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Note

# A note about the dominating circuit conjecture

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**Abstract**

The dominating circuit conjecture states that every cyclically 4-edge-connected cubic graph has a dominating circuit. We show that this is equivalent to the statement that any two edges of such a cyclically 4-edge-connected graph are contained in a dominating circuit.


*Keywords:* Dominating circuit; Dominating circuit conjecture

We deal with graphs without multiple edges and loops. A subgraph $H$ of a graph $G$ is called *dominating* if each edge of $G$ is incident with a vertex from $H$. A graph is *cyclically $k$-edge-connected* if deleting fewer than $k$ edges does not result in a graph having at least two components containing cycles.

Fleischner [[2]] conjectured that every cyclically 4-edge-connected cubic graph has either a dominating circuit or a 3-edge-coloring. As pointed out by Jaeger [[3]], this conjecture, if true, would be an interesting approach to the cycle double cover conjecture (every bridgeless graph has a family of circuits which together cover each edge twice). Ash and Jackson [[1]] conjectured that every cyclically 4-edge-connected cubic graph has a dominating circuit. Kochol [[4]] proved that the conjectures of Fleischner and Ash and Jackson are equivalent. Now we show that the conjecture of Ash and Jackson holds true if and only if any two edges of a cyclically 4-edge-connected cubic graph are contained in a dominating circuit.

Following the notation of [[5]], we call a pair $\mathcal{N} = (G, U)$ a *network*, where $G$ is a graph and $U \subseteq V(G)$. Any path in $G$ with both ends in $U$ is called *open* in $\mathcal{N}$.

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For technical reasons, we shall deal with networks where the set $U$ is partitioned into nonempty sets $U_1, \dots, U_n$. In this case we write $\mathcal{N} = (G, U_1, \dots, U_n)$ and call $\mathcal{N}$ a *partitioned network*, for which the sets $U_1, \dots, U_n$ are called *connectors* of $\mathcal{N}$. An open path in $\mathcal{N}$ is called *crossing* if its two end vertices belong to different connectors.

**Theorem 1.** *The following statements are equivalent*:

(a) *Every cyclically* 4-*edge-connected cubic graph has a dominating circuit.*
(b) *Any two edges of a cyclically* 4-*edge-connected cubic graph are contained in a dominating circuit.*

**Proof.** It suffices to show that (a) implies (b). Suppose that there is a cyclically 4-edge-connected cubic graph $G$ containing two edges $e_1$ and $e_2$ so that there is no dominating circuit of $G$ containing $e_1, e_2$. We construct a cyclically 4-edge-connected cubic graph having no dominating circuit.

*Case* 1: Suppose that $e_1$ and $e_2$ do not have a vertex in common. Let $x, y$ $(z, t)$ be the ends of $e_1$ $(e_2)$ and $H = G - \{e_1, e_2\}$. Then the partitioned network $(H, \{x, y\}, \{z, t\})$ does not contain a dominating subgraph consisting of two vertex-disjoint crossing paths. Add two copies $H'$ and $H''$ of $H$ so that $x', y', z', t' \in V(H')$ and $x'', y'', z'', t'' \in V(H'')$ correspond to $x, y, z, t$, respectively, and add edges $zx'$, $ty'$, $z'x''$, $t'y''$. We thus obtain a graph $F$ so that the network $\mathcal{M} = (F, \{x, y\}, \{z'', t''\})$ has the following properties:

(1) $\mathcal{M}$ does not contain a dominating path with ends $x$ and $y$;
(2) $\mathcal{M}$ does not contain a dominating path with ends $z''$ and $t''$;
(3) $\mathcal{M}$ has no dominating subgraph consisting of two vertex-disjoint open paths.

Take $F$ and copies $F_1$ and $F_2$ of $F$ so that $x_1, y_1, z_1'', t_1'' \in V(F_1)$ and $x_2, y_2, z_2'', t_2'' \in V(F_2)$ correspond to $x, y, z'', t''$, respectively. Add vertices $u, v, u'', v''$ and edges $ux$, $ux_1$, $ux_2$, $vy$, $vy_1$, $vy_2$, $u''z''$, $u''z_1''$, $u''z_2''$, $v''t''$, $v''t_1''$, $v''t_2''$. We obtain a cyclically 4-edge-connected cubic graph $G'$. Let $S$ be the edge cut in $G'$ consisting of the edges having exactly one end in $\{u, v\}$ (note that $S = \{ux, ux_1, ux_2, vy, vy_1, vy_2\}$). Let $C$ be a dominating circuit in $G'$. Because of properties (1)–(3) of $\mathcal{M}$, precisely one of the edges $ux$ and $vy$ belongs to $C$; the same applies for the edges $ux_1$ and $vy_1$ with respect to $F_1$, and $ux_2$ and $vy_2$ with respect to $F_2$. Therefore, $S$ contains exactly 3 edges of $C$—a contradiction to the fact that every circuit intersects every edge cut in an even number of edges. Thus $G'$ has no dominating circuit.

*Case* 2: Suppose that $e_1$ and $e_2$ have a common vertex $v$. Let $v_1$ and $v_2$ be the vertices incident to $e_1$ and $e_2$, respectively, and different from $v$, and $v_3$ be the vertex adjacent with $v$ and different from $v_1, v_2$. Take the graph $K = G - v$ and a copy $K'$ of $K$ with $v_1', v_2', v_3' \in V(K')$ corresponding to $v_1, v_2, v_3$, respectively. Add edges $v_1v_1'$, $v_2v_3'$, $v_3v_2'$. This yields a cubic graph $L$ having no dominating circuit containing the edge $v_1v_1'$. Let $u, w$ $(u', w')$ be the vertices of $L$ adjacent to $v_1$ $(v_1')$ and different from $v_1'$ $(v_1)$. Consider $I = L - \{v_1, v_1'\}$. Then the partitioned network $(I, \{u, w\}, \{u', w'\})$ does not contain a dominating crossing path. Add a copy $I_1$ of $I$ so that $u_1, w_1, u_1', w_1'$ correspond to $u, w, u', w'$, respectively. Add vertices $r, s, p, q$ and edges $rs$, $ru$, $ru_1$, $sw$,

$sw_1$, $pq$, $pu'$, $pu_1'$, $qw'$, $qw_1'$. The resulting graph $G''$ is cubic and cyclically 4-edge-connected. By construction of $G''$, any dominating circuit $C''$ of $G''$ containing both $rs$ and $pq$ must contain exactly one edge of each of the four pairs of edges $\{ru, sw\}$, $\{ru_1, sw_1\}$, $\{pu', qw'\}$, $\{pu_1', qw_1'\}$. Then $C''$ induces a dominating crossing path in $(I, \{u, w\}, \{u', w'\})$, which is impossible. Therefore, $C''$ contains at most one of $rs$ and $pq$. Now, applying the arguments of Case 1 with $G''$ in place of $G$ and setting $e_1 = rs$, $e_2 = pq$, we may again construct a cyclically 4-edge-connected cubic graph without a dominating circuit.

Thus, in both cases we concluded that if (b) does not hold for some graph, then (a) does not hold for another graph. Therefore, (a) and (b) are equivalent. □

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