_k t}}}{1 - q^{x - x_k}} \right)^{-1} \left(\frac{1 - q^{x - x_k + \sqrt{\mu_k t}}}{1 - q^{x - x_k}} \right)^{-1},$$

which immediately follows from (82) and (83). Now we have

$$\begin{aligned}
\left(\frac{1 - q^{x - x_k - \sqrt{\mu_k t}}}{1 - q^{x - x_k}} \right) \cdot \left(\frac{1 - q^{x - x_k + \sqrt{\mu_k t}}}{1 - q^{x - x_k}} \right) &= \frac{1 + q^{2(x - x_k)} - q^{(x - x_k)} (e^{\sqrt{\mu_k t} \ln q^{-1}} + e^{-\sqrt{\mu_k t} \ln q^{-1}})}{(1 - q^{x - x_k})^2} \\
&= \frac{1 + q^{2(x - x_k)} - 2q^{(x - x_k)} (1 + \frac{\mu_k t}{2} \ln^2 q^{-1} + o(t))}{(1 - q^{x - x_k})^2} \\
&= 1 - \frac{q^{(x - x_k)} \mu_k \ln^2 q^{-1}}{(1 - q^{x - x_k})^2} t + o(t).
\end{aligned}$$

Using this we can rewrite the relation between the functions $R_{w_t}(x;q)$ and $R_w(x;q)$ as follows

$$R_{w_t}(x;q) = R_w(x;q) \left[1 + \sum_{k=1}^{m+1} \frac{q^{x - x_k} \mu_k \ln^2 q^{-1}}{(1 - q^{x - x_k})^2} t + o(t) \right],$$

which clearly implies

$$\left. \frac{\partial R_{w_t}(x;q)}{\partial t} \right|_{t=0} = \sum_{k=1}^{m+1} \frac{q^{x - x_k} \mu_k \ln^2 q^{-1}}{(1 - q^{x - x_k})^2} R_w(x;q). \tag{85}$$

Differentiate the function $R_{w_t}(x;q)$ with respect to x and obtain

$$\begin{aligned}
\frac{\partial R_{w_t}(x;q)}{\partial x} &= \frac{\partial R_{\mu_{q,t}}(x;q)}{\partial x} \\
&= (1 - q) \sum_{k=1}^{2(m+1)} \nu_k \frac{\partial}{\partial x} \frac{1}{1 - q^{x - x_t^{(k)}}}
\end{aligned}$$

$$
\begin{aligned}
&= (1-q)\sum_{k=1}^{m+1} \nu_{2k-1}\frac{\partial}{\partial x}\frac{1}{1-q^{x-x_k-\sqrt{\mu_k t}}} \\
&\quad + (1-q)\sum_{k=1}^{m+1} \nu_{2k}\frac{\partial}{\partial x}\frac{1}{1-q^{x-x_k+\sqrt{\mu_k t}}} \\
&= -(1-q)\ln q^{-1}\sum_{k=1}^{m+1} \nu_{2k-1}\frac{q^{x-x_k-\sqrt{\mu_k t}}}{(1-q^{x-x_k-\sqrt{\mu_k t}})^2} \\
&\quad - (1-q)\ln q^{-1}\sum_{k=1}^{m+1} \nu_{2k-1}\frac{q^{x-x_k-\sqrt{\mu_k t}}}{(1-q^{x-x_k-\sqrt{\mu_k t}})^2}.
\end{aligned}
$$

In particular, from the expression just written above it follows that

$$
\begin{aligned}
\frac{\partial R_{w_t}(x;q)}{\partial x}\bigg|_{t=0} &= -(1-q)\ln q^{-1}\sum_{k=1}^{m+1}(\nu_{2k-1}+\nu_{2k})\frac{q^{x-x_k}}{(1-q^{x-x_k})^2} \\
&= -(1-q)\ln q^{-1}\sum_{k=1}^{m+1}\mu_k\frac{q^{x-x_k}}{(1-q^{x-x_k})^2},
\end{aligned}
\tag{86}
$$

where in the last equation we have used relation (84) between transition probabilities. We compare the right-hand sides of Eqs. (86) and (85), and obtain differential equation (54). $\square$

References

[1] P. Biane, Approximate factorization and concentration for characters of symmetric groups, Int. Math. Res. Not. 4 (2001) 179–192.

[2] A. Borodin, G. Olshanski, Harmonic functions on multiplicative graphs and interpolation polynomials, Electron. J. Combin. 7 (2000), paper R28.

[3] R. Carter, Finite Groups of Lie Type—Conjugacy Classes and Complex Characters, Wiley–Interscience, New York, 1985.

[4] C. Curtis, I. Reiner, Methods of Representation Theory—With Applications to Finite Groups and Orders, Volumes I and II, Wiley–Interscience, New York, 1981, and 1987.

[5] P. Diaconis, A. Ram, Analysis of systematic scan Metropolis algorithms using Iwahori–Hecke algebra techniques, Michigan Math. J. 48 (2000) 157–190.

[6] J. Fulman, $GL(n,q)$ and increasing subsequences in nonuniform random permutations, Ann. Combin. 6 (2002) 19–32.

[7] W. Fulton, J. Harris, Representation Theory: A First Course, Grad. Texts in Math., vol. 129, Springer, 1991.

[8] V. Ivanov, G. Olshanski, Kerov's central limit theorem for the Plancherel measure on Young diagrams, in: Symmetric Functions 2001: Surveys of Developments and Perspectives, in: NATO Sci. Ser. II Math. Phys. Chem., vol. 74, Kluwer Acad. Publ., Dordrecht, 2002, pp. 93–151.

[9] S.V. Kerov, Transition probabilities of continual Young diagrams and Markov moment problem, Funct. Anal. Appl. 27 (1993) 104–117.

[10] S.V. Kerov, Asymptotic separation of roots of orthogonal polynomials, St. Petersburg Math. J. 5 (1994) 925–941.

[11] S.V. Kerov, Interlacing measures, in: G. Olshanski (Ed.), Kirillov's Seminar on Representation Theory, Amer. Math. Soc., Providence, RI, 1998, pp. 35–83.

[12] S.V. Kerov, A differential model for the growth of Young diagrams, Amer. Math. Soc. Transl. Ser. 2 188 (1999) 111–130.

[13] S.V. Kerov, Asymptotic Representation Theory of the Symmetric Group and Its Applications in Analysis, Transl. Math. Monogr., vol. 219, American Mathematical Society, Providence, RI, 2003.
[14] S.V. Kerov, A.M. Vershik, The Grothendieck group of the infinite symmetric group and symmetric functions with the elements of the K_0-functor theory of AF-algebras, in: A.M. Vershik, D.P. Zhelobenko (Eds.), Representation of Lie Groups and Related Topics, in: Adv. Stud. Contemp. Math., vol. 7, Gordon and Beach, 1990, pp. 36–114.
[15] S.V. Kerov, N.V. Tsilevich, The Markov–Krein correspondence in several dimensions, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 283 (2001), Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 6, 98–122, 259–260; translation in J. Math. Sci. (N.Y.) 121 (3) (2004) 2345–2359.
[16] B.F. Logan, L.A. Shepp, A variational problem for random Young tableaux, Adv. Math. 26 (1977) 206–222.
[17] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second ed., Oxford Science Publications, 1995.
[18] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, San Diego, 1991.
[19] A.M. Vershik, Two lectures on the asymptotic representation theory and statistics of Young diagrams, in: Asymptotic Combinatorics with Applications to Mathematical Physics, St. Petersburg, 2001, in: Lecture Notes in Math., vol. 1815, Springer, Berlin, 2003, pp. 161–182.
[20] A.M. Vershik, S.V. Kerov, Asymptotic behavior of the Plancherel measure of the symmetric group and the limit form of Young tableaux, Dokl. Akad. Nauk SSSR 233 (6) (1977) 1024–1027 (in Russian).
[21] A.M. Vershik, S.V. Kerov, Asymptotic theory of characters of the symmetric group, Funct. Anal. Appl. 15 (1981) 246–255.
[22] A.M. Vershik, S.V. Kerov, Characters and realizations of representations of the infinite-dimensional Hecke algebra, and knot invariants, Dokl. Akad. Nauk SSSR 301 (1988) 777–780, English transl.: Soviet Math. Dokl. 38 (1989) 134–137.
[23] A.M. Vershik, M. Yor, N.V. Tsilevich, The Markov–Krein identity and the quasi-invariance of the gamma process, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 283 (2001) (in Russian), Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 6, 21–36, 258; translation in J. Math. Sci. (N.Y.) 121 (3) (2004) 2303–2310.