Discrete Mathematics 311 (2011) 1925–1939

Contents lists available at ScienceDirect

# Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

# The number of vertices of degree 5 in a contraction-critically 5-connected graph

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ARTICLE INFO

*Article history:*
Received 30 September 2008
Received in revised form 27 April 2011
Accepted 29 April 2011
Available online 12 June 2011

*Keywords:*
5-connected graph
Contraction-critically 5-connected
Degree 5 vertex

ABSTRACT

An edge of a 5-connected graph is said to be 5-contractible if the contraction of the edge results in a 5-connected graph. A 5-connected graph with no 5-contractible edge is said to be contraction-critically 5-connected. Let $V(G)$ and $V_5(G)$ denote the vertex set of a graph $G$ and the set of degree 5 vertices of $G$, respectively. We prove that each contraction-critically 5-connected graph $G$ has at least $|V(G)|/2$ vertices of degree 5. We also show that there is a sequence of contraction-critically 5-connected graphs $\{G_i\}$ such that $\lim_{i\to\infty} |V_5(G_i)|/|V(G_i)| = 1/2$.



## 1. Introduction

In this paper, we deal with finite undirected graphs with neither loops nor multiple edges. For a graph $G$, let $V(G)$ and $E(G)$ denote the set of vertices of $G$ and the set of edges of $G$, respectively. Let $V_k(G)$ denote the set of vertices of degree $k$ and let $V_{\geq k}(G)$ denote the set of vertices of degree greater than or equal to $k$. For an edge $e \in E(G)$, we denote the set of end vertices of $e$ by $V(e)$. Let $E_G(x) = \{e \in E(G) \mid x \in V(e)\}$. For a vertex $x \in V(G)$, we denote by $N_G(x)$ the neighborhood of $x$ in $G$. Moreover, for a subset $S \subset V(G)$, let $N_G(S) = \cup_{x\in S} N(x) - S$. We denote the degree of $x \in V(G)$ by $\deg_G(x)$. Then $\deg_G(x) = |E_G(x)| = |N_G(x)|$. When there is no ambiguity, we write $V_k$, $V_{\geq k}$, $E(x)$, $N(x)$, $N(S)$ and $\deg(x)$ for $V_k(G)$, $V_{\geq k}(G)$, $E_G(x)$, $N_G(x)$, $N_G(S)$ and $\deg_G(x)$, respectively. For $S \subset V(G)$, let $G[S]$ denote the subgraph induced by $S$ in $G$. Let $G$ be a connected graph. A subset $S \subset V(G)$ is said to be a *cutset* of $G$, if $G - S$ is not connected. A cutset $S$ is said to be a *k-cutset* if $|S| = k$. For a noncomplete connected graph $G$, the order of a minimum cutset of $G$ is said to be the vertex connectivity of $G$. We denote the vertex connectivity of $G$ by $\kappa(G)$.

Let $k$ be an integer such that $k \geq 2$ and let $G$ be a $k$-connected graph. An edge $e$ of $G$ is said to be *k-contractible* if the contraction of the edge results in a $k$-connected graph. Note that, in the contraction, we replace each resulting pair of double edges by a simple edge. If an edge is not $k$-contractible, then it is called a *noncontractible* edge. Note that an edge $e$ of $G$ is not $k$-contractible if and only if there is a $k$-cutset $S$ of $G$ such that $V(e) \subset S$. If a $k$-connected graph $G$ has no $k$-contractible edge, then $G$ is said to be *contraction-critically k-connected*.

It is known that every 3-connected graph of order 5 or more contains a 3-contractible edge [9]. There are infinitely many contraction-critically 4-connected graphs. It is known that a 4-connected graph $G$ is contraction-critical if and only if $G$ is 4-regular, and for each edge $e$ of it, there is a triangle which contains $e$. [3,6]. If $k \geq 4$, then there are infinitely many contraction-critically $k$-connected graphs [8].

Egawa determined the following sharp minimum degree condition for a $k$-connected graph to have a $k$-contractible edge.

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doi:10.1016/j.disc.2011.04.032

**Theorem A** (*Egawa [2]*)**.** *Let $k \geq 2$ be an integer, and let $G$ be a $k$-connected graph with $\delta(G) \geq \lfloor \frac{5k}{4} \rfloor$. Then $G$ has a $k$-contractible edge, unless $2 \leq k \leq 3$ and $G$ is isomorphic to $K_{k+1}$.*

Kriesell extended Egawa's Theorem and determined the following sharp degree sum condition for a $k$-connected graph to have a $k$-contractible edge.

**Theorem B** (*Kriesell [4]*)**.** *Let $k \geq 2$ be an integer, and let $G$ be a noncomplete $k$-connected graph. If $\deg_G(x) + \deg_G(y) \geq 2\lfloor \frac{5k}{4} \rfloor - 1$ for any pair of distinct vertices $x, y$ of $G$, then $G$ has a $k$-contractible edge.*

There is a contraction-critically 5-connected graph which is not 5-regular. However, we see from Theorem A that the minimum degree of a contraction-critically 5-connected graph is 5. Ando et al. [1] posed Problem D and proved Theorem C, which says that each contraction-critically 5-connected graph has many vertices of degree 5.

**Theorem C.** *Let $G$ be a contraction-critically 5-connected graph of order $n$. Then each vertex of $G$ has a neighbor of degree 5 and $G$ has at least $n/5$ vertices of degree 5.*

**Problem D.** Determine the maximum value of the constant $c$ such that the inequality $|V_5(G)| \geq c|V(G)|$ holds for each contraction-critically 5-connected graph $G$.

The following important result was showed by Su [7].

**Theorem E.** *Every vertex of a contraction-critically 5-connected graph has two neighbors of degree five.*

As an immediate consequence of Theorem E, we have the following.

**Theorem F.** *For every contraction-critically 5-connected graph $G$, $|V_5| \geq \frac{2}{5}|V(G)|$ holds.*

By more detailed investigation of contraction-critically 5-connected graphs, Yuan and others [10] proved the following Theorems H and G.

**Theorem G.** *Let $G$ be a contraction-critically 5-connected graph and let $x$ be a vertex of $G$ with $\deg_G(x) \geq 8$. If $x$ has adjacent two neighbors of degree five, then $x$ has three neighbors of degree five.*

**Theorem H.** *For every contraction-critically 5-connected graph $G$, $|V_5(G)| \geq \frac{4}{9}|V(G)|$ holds.*

On the other hands, there is a contraction-critically 5-connected graph $G$ such that $|V_5(G)| = \frac{8}{13}|V(G)|$ [1].
Ando posed the following conjecture.

**Conjecture I.** *The constant $c$ for Problem D is $\frac{8}{13}$.*

In this paper we prove the following stronger version of Theorem G (Proposition 1). And using Proposition 1, by detailed investigation on vertices not in $V_5(G)$ each of which has just two neighbors of degree 5, we show the constant $c$ in Problem D is not less than $\frac{1}{2}$ (Main Theorem). Moreover, we construct a sequence of contraction-critically 5-connected graphs $\{G_i\}$ such that $\lim_{i\to\infty} |V_5(G_i)|/|V(G_i)| = 1/2$.
This sequence disproves Conjecture I and, together with Main Theorem, it gives the answer for Problem D, that is $c = \frac{1}{2}$.

**Proposition 1.** *Let $G$ be a contraction-critically 5-connected graph and let $x$ be a vertex of $G$ such that $x \notin V_5(G)$. Suppose $|N_G(x) \cap V_5(G)| = 2$, say $N_G(x) \cap V_5(G) = \{y_1, y_2\}$. Then $y_1y_2 \notin E(G)$.*

Next we concentrate on vertices not in $V_5(G)$ each of which has just two neighbors of degree 5 and we find two specific configurations.

*Configuration of the first kind.* A subgraph $H$ on eight vertices (in degenerated case, on seven vertices) of a contraction-critically 5-connected graph $G$ is called a configuration of the first kind around $(x, y)$ if the following (1)–(4) hold (see Fig. 1).

(1) $V(H) = \{x, y, z_1, z_2, z_3, z_4, u_1, u_2\}$,
(2) $E(H) \supset \{yx, yz_1, yz_2, yz_3, yz_4, xz_4, z_1z_2, z_1z_3, z_1u_1, z_1u_2, z_2u_1, z_3u_2\}$,
(3) $\{y, z_1, z_2, z_3\} \subset V_5$ and $\{x, z_4\} \cap V_5 = \emptyset$,
(4) There is a 5-cutset $S$ such that $\{x, y, z_1\} \subset S$ and $S$ separates $\{u_1, z_2\}$ and $\{u_2, z_3, z_4\}$.

In a configuration of the first kind, if $z_4 = u_2$, then it is said to be a degenerated configuration of the first kind.

*Configuration of the second kind.* A subgraph $H$ on nine vertices of a contraction-critically 5-connected graph $G$ is called a configuration of the second kind around $(y, x)$ if the following (1)–(4) hold (see Fig. 2).

(1) $V(H) = \{x, y, z_1, z_2, z_3, z_4, u_1, w_1, w_2\}$,
(2) $E(H) \supset \{yx, yz_1, yz_2, yz_3, yz_4, xz_3, xz_4, z_1z_2, z_1z_3, z_1z_4, z_1u_1, z_2u_1, z_3z_4, z_3w_1, z_3w_2, z_4w_1, z_4w_2\}$,
(3) $\{y, z_1, z_2, w_1, w_2\} \subset V_5, x \notin V_5$.
(4) $\{x, y, z_1, w_1, w_2\}$ is a 5-cutset of $G$ which separates $\{z_2, u\}$ and $\{z_3, z_4\}$, and hence $\{z_3, z_4\} \subset V_6(G)$.

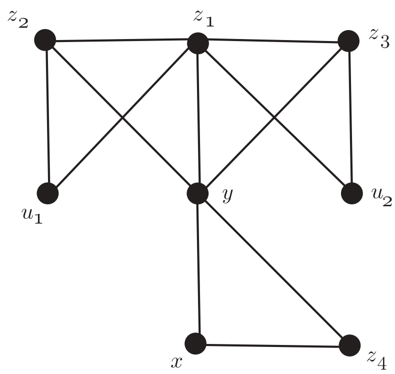


**Fig. 1.** A configuration of the first kind.

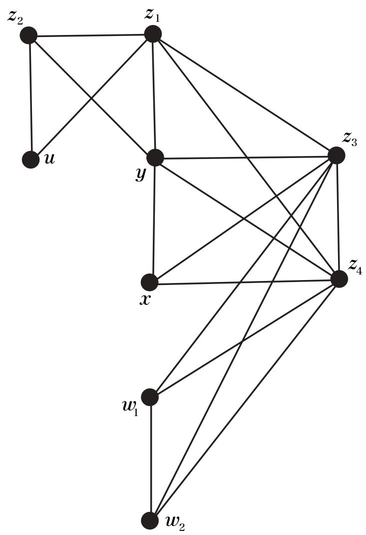


**Fig. 2.** Configuration of the second kind.

**Proposition 2.** *Let G be a contraction-critically* 5*-connected graph. Let x be a vertex of G such that* $x \notin V_5$ *and* $|N(x) \cap V_5(G)| = 2$. *Let* $y \in N(x) \cap V_5(G)$. *Then, around* $(y, x)$, *there is either a configuration of the first kind or a configuration of the second kind.*

By virtue of Proposition 2, we get the following result.

**Main Theorem.** *For every contraction-critically* 5*-connected graph G,* $|V_5(G)| \geq \frac{1}{2}|V(G)|$ *holds.*

Recently, Li and Su [5] proved the same bound of the constant $c$ in Problem D.

The organization of the paper is as follows. Section 2 contains preliminary results. In Section 3 we give a proof of Proposition 1. In Section 4 we give a proof of Proposition 2 and in Section 5 we give a proof of Main Theorem.

To conclude this section we give three contraction-critically 5-connected graphs. The first one has a configuration of the first kind. The second has a configuration of the second kind. The third shows that there is a sequence of contraction-critically 5-connected graphs $\{G_i\}$ such that $\lim_{i\to\infty} |V_5(G_i)|/|V(G_i)| = 1/2$.

**Example 1.** The graph $G_1$ illustrated in Fig. 3 is contraction-critically 5-connected, and we observe that it has a configuration of the first kind.

**Example 2.** The graph $G_2$ illustrated in Fig. 4 is contraction-critically 5-connected. We observe that $G_2$ has a configuration of the second kind.

**Example 3.** The graph $G_3$ illustrated in Fig. 5 is contraction-critically 5-connected. Adding pairs of vertices $(x_4, y_4)$, $(x_5, y_5)$, $\ldots$, $(x_i, y_i)$ to this graph by the similar way, we can construct a sequence of contraction-critically 5-connected graphs $\{G_i\}$. We see that $|V(G_i)| = 2i + 15$ and $|V_5(G_i)| = i + 10$ since $\{y_1, y_2, \ldots, y_i\} \subset V_5(G_i)$ and $\{x_1, x_2, \ldots, x_i\} \subset V_6(G_i)$. Hence we have $\lim_{i\to\infty} |V_5(G_i)|/|V(G_i)| = 1/2$.

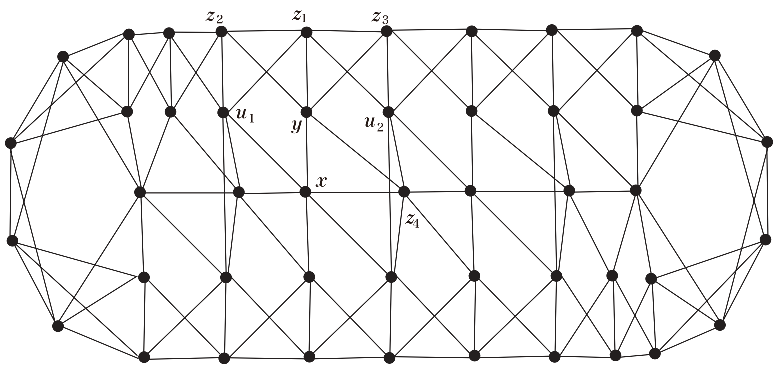


**Fig. 3.** $G_1$.

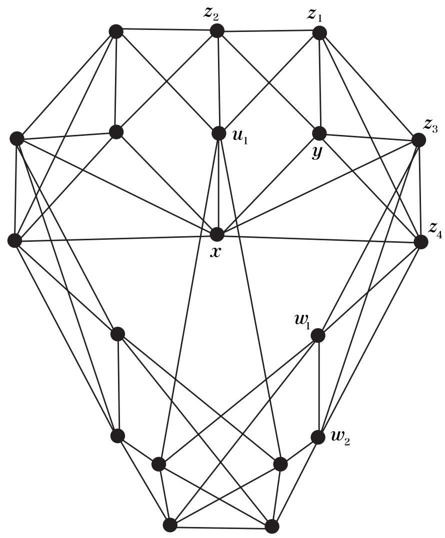


**Fig. 4.** $G_2$.

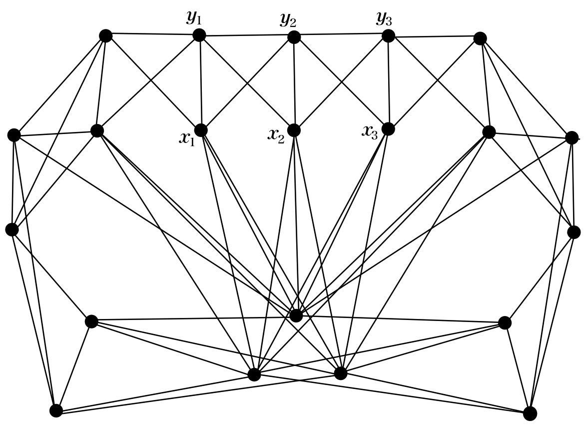


**Fig. 5.** $G_3$.

## 2. Preliminaries

In this section we give some more definitions and preliminary results.

For a graph $G$, we denote $|G|$ for $|V(G)|$. For a subgraphs $A$ and $B$ of a graph $G$, when there is no ambiguity, we write simply $A$ for $V(A)$ and $B$ for $V(B)$. So $N(A)$ and $A \cap B$ mean $N(V(A))$ and $V(A) \cap V(B)$, respectively. Also for a subgraph $A$ of $G$ and a subset $S$ of $V(G)$ we write $A \cap S$ and $A \cup S$ for $V(A) \cap S$ and $V(A) \cup S$, respectively. For $S \subset V(G)$, we let $G - S$ denote the graph obtained from $G$ by deleting the vertices in $S$ together with the edges incident with them; thus $G - S = G[V(G) - S]$. When there is no ambiguity, we write $E(S)$ for $E(G[S])$. For subsets $S$ and $T$ of $V(G)$, we denote the set of edges between $S$ and $T$ by $E_G(S, T)$. We write $E_G(x, S)$ for $E_G(\{x\}, S)$. Then $E_G(x) = E_G(x, V(G) - \{x\})$.

An induced subgraph $A$ of a $k$-connected graph $G$ is called a *fragment* if $|N(A)| = k$ and $V(G) - (A \cup N(A)) \neq \emptyset$. In other words, a fragment $A$ is a nonempty union of components of $G - S$ where $S$ is a $k$-cutset of $G$ such that $V(G) - (A \cup S) \neq \emptyset$. By the definition, if $A$ is a fragment of $G$, then $G - (A \cup N(A))$ is also a fragment of $G$. Let $\bar{A}$ stand for $G - (A \cup N(A))$.

Let $A$ be a fragment of a $k$-connected graph $G$ and let $e$ be an edge of $G$. Then $A$ is said to be a *fragment with respect to e* if $V(e) \subset N(A)$. For a set of edges $F \subset E(G)$, we say that $A$ is a *fragment with respect to F* if $A$ is a fragment with respect to some $e \in F$. Sometimes we write "an $F$-fragment" for "a fragment with respect to $F$". If $F = \{e\}$, then we write $e$-fragment instead of $\{e\}$-fragment. For $S \subset V(G)$, a fragment $A$ is said to be *S-free* and *S-opposite* if $A \cap S = \emptyset$ and $S \subset \bar{A}$, respectively. Hence, if $A$ is $S$-opposite, the $A$ is $S$-free. If $S = \{y\}$, then we write $y$-free and $y$-opposite instead of $\{y\}$-free and $\{y\}$-opposite, respectively. An $F$-fragment $A$ is said to be *minimum* (resp. *minimal*) if there is no $F$-fragment $B$ other than $A$ such that $|B| < |A|$ (resp. $B \subsetneq A$).

Hereafter, we consider 5-connected graphs. Let $A$ be a fragment of a 5-connected graph $G$ and let $S = N(A)$. Let $x \in S$ and let $y \in N(x) \cap A$. A vertex $z$ is said to be an *admissible vertex of* $(x, y; A)$, if the following two conditions hold.

(1) $z \in N(x) \cap N(y) \cap S \cap V_5$.
(2) $|N(z) \cap A| \geq 2$.

Moreover, if $|N(z) \cap \bar{A}| = 1$, then $z$ is said to be *strongly admissible*.

A vertex $z$ is said to be an *admissible vertex of* $(x; A)$ or a *strongly admissible vertex of* $(x; A)$, if $z$ is an admissible vertex of $(x, y; A)$ or a strongly admissible vertex of $(x, y; A)$ for some $y \in N(x) \cap A$. Let $\mathrm{Ad}(x, y; A)$ denote the set of admissible vertices of $(x, y; A)$ and let $\mathrm{Ad}(x; A)$ denote the set of admissible vertices of $(x; A)$. Let $A$ be a fragment of a 5-connected graph $G$ and let $x \in N(A)$. A fragment $B$ of $G$ is said to be *(x; A)-fit* if $\{x\} \cup A \subset N(B)$. A vertex $x \in N(A)$ is said to be *tractable with A* if there is an $(x; A)$-fit fragment $B$ such that $|S \cap B| = |S \cap \bar{B}| = 2$. If there is no ambiguity, we sometimes write "$A$-tractable" for "tractable with $A$".

We begin with the following two lemmas, which are both simple observations.

**Lemma 1.** *Let $A$ be a fragment of a 5-connected graph $G$ and let $S \subset N(A)$. If $|N(S) \cap A| < |S|$, then $A \subset N(S)$.*

**Proof.** Assume that $A \neq N(S) \cap A$. Let $A' = A - (N(S) \cap A)$. Since $A' \neq \emptyset$ and $N(A') \cap (\bar{A} \cup S) = \emptyset$, $(N(A) - S) \cup (N(S) \cap A)$ separates $A'$ and $\bar{A} \cup S$. Since $|N(S) \cap A| < |S|$, we see that $|(N(A) - S) \cup (N(S) \cap A)| = |N(A)| - |S| + |N(S) \cap A| < |N(A)| = 5$, which contradicts the assumption that $G$ is 5-connected. □

**Lemma 2.** *Let $G$ be a 5-connected graph, and let $A$ and $B$ be fragments of $G$ Let $S = N(A)$ and $T = N(B)$.*

| | | | |
|---|---|---|---|
| $B$ | $\bar{A} \cap B$ | $S \cap B$ | $A \cap B$ |
| $T$ | $\bar{A} \cap T$ | $S \cap T$ | $A \cap T$ |
| $\bar{B}$ | $\bar{A} \cap \bar{B}$ | $S \cap \bar{B}$ | $A \cap \bar{B}$ |
| | $\bar{A}$ | $S$ | $A$ |

*Then the following hold.*

(1) *If $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \geq 6$, then $|(\bar{A} \cap T) \cup (S \cap T) \cup (S \cap \bar{B})| \leq 4$ and $\bar{A} \cap \bar{B} = \emptyset$. In particular, if neither $A \cap B$ nor $\bar{A} \cap \bar{B}$ is empty, then both $A \cap B$ and $\bar{A} \cap \bar{B}$ are fragments of $G$.*
(2) *$|(S \cap B) \cup (S \cap T) \cup (A \cap T)| = 5 + |S \cap B| - |\bar{A} \cap T|$. In particular, if $A \cap B \neq \emptyset$, then $|S \cap B| \geq |\bar{A} \cap T|$.*
(3) *If $|\bar{A}| \geq 2$, then either $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \leq 5$ or $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| \leq 5$.*

**Proof.** (1) Since $S$ and $T$ are both 5-cutsets, $|S| + |T| = |(S \cap B) \cup (S \cap T) \cup (S \cap \bar{B})| + |(\bar{A} \cap T) \cup (S \cap T) \cup (A \cap T)| = 10$. Hence, if $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \geq 6$, then $|(\bar{A} \cap T) \cup (S \cap T) \cup (S \cap \bar{B})| \leq 4$, which implies that $\bar{A} \cap \bar{B} = \emptyset$, since $G$ is 5-connected. If neither $A \cap B$ nor $\bar{A} \cap \bar{B}$ is empty, then $|(S \cap B) \cup (S \cap T) \cup (A \cap T)|$, $|(\bar{A} \cap T) \cup (S \cap T) \cup (S \cap \bar{B})| \geq 5$, which implies $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| = |(\bar{A} \cap T) \cup (S \cap T) \cup (S \cap \bar{B})| = 5$. Hence, we see that both $A \cap B$ and $\bar{A} \cap \bar{B}$ are fragments of $G$.

(2) Since $|T| = |(\bar{A} \cap T) \cup (S \cap T) \cup (A \cap T)| = 5$, we see that $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| = |T| + |S \cap B| - |\bar{A} \cap T| = 5 + |S \cap B| - |\bar{A} \cap T|$. Next assume $A \cap B \neq \emptyset$. Then $(S \cap B) \cup (S \cap T) \cup (A \cap T)$ is a cutset of $G$ since $\bar{A} \cup \bar{B} \neq \emptyset$. Hence $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \geq 5$. Thus, we have $|S \cap B| \geq |\bar{A} \cap T|$.

(3) Assume $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \geq 6$ and $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| \geq 6$. Then, by (1), we have $\bar{A} \cap B = \bar{A} \cap \bar{B} = \emptyset$, which implies $|\bar{A} \cap T| = |\bar{A}| \geq 2$. Hence we see that $|(S \cap T) \cup (A \cap T)| = |T| - |\bar{A} \cap T| \leq 3$. On the other hand, since $|S| = 5$, we observe that either $|S \cap B| \leq 2$ or $|S \cap \bar{B}| \leq 2$. This together with the fact $|(S \cap T) \cup (A \cap T)| \leq 3$ implies either $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \leq 5$ or $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| \leq 5$, which contradicts the assumption. □

**Lemma 3.** *Let $x$ be a vertex of a contraction-critically 5-connected graph $G$. Let $A$ be a fragment with respect to $E(x)$ such that $|\bar{A}| \geq 2$ and $|A| \geq 3$. For each $y \in N(x) \cap A$, if $\mathrm{Ad}(x, y; A) = \emptyset$, then there is a fragment $A'$ with respect to $xy$ such that $A' \subsetneq A$.*

**Proof.** Assume that there is neither an admissible vertex of $(x, y; A)$ nor an $xy$-fragment $A'$ such that $A' \subsetneq A$. Let $B$ be an $xy$-fragment. Let $S = N(A)$ and let $T = N(B)$. Since $|\bar{A}| \geq 2$, by Lemma 2(3), we see that either $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \leq 5$ or $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| \leq 5$. Without loss of generality we may assume $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \leq 5$. Then, since there is no $xy$-fragment $A'$ such that $A' \subsetneq A$, we see that $A \cap B = \emptyset$.

Claim 3.1. $A \cap \bar{B} \neq \emptyset$.

**Proof.** Assume $A \cap \bar{B} = \emptyset$. Then, since $A \cap B = \emptyset$, we have $A = A \cap T$ and $|A| = |A \cap T| \geq 3$, which implies that $|\bar{A} \cap T| = |T| - |S \cap T| - |A \cap T| \leq 1$. Hence, since $|\bar{A}| \geq 2$, by symmetry, we may assume that $\bar{A} \cap \bar{B} \neq \emptyset$. Then, by Lemma 2(2), we observe that $|S \cap \bar{B}| \geq |A \cap T| \geq 3$, which implies that $|S \cap B| = |S| - |S \cap T| - |S \cap \bar{B}| \leq 1$. If $S \cap B = \emptyset$, then we have $B = \emptyset$, which contradicts the choice of $B$. Hence $|S \cap B| = 1$, say $S \cap B = \{z\}$. Then we observe that $z \in N(x) \cap N(y) \cap S \cap V_5$ and $|N(z) \cap A| = |A \cap T| = 3$. Now we see that $z \in \mathrm{Ad}(x, y; A)$, which contradicts the assumption. □

By Claim 3.1, we see that $A \cap \bar{B} \neq \emptyset$. If $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| = 5$, then $A \cap \bar{B}$ is an $xy$-fragment such that $A \cap \bar{B} \subsetneq A$, which contradicts the assumption. Hence we have $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| \geq 6$. Thus, by Lemma 2(1), we observe that $\bar{A} \cap B = \emptyset$, which implies $B = S \cap B$ since $A \cap B = \emptyset$. We show that $|B| = |S \cap B| = 1$. Assume that $|S \cap B| \geq 2$. Since $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| \geq 6$, applying Lemma 2(2) with the roles $S \cap B$ and $\bar{A} \cap T$ replaced by $A \cap T$ and $S \cap B$, respectively, we see that $|A \cap T| \geq |S \cap B| + 1 \geq 3$, which implies that $|\bar{A} \cap T| = |T| - |S \cap T| - |A \cap T| \leq 1$ since $x \in S \cap T$. Hence, since $|\bar{A} \cap T| < |S \cap B|$, applying Lemma 2(2), we see that $\bar{A} \cap \bar{B} = \emptyset$, which implies $|\bar{A}| = |\bar{A} \cap T| \leq 1$. This contradicts the assumption and it is shown that $|S \cap B| = 1$, say $B = S \cap B = \{z\}$. Then we observe that $z \in N(x) \cap N(y) \cap S \cap V_5$ and $|N(z) \cap A| = |A \cap T| \geq |S \cap B| + 1 = 2$. Hence $z$ is an admissible vertex of $(x, y; A)$, which contradicts the assumption. This contradiction proves Lemma 3. □

The following corollary is an immediate consequence of Lemma 3.

**Corollary 4.** *Let $x$ be a vertex of a contraction-critically 5-connected graph $G$. Let $A$ be a fragment with respect to $E(x)$ such that $|\bar{A}| \geq 2$, $|A| \geq 3$. Suppose $|N(x) \cap A| = 1$, say $N(x) \cap A = \{y\}$. Then there is an admissible vertex of $(x, y; A)$.*

**Proof.** Assume that there is no admissible vertex of $(x, y; A)$. Then, Lemma 3 assure us that there is an $xy$-fragment $A'$ such that $A' \subsetneq A$. Since $N(x) \cap A = \{y\}$, we observe that $N(x) \cap A' = \emptyset$, which contradicts the fact that $A'$ is an $xy$-fragment. This contradiction proves Corollary 4. □

Let $A$ be a fragment of a 5-connected graph and let $x \in N(A)$. Recall that a fragment $B$ is $(x; A)$-fit if $A \cup \{x\} \subset N(B)$ and a vertex $x \in N(A)$ is tractable with $A$ if there is an $(x; A)$-fit fragment $B$ such that $|S \cap B| = |S \cap \bar{B}| = 2$.

**Lemma 5.** *Let $G$ be a contraction-critically 5-connected graph. Let $A$ be a fragment such that $|\bar{A}| \geq 2$, $|A| = 2$ and $A \cap V_{\geq 6} \neq \emptyset$. Then the following (1), (2) and (3) hold.*

(1) $|\{x \in N(A) \mid \mathrm{Ad}(x; A) \neq \emptyset\}| \geq 4$.
(2) *If* $\mathrm{Ad}(x; A) = \emptyset$ *for* $x \in N(A)$, *then* $N(x) \cap N(A) = \emptyset$.
(3) $|N(A) \cap V_5| \geq 4$.

**Proof.** Let $S = N(A)$ and let $A = \{y_1, y_2\}$. We may assume that $\deg_G(y_1) \leq \deg_G(y_2)$, then we observe that $\deg_G(y_2) = 6$ and $S \subset N(y_2)$ since $A \cap V_{\geq 6} \neq \emptyset$.

Claim 5.1. *For each $x \in S$, there is an $(x; A)$-fit fragment.*

**Proof.** At first we consider the case that $xy_1 \in E(G)$. Let $B$ be an $xy_1$-fragment. Then, since $S \subset N(y_2)$, we observe that $y_2 \in N(B)$, which implies that $B$ is an $(x; A)$-fit fragment. Next assume that $xy_1 \notin E(G)$. Then we observe that $S - \{x\} \subset N(y_1)$. Let $B$ be an $xy_2$-fragment. Then, since $S - \{x\} \subset N(y_1)$, we see that $y_1 \in N(B)$, which implies that $B$ is an $(x; A)$-fit fragment. Now Claim 5.1 is proved. □

Claim 5.2. *If $x \in S$ is not tractable with $A$, then $\mathrm{Ad}(x; A) \neq \emptyset$.*

**Proof.** Assume $x$ is not tractable with $A$. By Claim 5.1, let $B$ an $(x; A)$-fit fragment and let $T = N(B)$. Since $x$ is not tractable with $A$, we know that either $|S \cap B| = 1$ or $|S \cap \bar{B}| = 1$. Without loss of generality, we may assume that $|S \cap B| = 1$, say $S \cap B = \{z\}$. Then, since $|S \cap B| < |A \cap T|$, by Lemma 2(2), we see that $\bar{A} \cap B = \emptyset$, which implies that $B = S \cap B = \{z\}$. Hence, we observe that $z \in N(x) \cap N(y) \cap S \cap V_5$ and $|N(z) \cap A| = |A| = 2$, which implies that $z \in \mathrm{Ad}(x; A)$ and Claim 5.2 is proved. □

Let $S = \{x_1, x_2, x_3, x_4, x_5\}$.

Claim 5.3. *If both $x_1$ and $x_2$ are tractable with $A$, then neither $\mathrm{Ad}(x_1; A)$ nor $\mathrm{Ad}(x_2; A)$ is empty.*

**Proof.** Since both $x_1$ and $x_2$ are tractable with $A$, there are an $(x_1; A)$-fit fragment $B_1$ and an $(x_2; A)$-fit fragment $B_2$ such that $|S \cap B_1| = |S \cap \bar{B}_1| = |S \cap B_2| = |S \cap \bar{B}_2| = 2$. Let $T_1 = N(B_1)$ and $T_2 = N(B_2)$. Then, we observe that $A \subset T_1 \cap T_2$, $S \cap T_1 = \{x_1\}$ and $S \cap T_2 = \{x_2\}$. Without loss of generality, we may assume that $S \cap B_1 = \{x_2, x_3\}$ and $S \cap \bar{B}_1 = \{x_4, x_5\}$. Furthermore, without loss of generality, we may assume that $x_1 \in S \cap B_2$.

At first we consider the case that $x_3 \in S \cap B_2$. In this case $S \cap B_2 = \{x_1, x_3\}$ and $S \cap \bar{B}_2 = \{x_4, x_5\}$. Then we observe that $x_3 \in B_1 \cap B_2$ and $\{x_4, x_5\} \subseteq \bar{B}_1 \cap \bar{B}_2$, which implies that neither $B_1 \cap B_2$ nor $\bar{B}_1 \cap \bar{B}_2$ is empty. Then, Lemma 2(1) assures us that both $B_1 \cap B_2$ and $\bar{B}_1 \cap \bar{B}_2$ are fragments of $G$. Moreover, we observe that $\{y_1, y_2\} \subset T_1 \cap T_2$, $x_1 \in T_1 \cap B_2$ and $x_2 \in B_1 \cap T_2$, which implies that $N(\{y_1, y_2\}) \cap (B_1 \cap B_2) = \{x_3\}$. Hence, applying Lemma 1 with the roles $S$ and $A$ replaced by $\{y_1, y_2\}$ and $B_1 \cap B_2$, respectively, we see that $B_1 \cap B_2 = \{x_3\}$. Now we have $x_3 \in V_5$, $A = \{y_1, y_2\} \subset N(x_3)$, and $x_1, x_2 \in N(x_3)$, which implies $x_3 \in \mathrm{Ad}(x_1; A)$ and $x_3 \in \mathrm{Ad}(x_2; A)$. Hence we have the desired conclusion that neither $\mathrm{Ad}(x_1; A)$ nor $\mathrm{Ad}(x_2; A)$ is empty.

Next we consider the case that $x_3 \notin S \cap B_2$. In this case, without loss of generality, we may assume that $S \cap B_2 = \{x_1, x_4\}$ and $S \cap \bar{B}_2 = \{x_3, x_5\}$. Then we observe that $x_3 \in B_1 \cap \bar{B}_2$, $x_4 \in \bar{B}_1 \cap B_2$ and $x_5 \in \bar{B}_1 \cap \bar{B}_2$, which implies that neither $B_1 \cap \bar{B}_2$ nor $\bar{B}_1 \cap B_2$ is empty. Then, Lemma 2(1) again assures us that both $B_1 \cap \bar{B}_2$ and $\bar{B}_1 \cap B_2$ are fragments of $G$. Moreover, we observe that $\{y_1, y_2\} \subset T_1 \cap T_2$, $x_1 \in T_1 \cap B_2$ and $x_2 \in B_1 \cap T_2$. Since $N(\{y_1, y_2\}) \cap (B_1 \cap \bar{B}_2) = \{x_3\}$, applying Lemma 1 with the roles $S$ and $A$ replaced by $\{y_1, y_2\}$ and $B_1 \cap \bar{B}_2$, respectively, we see that $B_1 \cap \bar{B}_2 = \{x_3\}$, which implies $x_3 \in V_5$, $A = \{y_1, y_2\} \subset N(x_3)$ and $x_3 \in N(x_2)$. Hence we see that $x_3 \in \mathrm{Ad}(x_2; A)$. We can similarly prove $x_4 \in \mathrm{Ad}(x_1; A)$. Now we obtain that neither $\mathrm{Ad}(x_1; A)$ nor $\mathrm{Ad}(x_2; A)$ is empty.

In both cases, we have the desired conclusion that neither $\mathrm{Ad}(x_1; A)$ nor $\mathrm{Ad}(x_2; A)$ is empty and Claim 5.3 is proved. □

By virtue of Claim 5.3, we show (1). Assume $\mathrm{Ad}(x_1; A) = \emptyset$. Then, by Claim 5.2, we see that $x_1$ is tractable with $A$. If there is a $A$-tractable vertex other than $x_1$, then Claim 5.3 assures us that $\mathrm{Ad}(x_1; A) \neq \emptyset$, which contradicts the assumption that $\mathrm{Ad}(x_1; A) = \emptyset$. Hence we see that none of $x_2, x_3, x_4$ and $x_5$ is $A$-tractable. Hence, again Claim 5.2 assures us that $\mathrm{Ad}(x_i; A) \neq \emptyset$ for $i = 2, 3, 4, 5$. Now (1) is proved.

Next we prove (2). Assume $\mathrm{Ad}(x_1; A) = \emptyset$ and $N(x_1) \cap S \neq \emptyset$. Then, by (1), we know that $\mathrm{Ad}(x_i; A) \neq \emptyset$ for $i = 2, 3, 4, 5$. Then, since $\mathrm{Ad}(x_1; A) = \emptyset$, Claim 5.2 assures us that $x_1$ is $A$-tractable. Hence there is an $(x_1; A)$-fit fragment $B_1$ such that $|S \cap B_1| = |S \cap \bar{B}_1| = 2$. Since $|S \cap B_1| = |S \cap \bar{B}_1| = 2$, we observe that $S \cap T_1 = \{x_1\}$, which implies that $|\bar{A} \cap T_1| = |T_1| - |S \cap T_1| - |A \cap T_1| = 2$. By symmetry, we may assume that $S \cap B_1 = \{x_2, x_3\}$ and $S \cap \bar{B}_1 = \{x_4, x_5\}$. Note that, in this situation, $E_G(\{x_2, x_3\}, \{x_4, x_5\}) = \emptyset$. Since $N(x_1) \cap S \neq \emptyset$, without loss of generality, we may assume $x_2 \in N(x_1) \cap S$. Then, since $\mathrm{Ad}(x_1; A) = \emptyset$, we observe that either $x_2 \in V_{\geq 6}$ or $|N(x_2) \cap A| = 1$, which implies that $x_2$ cannot be an admissible vertex of $(x_3; A)$. Since $N(x_3) \cap S \subset \{x_1, x_2\}$, $x_2 \notin \mathrm{Ad}(x_3; A)$, and $\mathrm{Ad}(x_3; A) \neq \emptyset$, we see that $x_1$ is an admissible vertex of $(x_3; A)$, which implies $\{y_1, y_2, x_2, x_3\} \subset N(x_1)$. Let $N(x_1) = \{y_1, y_2, x_2, x_3, v\}$. Then, since neither $N(x_1) \cap \bar{A}$ nor $N(x_1) \cap \bar{B}_1$ is empty, we see that $v \in \bar{A} \cap \bar{B}_1$, which implies that $|\bar{A}| \geq |\bar{A} \cap T_1| + |\bar{A} \cap \bar{B}_1| \geq 3$. Since $|\bar{A}| \geq 3$, $|A| = 2$ and $|N(x_1) \cap \bar{A}| = 1$, applying Corollary 4 with the roles of $x$, $y$ and $A$ replaced by $x_1$, $v$ and $\bar{A}$, respectively, we see that $\mathrm{Ad}(x_1, v; \bar{A}) \neq \emptyset$. Since $N(\{x_2, x_3\}) \cap \bar{B}_1 = \emptyset$, we observe that $v \notin N(\{x_2, x_3\})$, which implies that neither $x_2$ nor $x_3$ is an admissible vertex of $(x_1, v; \bar{A})$. Since $N(x_1) \cap S = \{x_2, x_3\}$, we have $\mathrm{Ad}(x_1, v; \bar{A}) = \emptyset$, which contradicts the previous assertion. This contradiction proves (2).

At last we show (3). By (1), we know that $|\{x \in N(A) \mid \mathrm{Ad}(x; A) \neq \emptyset\}| \geq 4$. To begin with the case that $|\{x \in N(A) \mid \mathrm{Ad}(x; A) \neq \emptyset\}| = 4$, let $\mathrm{Ad}(x_1; A) = \emptyset$ and $\mathrm{Ad}(x_i; A) \neq \emptyset$ for $i = 2, 3, 4, 5$. By Claim 5.2, let $B_1$ be an $(x_1; A)$-fit fragment such that $|S \cap B_1| = |S \cap \bar{B}_1| = 2$. By symmetry, we may assume that $S \cap B_1 = \{x_2, x_3\}$ and $S \cap \bar{B}_1 = \{x_4, x_5\}$. By (2), we see that $N(x_1) \cap S = \emptyset$. Then, since neither $\mathrm{Ad}(x_2; A)$ nor $\mathrm{Ad}(x_3; A)$ is empty, we see that $x_2$ is an admissible vertex of $(x_3; A)$ and $x_3$ is an admissible vertex of $(x_2; A)$. Similarly we see that $x_4$ is an admissible vertex of $(x_5; A)$ and $x_5$ is an admissible vertex of $(x_4; A)$. Now we have $|S \cap V_5| \geq 4$.

Hereafter, we assume that $\mathrm{Ad}(x_i; A) \neq \emptyset$ for $i = 1, 2, 3, 4, 5$. Assume $|S \cap V_5| \leq 3$, say $x_4, x_5 \in V_{\geq 6}$. Since $\mathrm{Ad}(x_i; A) \neq \emptyset$ for $i = 1, 2, 3$, we observe that $N(x_i) \cap S \cap V_5 \neq \emptyset$ for $i = 1, 2, 3$. Hence we can find a path of length 2 in $G[\{x_1, x_2, x_3\}]$ whose center vertex has degree 5. Without loss of generality, we may assume that $x_1x_2, x_2x_3 \in E(G)$ and $v_2 \in V_5$. We show $x_2 \notin \mathrm{Ad}(x_4; A)$. Assume $x_2 \in \mathrm{Ad}(x_4; A)$. Then, since $x_2 \in V_5$, $\{x_1, x_3\} \subset N(x_2)$ and $N(x_2) \cap \bar{A} \neq \emptyset$, we observe that $|N(x_2) \cap (A \cup \{x_4, x_5\})| \leq 2$. Hence, we observe that either $N(x_2) \cap \{x_4, x_5\} = \emptyset$ or $|N(x_2) \cap A| = 1$. If $N(x_2) \cap \{x_4, x_5\} = \emptyset$, then we observe that neither $x_2 \in \mathrm{Ad}(x_4; A)$ nor $x_2 \in \mathrm{Ad}(x_5; A)$. Otherwise, if $|N(x_2) \cap A| = 1$, then we also observe that neither $x_2 \in \mathrm{Ad}(x_4; A)$ nor $x_2 \in \mathrm{Ad}(x_5; A)$. This contradicts the assumption and it is shown that $x_2 \notin \mathrm{Ad}(x_4; A)$. Since $\mathrm{Ad}(x_4; A) \neq \emptyset$ and $x_2, x_5 \notin \mathrm{Ad}(x_4; A)$, by symmetry, we may assume that $x_1 \in \mathrm{Ad}(x_4; A)$, which implies that $x_1 \in V_5$, $\{y_1, y_2, x_2, x_4\} \subset N(x_1)$ and $|N(x_1) \cap \bar{A}| = 1$, say $N(x_1) \cap \bar{A} = \{v_1\}$. Since $N(x_1) \cap S = \{x_2, x_4\}$ and $x_4 \in V_{\geq 6}$, we see that $x_2 \in \mathrm{Ad}(x_1; A)$, which implies that $x_2 \in V_5$, $\{y_1, y_2, x_1, x_4\} \subset N(x_2)$ and $|N(x_2) \cap \bar{A}| = 1$, say $N(x_2) \cap \bar{A} = \{v_2\}$. Since $\mathrm{Ad}(x_5; A) \neq \emptyset$ and $N(x_5) \cap \{x_1, x_2\} = \emptyset$, we see that $x_3 \in \mathrm{Ad}(x_5; A)$, which implies that $x_3 \in V_5$, $\{y_1, y_2, x_2, x_5\} \subset N(x_3)$ and $|N(x_3) \cap \bar{A}| = 1$, say $N(x_3) \cap \bar{A} = \{v_3\}$. If $v_1 = v_2$, then, applying Lemma 1 with the roles of $S$ and $A$ replaced by $\{x_1, x_2\}$ and $\bar{A}$, respectively, we see that $|\bar{A}| = 1$, which contradicts the assumption that $|\bar{A}| \geq 2$. Hence $v_1 \neq v_2$. By similar arguments, we know that $v_1, v_2, v_3$ are distinct, which implies that $|\bar{A}| \geq 3$. Since $|\bar{A}| \geq 3$, $|A| = 2$ and $|N(x_1) \cap \bar{A}| = 1$, applying Corollary 4 with the roles of $x$ and $A$ replaced by $x_1$ and $\bar{A}$, respectively, we see that $\mathrm{Ad}(x_1; \bar{A}) \neq \emptyset$. However, we already know that $N(x_1) \cap S = \{x_2, x_4\}$, $|N(x_2) \cap \bar{A}| = 1$ and $x_4 \in V_{\geq 6}$, which implies that there is no admissible vertex of $(x_1; \bar{A})$. This contradicts the previous assertion and this contradiction proves (3) and Lemma 5 is proved. □

The following is an immediate corollary from Lemmas 3 and 5.

**Lemma 6.** *Let $x$ be a vertex of a contraction-critically 5-connected graph. Let $A$ be a fragment with respect to $E(x)$ such that $|\bar{A}| \geq 2$. If $N(x) \cap A \cap V_5 = \emptyset$, then there is an admissible vertex of $(x; A)$.*

**Proof.** We prove Lemma 6 by induction on $|A|$. Note that $|A| \geq 2$, since $N(x) \cap A \cap V_5 = \emptyset$. If $|A| = 2$, then, since $|\bar{A}| \geq 2$ and $N(x) \cap N(A) \neq \emptyset$, Lemma 5(2) assures us that there is an admissible vertex of $(x; A)$. Now the initial step is completed.

Next assume $|A| \geq 3$ and let $y \in N(x) \cap A$. If there is an admissible vertex of $(x, y; A)$, then we are done. Hence, assume that $\mathrm{Ad}(x, y; A) = \emptyset$. Then, by Lemma 3, we see that there is an $xy$-fragment $A'$ such that $A' \subsetneq A$. Then we see that $A'$ is an $E(x)$-fragment, $|\bar{A}'| > |\bar{A}| \geq 2$, $N(x) \cap A' \cap V_5 = \emptyset$ and $|A'| < |A|$. Hence, by the induction hypothesis, we see that $\mathrm{Ad}(x; A') \neq \emptyset$. Since $A' \subsetneq A$ and $N(x) \cap A \cap V_5 = \emptyset$, we see that an admissible vertex of $(x; A')$ is an admissible vertex of $(x; A)$. The induction step is now completed and Lemma 6 is proved. $\square$

**Lemma 7.** *Let $x$ be a vertex of a contraction-critically 5-connected graph $G$. Let $A$ be a fragment with respect to $E(x)$ such that $|\bar{A}| \geq 2$ and $|A| \geq 3$. Suppose $|N(x) \cap A| = 1$, say $N(x) \cap A = \{y\}$. If $y \notin V_5$, then there is a strongly admissible vertex of $(x, y; A)$.*

**Proof.** Assume that there is no strongly admissible vertex of $(x, y; A)$. By Corollary 4, we know that $\mathrm{Ad}(x, y; A) \neq \emptyset$, say $z \in \mathrm{Ad}(x, y; A)$. Let $B = \{z\}$ and $T = N(z)$. By the assumption, $z$ is not strongly admissible, which implies $|N(z) \cap \bar{A}| \geq 2$. Since $|N(z) \cap A| \geq 2$ and $|N(z) \cap \bar{A}| \geq 2$, we observe that $|N(z) \cap A| = |N(z) \cap \bar{A}| = 2$ and $S \cap T = \{x\}$. Let $N(z) \cap A = \{y, u\}$ and $S \cap \bar{B} = \{v_1, v_2, v_3\}$.

Claim 7.1. $|A| = 3$.

**Proof.** Assume $|A| \geq 4$. Let $A' = A - \{y\}$ and $S' = (S - \{x\}) \cup \{y\}$. Then we observe that $A'$ is a $zy$-fragment, $|\bar{A}'| > |\bar{A}| \geq 2$, $|A'| = |A| - 1 \geq 3$ and $N(z) \cap A' = \{u\}$. Hence, by Corollary 4, we see that there is an admissible vertex of $(z, u; A')$. But we know that $N(z) \cap S' = \{y\}$ and $y \in V_{\geq 6}$, which implies that there is no admissible vertex of $(z, u; A')$. This contradicts the previous assertion and it is shown that $|A| = 3$. $\square$

Let $A = \{y, u, w\}$, then $A \cap T = \{y, u\}$ and $A \cap \bar{B} = \{w\}$. In this situation, since $N(x) \cap (A \cap \bar{B}) = \emptyset$, we observe that $w \in V_5$ and $N(w) = \{y, u, v_1, v_2, v_3\}$. Let $A' = \{u, w\}$ and let $S' = N(A') = \{z, y, v_1, v_2, v_3\}$.

Claim 7.2. *$z$ is tractable with $A'$.*

**Proof.** Assume $z$ is not tractable with $A'$. Let $C$ be a $zu$-fragment. Since $S' - \{z\} = \{y, v_1, v_2, v_3\} \subset N(w)$, we observe that $C$ is a $(z; A')$-fit fragment. Since $z$ is not tractable with $A'$, we see that either $|S' \cap C| \leq 1$ or $|S' \cap \bar{C}| \leq 1$. Without loss of generality, we may assume that $|S' \cap C| \leq 1$. Since $C$ is $(z; A')$-fit, we know that $S' \cap C \neq \emptyset$, which implies $|S' \cap C| = 1$, say $S' \cap C = \{y'\}$. Then, since $|S' \cap C| < |A' \cap N(C)| = |A'| = 2$, Lemma 2(2) assures us that $\bar{A}' \cap C = \emptyset$, which implies $C = S' \cap C = \{y'\}$. Then, we observe that $y' \in N(z) \cap S' \cap V_5$, which contradicts the fact that $N(z) \cap S' = \{y\}$ and $y \notin V_5$. This contradiction proves Claim 7.2. $\square$

Let $C$ be a $zu$-fragment. By Claim 7.2, we know that $A' \subset \underline{N}(C)$, $|S' \cap C| = |S' \cap \bar{C}| = 2$ and $S' \cap N(C) = \{z\}$. Without loss of generality, we may assume that $S' \cap C = \{y, v_1\}$ and $S' \cap \bar{C} = \{v_2, v_3\}$. Then, we observe that $N(y) \cap \{v_2, v_3\} = \emptyset$, which implies that $N(y) \subset (S \cup A) - \{y, v_2, v_3\} = \{x, z, u, w, v_1\}$. Now we have $\deg_G(y) = |N(y)| \leq 5$, which contradicts the fact that $y \in V_{\geq 6}$. This contradiction proves Lemma 7. $\square$

## 3. Proof of Proposition 1

Let $G$ be a contraction-critically 5-connected graph and let $x \in V(G)$ such that $x \notin V_5$ and $|N(x) \cap V_5| = 2$, say $N(x) \cap V_5 = \{y_1, y_2\}$. By way of contradiction, assume $y_1y_2 \in E(G)$. Let $E'(x) = E(x) - \{xy_1, xy_2\}$. Let $A$ be a $E'(x)$-fragment of $G$. Then, since $y_1y_2 \in E(G)$, we observe that either $A \cap \{y_1, y_2\} = \emptyset$ or $\bar{A} \cap \{y_1, y_2\} = \emptyset$. Hence there is a $\{y_1, y_2\}$-free $E'(x)$-fragment of $G$.

Claim 1. *Let $xz \in E'(x)$ and let $A$ be a minimal $\{y_1, y_2\}$-free $xz$-fragment.*
*Then,* (1) $\bar{A} \cap \{y_1, y_2\} \neq \emptyset$ *and* (2) if $|\bar{A}| \geq 2$, *then* $|A| \geq 3$.

**Proof.** (1) Assume that $\bar{A} \cap \{y_1, y_2\} = \emptyset$. Then $\{y_1, y_2\} \subset N(A)$. Then, since $N(x) \cap A \cap V_5 = \emptyset$, we observe that $|A| \geq 2$. Since $N(x) \cap \bar{A} \cap V_5 = \emptyset$, we also see that $|\bar{A}| \geq 2$. We show that $|N(y_1) \cap A| \geq 2$. Assume $|N(y_1) \cap A| = 1$, say $N(y_1) \cap A = \{u\}$. Let $A' = A - \{u\}$. Then we see that $A'$ is a $\{y_1, y_2\}$-free $xz$-fragment of $G$ and $A' \subsetneq A$, which contradicts the minimality of $A$. This contradiction proves that $|N(y_1) \cap A| \geq 2$. By symmetry, we have $|N(y_2) \cap A| \geq 2$. Hence $|N(y_1) \cap (N(A) \cup A)| \geq |\{x, y_2\}| + |N(y_1) \cap A| \geq 4$, which implies $|N(y_1) \cap \bar{A}| = 1$. Similarly we have $|N(y_2) \cap \bar{A}| = 1$. Hence, we see that $\{y_1, y_2\} \cap \mathrm{Ad}(x; \bar{A}) = \emptyset$. Since $N(x) \cap V_5 = \{y_1, y_2\}$, this implies that $\mathrm{Ad}(x; \bar{A}) = \emptyset$. On the other hand, since $|A| \geq 2$ and $N(x) \cap \bar{A} \cap V_5 = \emptyset$, Lemma 6 assures us that there is an admissible vertex of $(x; \bar{A})$, which contradicts the previous assertion. This contradiction proves (1).

(2) Assume that $|\bar{A}| \geq 2$ and $|A| \leq 2$. Since $N(x) \cap V_5 = \{y_1, y_2\}$ and $A$ is $\{y_1, y_2\}$-free, we observe that $N(x) \cap A \cap V_5 = \emptyset$, which implies $A \cap V_{\geq 6} \neq \emptyset$ and $|A| \geq 2$. Hence we see that $|A| = 2$. Since $|\bar{A}| \geq 2$, $|A| = 2$ and $A \cap V_{\geq 6} \neq \emptyset$, applying Lemma 5(3), we see that $|N(A) \cap V_5| \geq 4$. However, since $\{x, z\} \subset N(A) \cap V_{\geq 6}$, we observe that $|N(A) \cap V_5| \leq 3$, which contradicts the previous assertion. This contradiction shows (2) and Claim 1 is proved. $\square$

Claim 2. *There is a $y_i$-opposite $E'(x)$-fragment of $G$ for each $i \in \{1, 2\}$.*

**Proof.** Let $A$ be a minimal $\{y_1, y_2\}$-free $E'(x)$-fragment of $G$ and let $S = N(A)$. Then Claim 1 assures us that $\bar{A} \cap \{y_1, y_2\} \neq \emptyset$. Hence, by symmetry, we may assume that $A$ is a minimal $y_1$-opposite $E'(x)$-fragment of $G$. Say $xz \in E(S) \cap E'(x)$. Since $N(x) \cap A \cap V_5 = \emptyset$, we see that $|A| \geq 2$. Assume that there is no $y_2$-opposite $E'(x)$-fragment. Let $u \in N(x) \cap A$ and let $B$ be a minimal $\{y_1, y_2\}$-free $xu$-fragment of $G$. Then, since $N(x) \cap B \cap V_5 = \emptyset$, we observe that $|B| \geq 2$. Applying Claim 1 with the roles of $xz$ and $A$ replaced by $xu$ and $B$, respectively, we see that $\{y_1, y_2\} \cap \bar{B} \neq \emptyset$, which implies $y_1 \in \bar{A} \cap \bar{B}$ since $B$ is not a $y_2$-opposite $E'(x)$-fragment. Since both $A$ and $B$ are $\{y_1, y_2\}$-free and neither $A$ nor $B$ is $y_2$-opposite, we observe that $y_2 \in S \cap T$.

Subclaim 2.1. (1) $A \cap B = \emptyset$ *and* (2) $|S \cap B| = 1$.

**Proof.** (1) Assume $A \cap B \neq \emptyset$. Then, since neither $A \cap B$ nor $\bar{A} \cap \bar{B}$ is empty, Lemma 2(1) assures us that $A \cap B$ is $y_1$-opposite $E'(x)$-fragment, which contradicts the minimality of $A$ since $u \notin A \cap B$. This contradiction proves (1).

(2) Assume $|S \cap B| \geq 2$. Then, since $\bar{A} \cap \bar{B} \neq \emptyset$, Lemma 2(2) assures us $|\bar{A} \cap T| \geq |S \cap B| \geq 2$, which implies $|A \cap T| = 1$ since $\{x, y_2\} \subset S \cap T$. Hence, we observe that $|A \cap T| < |S \cap B|$ and again Lemma 2(2) assures us that $A \cap \bar{B} = \emptyset$, which implies that $|A| = |A \cap T| = 1$. This contradicts the fact that $|A| \geq 2$ and it is shown that $|S \cap B| = 1$. □

By Subclaim 2.1, we know that $A \cap B = \emptyset$ and $|S \cap B| = 1$, which implies $\bar{A} \cap B \neq \emptyset$ since $|B| \geq 2$. Hence, by Lemma 2(2), we see that $|A \cap T| \leq |S \cap B| = 1$, which implies $A \cap \bar{B} \neq \emptyset$ since $|A| \geq 2$. Now we observe that neither $\bar{A} \cap B$ nor $A \cap \bar{B}$ is empty. Hence, by Lemma 2(1), we see that $A \cap \bar{B}$ is a $y_1$-opposite $E'(x)$-fragment, which contradicts the minimality of $A$ since $u \notin A \cap \bar{B}$. This contradiction shows the existence of a $y_2$-opposite $E'(x)$-fragment and Claim 2 is proved. □

Let $A$ be a minimum $y_1$-opposite $E'(x)$-fragment and let $B$ be a minimum $y_2$-opposite $E'(x)$-fragment. Let $S = N(A)$ and let $T = N(B)$.

Claim 3. $|\bar{A}| \geq 2$ *and* $|\bar{B}| \geq 2$.

**Proof.** We show $|\bar{A}| \geq 2$. Assume $|\bar{A}| = 1$. Then $\bar{A} = \{y_1\}$ and $S = N(y_1)$. Let $S = N(y_1) = \{x, y_2, z_1, z_2, z_3\}$. Let $B' = G - N(y_2) \cup \{y_2\}$ and let $T' = N(B')$. Then $\bar{B}' = \{y_2\}$. Let $T' = N(y_2) = \{x, y_1, u_1, u_2, u_3\}$. By Theorem E, we know that $|N(y_1) \cap V_5| \geq 2$ and $|N(y_2) \cap V_5| \geq 2$. Hence, since $x \notin V_5$, we observe that neither $\{z_1, z_2, z_3\} \cap V_5$ nor $\{u_1, u_2, u_3\} \cap V_5$ is empty. Without loss of generality, we may assume $z_3, u_3 \in V_5$.

Subclaim 3.1. $N(x) \subset S \cup T' \cup \{y_1, y_2\}$.

**Proof.** Assume $N(x) \cap (A \cap B') \neq \emptyset$. Let $E''(x) = E_G(x, A \cap B')$. Then $E''(x) \neq \emptyset$. Let $C$ be a minimum $\{y_1, y_2\}$-free $E''(x)$-fragment. Say $xv \in E(N(C)) \cap E_G(x, A \cap B')$. Note that $v \in V_{\geq 6}$ since $N(x) \cap V_5 = \{y_1, y_2\}$. Since $v \in A \cap B'$, $\bar{A} = \{y_1\}$ and $\bar{B}' = \{y_2\}$, we observe that $N(v) \cap \{y_1, y_2\} = \emptyset$. Then, applying Claim 1 with the role of $A$ replaced by $C$, we see that $\{y_1, y_2\} \cap \bar{C} \neq \emptyset$. Since $v \in N(C)$ and $N(v) \cap \{y_1, y_2\} = \emptyset$, we see that $|\bar{C}| \geq 2$. If $y_1 \in \bar{C}$, then we observe that $\bar{A} \cap N(C) = \bar{A} \cap C = S \cap C = \emptyset$, which implies that $C$ is a $y_1$-opposite $E'(x)$-fragment such that $C \subset A - \{v\}$. This contradicts the minimality of $A$. Hence $y_1 \notin \bar{C}$, which implies that $y_2 \in S \cap \bar{C}$ and $y_1 \in N(C)$. Since $|\bar{C}| \geq 2$, applying Claim 1(2) with the roles of $A$ and $E'(x)$ replaced by $C$ and $E''(x)$, respectively, we see that $|C| \geq 3$. Since $|\bar{C}| \geq 2$, $|C| \geq 3$ and $N(x) \cap C \cap V_5 = \emptyset$, applying Lemma 6 with the role of $A$ replaced by $C$, we see that $\mathrm{Ad}(x; C) \neq \emptyset$. Since $N(x) \cap N(C) \cap V_5 = \{y_1\}$, we observe that $y_1$ is an admissible vertex of $(x; C)$, which implies that $|N(y_1) \cap C| = |S \cap C| \geq 2$.

Subsubclaim 3.1.1. $|S \cap \bar{C}| \geq 2$.

**Proof.** Assume $|S \cap \bar{C}| = 1$. Then, since $|\bar{C}| \geq 2$, we observe that $A \cap \bar{C} \neq \emptyset$. Then, since $|S \cap \bar{C}| = |\bar{A} \cap N(C)|$, applying Lemma 2(2), we see that $|(S \cap \bar{C}) \cup (S \cap N(C)) \cup (A \cap N(C))| = 5$. Hence we observe that $A \cap \bar{C}$ is a $y_1$-opposite $E'(x)$-fragment, which contradicts the minimality of $A$ since $v \notin A \cap \bar{C}$. This contradiction proves Subsubclaim 3.1.1. □

Since $|N(y_1) \cap C| = |S \cap C| \geq 2$, Subsubclaim 3.1.1 assures us that $|S \cap C| = |S \cap \bar{C}| = 2$ and $S \cap N(C) = \{x\}$.

Subsubclaim 3.1.2. $N(x) \cap (A \cap C) = \emptyset$.

**Proof.** Assume $v' \in N(x) \cap (A \cap C)$. Then, there is no admissible vertex of $(x, v'; C)$ since $N(x) \cap N(C) \cap V_5 = \{y_1\}$ and $y_1 v' \notin E(G)$. Then, since $|\bar{C}| \geq 2$, $|C| \geq 3$ and $\mathrm{Ad}(x, v'; C) = \emptyset$, applying Lemma 3 with the roles of $y$ and $A$ replaced by $v'$ and $C$, respectively, we see that there is an $xv'$-fragment $C'$ such that $C' \subsetneq C$, which contradicts the minimality of $C$. This contradiction shows that $N(x) \cap (A \cap C) = \emptyset$. □

Subsubclaim 3.1.2 assures us that $N(x) \cap C \subset S \cap C$.

Subsubclaim 3.1.3. $N(x) \cap C = S \cap C$.

**Proof.** Assume $N(x) \cap C \subsetneq S \cap C$. Then, since $|S \cap C| = 2$, we observe that $|N(x) \cap C| = 1$. Since $|\bar{C}| \geq 2$, $|C| \geq 3$, $|N(x) \cap C| = 1$ and $N(x) \cap C \subset V_{\geq 6}$, applying Lemma 7 with the role of $A$ replaced by $C$, we see that $y_1$ is a strongly admissible vertex of $(x; C)$. Hence $|N(y_1) \cap \bar{C}| = 1$, which contradicts Subsubclaim 3.1.1. This contradiction proves $N(x) \cap C = S \cap C$. □

We are in a position to complete the proof of Subclaim 3.1. By Subsubclaim 3.1.3, we know $N(x) \cap C = N(y_1) \cap C$, which implies that $N(y_1) \cap C \cap V_5 = \emptyset$ since $N(x) \cap C \cap V_5 = \emptyset$. Since $|\bar{C}| \geq 2$, $|C| \geq 3$ and $N(y_1) \cap C \cap V_5 = \emptyset$, applying Lemma 6 with the roles of $x$ and $A$ replaced by $y_1$ and $C$, respectively, we see that $\mathrm{Ad}(y_1; C) \neq \emptyset$. On the other hand, since $N(y_1) \cap N(C) = \{x\}$ and $x \in V_{\geq 6}$, we see that there is no admissible vertex of $(y_1; C)$, which contradicts the previous assertion. This contradiction proves Subclaim 3.1. □

By Subclaim 3.1 we know that $N(x) \subset (S \cup T') \cup \{y_1, y_2\}$. Since $N(x) \cap V_5 = \{y_1, y_2\}$ and $z_3, u_3 \in V_5$, we see that $z_3, u_3 \notin N(x)$. Hence, we observe that $y_1, y_2, z_1, z_2, u_1, u_2$ are distinct and $N(x) = \{y_1, y_2, z_1, z_2, u_1, u_2\}$ because $|N(x)| \geq 6$. Hence, we observe that $\{z_1, z_2, u_1, u_2\} \cap V_5 = \emptyset$ since $N(x) \cap V_5 = \{y_1, y_2\}$.

Subclaim 3.2. *Let $C$ be a $y_1x$-fragment such that $|S \cap C| \geq |S \cap \bar{C}|$ and $|S \cap N(C)| \geq 2$. Then, either* (1) $\bar{C} = \{y_2\}$ *or* (2) $\{z_1, z_2\} \subset C$ *and* $z_3 = u_3$.

**Proof.** Since $\bar{A} = \{y_1\}$ and $y_1 \in N(C)$, we observe that neither $S \cap C$ nor $S \cap \bar{C}$ is empty. Hence, since $|S \cap C| \geq |S \cap \bar{C}|$ and $|S \cap N(C)| \geq 2$, we see that $|S \cap \bar{C}| = 1$, say $S \cap \bar{C} = \{v\}$.

At first we consider the case that $A \cap \bar{C} = \emptyset$. In this case $\bar{C} = S \cap \bar{C} = \{v\}$. Hence $v \in N(x) \cap S \cap V_5$, which implies that $v = y_2$. Now it is shown that if $A \cap \bar{C} = \emptyset$, then (1) holds.

Next we assume $A \cap \bar{C} \neq \emptyset$. In this case, since $|\bar{A} \cap N(C)| = |S \cap \bar{C}|$, we see that $A \cap \bar{C}$ is a fragment of $G$. If $|A \cap \bar{C}| = 1$, then $N(x) \cap (A \cap \bar{B}) \cap V_5 \neq \emptyset$, which contradicts the fact that $A$ is $\{y_1, y_2\}$-free. Hence $|A \cap \bar{C}| \geq 2$, which implies $|\bar{C}| \geq 3$. Since $S \cap C \neq \emptyset$ and $|A \cap N(C)| = |S \cap C|$, we observe that $A \cap \bar{C} \subsetneq A$. If $\{z_1, z_2\} \cap (N(C) \cup \bar{C}) \neq \emptyset$, then we observe that $A \cap \bar{C}$ is a $y_1$-opposite $E'(x)$-fragment, which contradicts the minimality of $A$. Hence we have $\{z_1, z_2\} \subset C$. Since $\{z_1, z_2\} \subset C$, we observe that $|C| \geq 2$ and either $w = y_2$ or $w = z_3$. Now we know that $|C| \geq 2$, $|\bar{C}| \geq 3$ and $N(y_1) \cap \bar{C} = \{w\}$. Hence, applying Corollary 4 with the roles of $x$ and $A$ replaced by $y_1$ and $\bar{C}$, respectively, we see that $\mathrm{Ad}(y_1, w; \bar{C}) \neq \emptyset$. If $w = z_3$, then $y_2$ is an admissible vertex of $(y_1, z_3; \bar{C})$, which implies that $y_2z_3 \in E(G)$. If $w = y_2$, then $z_3$ is an admissible vertex of $(y_1, y_2; \bar{C})$, which again implies that $y_2z_3 \in E(G)$. Thus we have $z_3 = u_3$. Now it is shown that if $A \cap \bar{C} \neq \emptyset$, then (2) holds and Subclaim 3.2 is proved. $\square$

Subclaim 3.3. $z_3 = u_3$.

**Proof.** Let $C$ be a $y_1z_3$-fragment. Then, if $x \in C$, then, since $N(y_1) \subset N(x) \cup \{x, z_3\}$, we observe that $N(y_1) \cap \bar{C} = \emptyset$, which contradicts the choice of $C$. Hence $x \notin C$. By symmetry, we see that $x \notin \bar{C}$ and hence $x \in N(C)$, which implies that $C$ is a $y_1x$-fragment and $|S \cap N(C)| \geq 2$. Without loss of generality, we may assume that $|S \cap C| \geq |S \cap \bar{C}|$. Now we can apply Subclaim 2.2. If Subclaim (2) holds, then $z_3 = u_3$ and we are done. Hence, we may assume that Subclaim 2.2(1) holds, that is $\bar{C} = \{y_2\}$. Thus $y_2z_3 \in E(G)$, which implies again $z_3 = u_3$. Subclaim 3.3 is proved. $\square$

We proceed with the proof of Claim 3. By Subclaim 3.3, we know that $z_3 = u_3$, say $w = z_3 = u_3$. Since $w \in V_5$, and $\{y_1, y_2\} \subset N(w)$, we see that $|\{z_1, z_2, u_1, u_2\} \cap N(w)| \leq 3$. Without loss of generality, we may assume that $z_1 \notin N(w)$. Let $C$ be a fragment with respect to $y_1z_1$. We show that $x \in N(C)$. Assume $x \in C$. Then we observe that $N(y_1) \cap \bar{C} = \{w\}$ since $N(y_1) \subset N(x) \cup \{x, w\}$. Since $xy_2, y_2w \in E(G)$, we observe that $y_2 \in N(C)$. Furthermore, since $\{z_1, z_2, u_1, u_2\} \subset N(x)$, we see that $N(\{y_1, y_2\}) \cap \bar{C} = \{w\}$. Hence, applying Lemma 1 with the roles of $S$ and $A$ replaced by $\{y_1, y_2\}$ and $\bar{C}$, respectively, we see that $\bar{C} = \{w\}$ and $N(C) = N(w)$. This contradicts the fact that $z_1 \notin N(w)$, and this contradiction proves $x \notin C$. By symmetry, we see that $x \notin \bar{C}$. Now it is shown that $x \in N(C)$.

Without loss of generality, we may assume that $|S \cap C| \geq |S \cap \bar{C}|$. Applying Subclaim 3.2, we see that $\bar{C} = \{y_2\}$ since the fact that $z_1 \in N(C)$ assures us that Subclaim 3.2(2) does not occur. Hence, we observe that $y_2z_1 \in E(G)$. However $N(y_2) = \{x, y_1, u_1, u_2, w\}$ and $y_1, y_2, z_1, z_2, u_1, u_2$ are distinct, which implies $y_2z_1 \notin E(G)$. This contradicts the previous assertion and we have shown that $|\bar{A}| \geq 2$.

Using the same arguments with the roles of $A$ and $B'$ replaced by $B$ and $G - N(y_1) \cup \{y_1\}$, respectively, we can show that $|\bar{B}| \geq 2$. Now Claim 3 is proved. $\square$

Recall that $A$ and $B$ are a minimum $y_1$-opposite $E'(x)$-fragment and a minimum $y_2$-opposite $E'(x)$-fragment, respectively, and $S = N(A)$ and $T = N(B)$. By Claim 3, we know that $|\bar{A}|, |\bar{B}| \geq 2$. Then, applying Claim 1(2), we see that $|A|, |B| \geq 3$.

Claim 4. (1) $N(x) \cap A \subset N(y_2) \cap A$ *and* (2) $N(x) \cap B \subset N(y_1) \cap B$.

**Proof.** We show (1). Assume that there is a vertex $v \in N(x) \cap A$ such that $v \notin N(y_2)$. Since $N(x) \cap S \cap V_5 = \{y_2\}$, there is no admissible vertex of $(x, v; A)$. Then, since $|\bar{A}| \geq 2$ and $|A| \geq 3$, applying Lemma 3 with the role $y$ replaced by $v$, we see that there is an $xv$-fragment $A'$ such that $A' \subsetneq A$. Then $A'$ is a $y_1$-opposite $E'(x)$-fragment such that $A' \subsetneq A$, which contradicts that minimality of $A$. This contradiction shows (1).

By the similar arguments, we can show (2) and Claim 4 is proved. $\square$

Since $A$ is $y_1$-opposite and $B$ is $y_2$-opposite, Claim 4 assures us that $N(x) \cap (A \cap B) = \emptyset$.

Claim 5. *Neither $\bar{A} \cap B$ nor $A \cap \bar{B}$ is empty.*

**Proof.** Assume that either $\bar{A} \cap B = \emptyset$ or $A \cap \bar{B} = \emptyset$. Without loss of generality, we may assume that $\bar{A} \cap B = \emptyset$. We show $A \cap B = \emptyset$. Assume $A \cap B \neq \emptyset$. Then, since $N(x) \cap (A \cap B) = \emptyset$, we observe that $|(S \cap B) \cup (S \cap T) \cup (S \cap T)| \geq 6$. Then Lemma 2(1) assures us that $\bar{A} \cap \bar{B} = \emptyset$, which implies $\bar{A} = \bar{A} \cap T$. Since $|\bar{A} \cap T| = |\bar{A}| \geq 2$, by Lemma 2(2), we observe that $|S \cap B| \geq |\bar{A} \cap T| + 1 = 3$, which implies $|S \cap \bar{B}| = |S| - |S \cap T| - |S \cap B| \leq 1$. Then we observe that $|S \cap \bar{B}| < |\bar{A} \cap T|$ and Lemma 2(2) again assures us that $A \cap \bar{B} = \emptyset$, which implies that $|\bar{B}| = |S \cap \bar{B}| \leq 1$. This contradicts Claim 3 and it is shown that $A \cap B = \emptyset$. Since $A \cap B = \emptyset$, $\bar{A} \cap B = \emptyset$ and $|B| \geq 3$, we observe that $|S \cap B| = |B| \geq 3$. Next we show that $|A \cap T| \geq 3$. Assume $|A \cap T| \leq 2$. Then, $|A \cap T| < |S \cap B|$ and Lemma 2(2) assures us that $A \cap \bar{B} = \emptyset$, which implies $|A| = |A \cap T| \leq 2$. This contradicts the assertion before Claim 4, and it is shown that $|A \cap T| \geq 3$. Since $|A \cap T| \geq 3$, we observe that $|\bar{A} \cap T| = |T| - |S \cap T| - |A \cap T| \leq 1$. Hence, since $|\bar{A} \cap T| < |S \cap B|$, we see that $\bar{A} \cap \bar{B} = \emptyset$, which implies that $\bar{A} = \bar{A} \cap T$. Thus $|\bar{A}| = |\bar{A} \cap T| \leq 1$, which contradicts Claim 3. This contradiction proves Claim 5. $\square$

By Claim 5 we know that neither $\bar{A} \cap B$ nor $A \cap \bar{B}$ is empty. Then, Lemma 2(1) assures us that both $\bar{A} \cap B$ and $A \cap \bar{B}$ are fragments of $G$. We show that $S \cap B = A \cap B = A \cap T = \emptyset$. Since $A$ is an $E'(x)$-fragment, $E(S) \cap E'(x) \neq \emptyset$, say $xz \in E(S) \cap E'(x)$. If $z \in S \cap (T \cup \bar{B})$, then $A \cap \bar{B}$ is a $y_1$-opposite $E'(x)$-fragment. Then, the minimality of $A$ assures us $A = A \cap \bar{B}$, which implies that $S \cap B = A \cap B = A \cap T = \emptyset$. Hence, $z \in S \cap B$, which implies that $\bar{A} \cap B$ is a $y_2$-opposite $E'(x)$-fragment. In this case, the minimality of $B$ assures us $B = \bar{A} \cap B$, which implies again that $S \cap B = A \cap B = A \cap T = \emptyset$. Now we know that $A = A \cap \bar{B}$, $B = \bar{A} \cap B$ and $S \cap B = A \cap B = A \cap T = \emptyset$. Since $A = A \cap \bar{B}$, $B = \bar{A} \cap B$ and $|A|, |B| \geq 3$, we see that $|\bar{A}|, |\bar{B}| \geq 3$.

Claim 6. $N(y_2) \cap \bar{A} = \{y_1\}$, $N(y_1) \cap \bar{B} = \{y_2\}$,

**Proof.** We show that $N(y_2) \cap \bar{A} = \{y_1\}$. Since $y_1 \in N(y_2) \cap \bar{A}$, it suffices to show that $|N(y_2) \cap \bar{A}| = 1$. If $|N(y_2) \cap A| = 3$, then $|N(y_2) \cap \bar{A}| = |N(y_2)| - |N(y_2) \cap S| - |N(y_2) \cap A| = 1$ and we are done. Hence we may assume that $|N(y_2) \cap A| = 2$. Then, by Claim 4, we observe that $|N(x) \cap A| \leq |N(y_2) \cap A| = 2$. If $|N(x) \cap A| = 1$, then we see that $y_2$ is a strongly admissible vertex of $(x; A)$, which implies that $|N(y_2) \cap \bar{A}| = 1$. Hence we may assume that $|N(x) \cap A| = 2$. In this situation, by Claim 4, we see that $N(x) \cap A = N(y_2) \cap A$, which implies that $N(y_2) \cap A \cap V_5 = \emptyset$ since $N(x) \cap A \cap V_5 = \emptyset$. Since $|\bar{A}| \geq 2$ and $N(y_2) \cap A \cap V_5 = \emptyset$, applying Lemma 6 with the role of $x$ replaced by $y_2$, we see that $\mathrm{Ad}(y_2; A) \neq \emptyset$, which implies that $y_2$ has a neighbor other than $x$ in $S$. Thus we observe that $|N(y_2) \cap S| \geq 2$, which implies that $|N(y_2) \cap \bar{A}| = 1$. Hence, it is shown $N(y_2) \cap \bar{A} = \{y_1\}$.

By the similar arguments, we can show that $N(y_1) \cap \bar{B} = \{y_2\}$, and Claim 6 is proved. $\square$

Since $|\bar{A}|, |A| \geq 3$ and $N(y_2) \cap \bar{A} = \{y_1\}$, applying Corollary 4 with the roles of $x, y$ and $A$ replaced by $y_2, y_1$ and $\bar{A}$, respectively, we see that $\mathrm{Ad}(y_2, y_1; \bar{A}) \neq \emptyset$. Since $|\bar{B}|, |B| \geq 3$ and $N(y_1) \cap \bar{B} = \{y_2\}$, applying Corollary 4 with the roles of $x, y$ and $A$ replaced by $y_1, y_2$ and $\bar{B}$, respectively, we also see that $\mathrm{Ad}(y_1, y_2; \bar{B}) \neq \emptyset$. Say $w \in \mathrm{Ad}(y_2, y_1; \bar{A})$. Then, since $w \in N(y_1) \cap N(y_2)$, we observe that $N(y_2) = (N(y_2) \cap A) \cup \{x, w, y_1\}$ and $N(y_1) = (N(y_1) \cap B) \cup \{x, w, y_2\}$, which implies $w \in N(y_1) \cap N(y_2)$ and $\mathrm{Ad}(y_2, y_1; \bar{A}) = \mathrm{Ad}(y_1, y_2; \bar{A}) = \{w\}$.

Claim 7. $|N(w) \cap A| \geq 2$ and $|N(w) \cap B| \geq 2$.

**Proof.** We show $|N(w) \cap A| \geq 2$. Assume $|N(w) \cap A| = 1$, say $N(w) \cap A = \{u_1\}$.

Subclaim 7.1. $|N(x) \cap A| = 1$.

**Proof.** Assume $|N(x) \cap A| = 2$. Then, since $|N(y_2) \cap A| = 2$, Claim 4 assures us that $N(y_2) \cap A = N(x) \cap A$, which implies $N(y_2) \cap A \cap V_5 = \emptyset$ since $N(x) \cap A \cap V_5 = \emptyset$. Since $|\bar{A}| \geq 3$, applying Lemma 6 with the role of $x$ replaced by $y_2$, we see that $\mathrm{Ad}(y_2; A) \neq \emptyset$. Since $N(y_2) \cap S \cap V_5 = \{w\}$, we observe that $w$ is an admissible vertex of $(y_2; A)$, which implies $|N(w) \cap A| \geq 2$. This contradicts the assumption and Subclaim 7.1 is proved. $\square$

By Subclaim 7.1, we know that $|N(x) \cap A| = 1$, say $N(x) \cap A = \{u_2\}$. We show that $u_1 \neq u_2$. Assume $u_1 = u_2$. Then, since $N(\{x, w\}) \cap A = \{u_1\}$, applying Lemma 1 with the role $S$ replaced by $\{x, w\}$, we see that $A = \{u_1\}$, which contradicts the fact that $|A| \geq 3$. Now it is shown that $u_1 \neq u_2$.

Subclaim 7.2. $|A| \geq 4$.

**Proof.** Assume $|A| = 3$. Recall that $zx \in E(S) \cap E'(x)$. Since $w \in V_5$, we observe that $z \neq w$. Let $A_1 = A - \{u_1\}$. Then, since $N(w) \cap A = \{u_1\}$, we observe that $N(A_1) = (S - \{w\}) \cup \{u_1\}$. This implies that $A_1$ is a fragment of $G$. Since $|A| = 3$ and $u_2 \in A_1 \cap V_{\geq 6}$, we observe that $|A_1| = 2$ and $A_1 \cap V_{\geq 6} \neq \emptyset$. Then, since $|\bar{A}_1| \geq 3$, $|A_1| = 2$ and $A_1 \cap V_{\geq 6} \neq \emptyset$, applying Lemma 5(3) with the role of $A$ replaced by $A_1$, we see that $|N(A_1) \cap V_5| \geq 4$. However, since $\{x, z\} \subset N(A_1) \cap V_{\geq 6}$, we observe that $|N(A_1) \cap V_{\geq 6}| \geq 2$, which implies that $|N(A_2) \cap V_5| \leq 3$. This contradicts the previous assertion. and Subclaim 7.2 is proved. $\square$

Recall that $N(x) \cap A = \{u_2\}$ and $y_2$ is an admissible vertex of $(x, u_2; A)$. Hence, we observe that $u_2 \in N(x) \cap N(y_2) \cap A$. Let $A_2 = A - \{u_2\}$. Then $A_2$ is a fragment of $G$ since $N(x) \cap A = \{u_2\}$. Subclaim 7.2 assures us that $|A| \geq 4$, which implies that $|A_2| \geq 3$. Since $|N(y_2) \cap A| = 2$ and $u_2 \in N(y_2) \cap A$, we observe that $|N(y_2) \cap A_2| = 1$. Then, since $|\bar{A}_2| \geq 2$, $|A_2| \geq 3$ and $|N(y_2) \cap A_2| = 1$, applying Corollary 4 with the roles $x$ and $A$ replaced by $y_2$ and $A_2$, respectively, we see that $\mathrm{Ad}(y_2; A_2) \neq \emptyset$. Then, since $N(y_2) \cap S \cap V_5 = \{w\}$, $w$ is an admissible vertex of $(y_2; A_2)$, which implies $|N(w) \cap A_2| \geq 2$. This contradicts the assumption that $|N(w) \cap A| = 1$. This contradiction proves $|N(w) \cap A| \geq 2$.

By the similar arguments, we can show $|N(w) \cap B| \geq 2$ and Claim 7 is proved. $\square$

We are in a position to complete the proof of Proposition 1. Claim 7 assures us that $|N(w) \cap A| \geq 2$ and $|N(w) \cap B| \geq 2$. Since $A \cap B = \emptyset$, we observe that $|N(w) \cap (A \cup B)| = |N(w) \cap A| + |N(w) \cap B| \geq 4$. Since $\{y_1, y_2\} \subset N(w)$ and both $A$ and $B$ are $\{y_1, y_2\}$-free, we see that $|N(w)| \geq |N(w) \cap (A \cup B \cup \{y_1, y_2\})| = |N(w) \cap (A \cup B)| + |N(w) \cap \{y_1, y_2\}| \geq 6$, which contradicts the fact that $w \in V_5$. This is the final contradiction and the proof of Proposition 1 is completed. $\square$

## 4. Proof of Proposition 2

In this section we prove Proposition 2.

Let $G$ be a contraction-critically 5-connected graph. Let $x$ be a vertex of $G$ such that $x \notin V_5$ and $|N(x) \cap V_5| = 2$. Let $N(x) \cap V_5 = \{y_1, y_2\}$. Then, Proposition 1 assures us that $y_1y_2 \notin E(G)$.

Claim 1. $|G| \geq 10$.

**Proof.** Assume $|G| \leq 9$. Let $A$ be a $\{y_2\}$-free $xy_1$-fragment and let $S = N(A)$. Then, since $N(x) \cap A \cap V_5 = \emptyset$, we observe that $|A| \geq 2$. If $\bar{A} \cap \{y_2\} = \emptyset$, then, by the same reason, we see that $|\bar{A}| \geq 2$. Otherwise, if $y_2 \in \bar{A}$, then we also see that $|\bar{A}| \geq 2$ since $y_1y_2 \notin E(G)$. Hence, since $|G| \leq 9$, we observe that $|A| = |\bar{A}| = 2$. Let $A = \{u_1, u_2\}$ and $\bar{A} = \{v_1, v_2\}$. Then, since $|A| = |\bar{A}| = 2$ and $A \cap V_{\geq 6} \neq \emptyset$, applying Lemma 5, we see that $|S \cap V_5| \geq 4$, which implies that $S - \{x\} \subset V_5$. Since $x \notin V_5$ and $|A \cup \bar{A}| = 4$, we observe that $|N(x) \cap S| \geq 2$, which implies $N(x) \cap S = \{y_1, y_2\}$ and $N(x) = \{y_1, y_2, u_1, u_2, v_1, v_2\}$. Let $S = \{x, y_1, y_2, w_1, w_2\}$. Since $N(x) \cap V_5 = \{y_1, y_2\}$, we observe that $\{u_1, u_2, v_1, v_2\} \subset V_{\geq 6}$, which implies that $S \subset N(u_1) \cap N(u_2) \cap N(v_1) \cap N(v_2)$. Hence, we see that $N(y_1) = N(y_2) = \{x, u_1, u_2, v_1, v_2\}$, which implies $N(\{w_1, w_2\}) = \{u_1, u_2, v_1, v_2\}$. This contradicts the assumption that $G$ is 5-connected. This contradiction proves Claim 1. □

We start with the following observation, which has a somewhat technical appearance but is useful.

Claim 2. *Let $y \in \{y_1, y_2\}$. Let $A$ be an xy-fragment such that $|\bar{A}| \geq 2$ and $|A| \geq 3$. Suppose $|N(y) \cap A| = 2$, $N(y) \cap N(A) = \{x\}$ and $N(x) \cap N(y) \cap A \neq \emptyset$. Then, for each $u \in N(x) \cap N(y) \cap A$, there is an xy-fragment $A'$ such that $A' \subsetneq A$ and $N(y) \cap A' = \{u\}$.*

**Proof.** Let $S = N(A)$. Let $N(y) \cap A = \{u, u'\}$ and $u \in N(x) \cap N(y) \cap A$. Since $N(y) \cap S = \{x\}$ and $x \notin V_5$, there is no admissible vertex of $(y, u'; A)$. Hence, since $|\bar{A}| \geq 2$, $|A| \geq 3$ and $\mathrm{Ad}(y, u'; A) = \emptyset$, applying Lemma 3 with the roles of $x$ and $y$ replaced by $y$ and $u'$, respectively, we see that there is a $yu'$-fragment $A'$ such that $A' \subsetneq A$. Since $N(y) \cap A' \neq \emptyset$, $N(y) \cap A = \{u, u'\}$ and $A' \subsetneq A$, we observe that $N(y) \cap A' = \{u\}$. Since $x \notin A$, $A' \subsetneq A$, $u \in A'$ and $xu \in E(G)$, we see that $x \in N(A')$, which implies that $A'$ is an $xy$-fragment. Hence $A'$ is a desired fragment and Claim 2 is proved. □

Claim 3. *There is a $y_2$-opposite $xy_1$-fragment.*

**Proof.** Assume that there is no $y_2$-opposite $xy_1$-fragment. Let $A$ be a fragment with respect to $xy_1$ and let $S = N(A)$. Then, since neither $A$ nor $\bar{A}$ is $y_2$-opposite, we observe that $\{x, y_1, y_2\} \subset S$, which implies $N(x) \cap V_5 \subset S$. Hence we see that $N(x) \cap A \cap V_5 = N(x) \cap \bar{A} \cap V_5 = \emptyset$, which implies $|A|, |\bar{A}| \geq 2$. We choose a fragment $A$ with respect to $xy_1$ so that $|N(y_1) \cap A|$ is as small as possible. Furthermore, subject to the above condition, we choose $A$ so that $|A|$ is as large as possible.

Subclaim 3.1. $|N(y_2) \cap A| = |N(y_2) \cap \bar{A}| = 2$.

**Proof.** Assume $|N(y_2) \cap A| = 1$, say $N(y_2) \cap A = \{u\}$. Then, since $|A| \geq 2$, $S' = (S - \{y_2\}) \cup \{u\}$ is a 5-cutset of $G$ and $A - \{u\}$ is a $y_2$-opposite $xy_1$-fragment, which contradicts the assumption. Hence $|N(y_2) \cap A| \geq 2$. By symmetry, we see that $|N(y_2) \cap \bar{A}| \geq 2$. Since $x \in N(y_2) \cap S$ and $|N(y_2)| = 5$, we have the desired conclusion. □

Subclaim 3.2. (1) $|N(y_1) \cap A| \leq 2$. *Furthermore, if* $|N(y_1) \cap A| = 2$, *then* $|N(y_1) \cap \bar{A}| = 2$ *and* $|A| \geq |\bar{A}|$, (2) $|A| \geq 3$.

**Proof.** (1) By the choice of $A$, we know that $|N(y_1) \cap A| \leq |N(y_1) \cap \bar{A}|$. Since $x \in N(y_1) \cap S$ and $|N(y_1)| = 5$, we observe that $|N(y_1) \cap A| + |N(y_1) \cap \bar{A}| \leq 4$. Hence, since $|N(y_1) \cap A| \leq |N(y_1) \cap \bar{A}|$, we see that $|N(y_1) \cap A| \leq 2$. Now the former part of (1) is shown. Next assume that $|N(y_1) \cap A| = 2$. Then, since $|N(y_1) \cap A| \leq |N(y_1) \cap \bar{A}|$ and $x \in N(y_1) \cap S$, we observe that $|N(y_1) \cap A| = |N(y_1) \cap \bar{A}| = 2$. Hence, by the choice of $A$, we see that $|A| \geq |\bar{A}|$ and the latter part of (1) is proved.

(2) Assume $|A| = 2$, say $A = \{u, u'\}$. If $y_1u \notin E(G)$, then $u \in V_5$ and $xu \in E(G)$, which contradicts the assumption that $N(x) \cap V_5 = \{y_1, y_2\}$. Hence $y_1u \in E(G)$. Similarly we observe that $y_1u' \in E(G)$. Thus we have $|N(y_1) \cap A| = 2$. Hence, (1) assures us that $|A| \geq |\bar{A}|$. On the other hand, by Claim 1, we know that $|G| \geq 10$, which implies that $|A| + |\bar{A}| = |G| - |S| \geq 5$. This together with the fact $|A| \geq |\bar{A}|$ implies $|A| \geq 3$, which contradicts the assumption that $|A| = 2$. This contradiction proves (2). □

Subclaim 3.3. $y_1 \notin \mathrm{Ad}(x; A)$.

**Proof.** Assume $y_1$ is an admissible vertex of $(x; A)$. Then $N(x) \cap N(y_1) \cap A \neq \emptyset$ and $|N(y_1) \cap A| \geq 2$. Since $|N(y_1) \cap A| \geq 2$, Subclaim 3.2(1) assures us that $|N(y_1) \cap A| = |N(y_1) \cap \bar{A}| = 2$, $N(y_1) \cap N(A) = \{x\}$ and $|A| \geq |\bar{A}|$. Also Subclaim 3.2(2) assures us that $|A| \geq 3$. Now we know that $|\bar{A}| \geq 2$, $|A| \geq 3$, $|N(y_1) \cap A| = 2$, $N(y_1) \cap N(A) = \{x\}$ and $N(x) \cap N(y_1) \cap A \neq \emptyset$. Applying Claim 2 with the role of $y$ replaced by $y_1$, we see that there is an $xy_1$-fragment $A'$ such that $|N(y_1) \cap A'| = 1$, which contradicts the choice of $A$. This contradiction proves Subclaim 3.3. □

We proceed with the proof of Claim 3. Subclaim 3.2(2) assures us $|A| \geq 3$, hence now we know $|\bar{A}| \geq 2$, $|A| \geq 3$ and $N(x) \cap A \cap V_5 = \emptyset$. Applying Lemma 6, we see that there is an admissible vertex of $(x; A)$. Since $N(x) \cap V_5 = \{y_1, y_2\}$, either $y_1$ or $y_2$ is an admissible vertex of $(x; A)$. By Subclaim 3.3, we know that $y_1 \notin \mathrm{Ad}(x; A)$. Hence $y_2$ is an admissible vertex of $(x; A)$, which implies $N(x) \cap N(y_2) \cap A \neq \emptyset$. Let $u \in N(x) \cap N(y_2) \cap A$. By Subclaim 3.1 we know that $|N(y_2) \cap A| = 2$. Thus we have $|\bar{A}| \geq 2$, $|A| \geq 3$, $|N(y_2) \cap A| = 2$, $N(y_2) \cap N(A) = \{x\}$ and $u \in N(x) \cap N(y) \cap A$. Applying Claim 2 with the role of $y$ replaced by $y_2$, we see that there is an $xy_2$-fragment $A'$ such that $A' \subsetneq A$ and $N(y_2) \cap A' = \{u\}$. Since $N(x) \cap A' \cap V_5 = \emptyset$, we observe that $|A'| \geq 2$. We show $|A'| \geq 3$. Assume $|A'| = 2$, say $A' = \{u, w\}$. Since $N(y_2) \cap A' = \{u\}$ and $x \in N(A')$, we see that $w \in V_5$ and $xw \in E(G)$, which contradicts the assumption that $N(x) \cap V_5 = \{y_1, y_2\}$. Hence, it is shown that $|A'| \geq 3$. Since $|\bar{A}'| \geq 2$, $|A'| \geq 3$ and $N(x) \cap A' \cap V_5 = \emptyset$, applying Lemma 6 with the role of $A$ replaced by $A'$, we see that $\mathrm{Ad}(x; A') \neq \emptyset$. By Subclaim 3.3, we observe that $y_1 \notin \mathrm{Ad}(x; A)$, which implies $y_1$ is not an admissible vertex of $(x; A')$ since $A' \subsetneq A$. Since $|N(y_2) \cap A'| = 1$, $y_2$ is not an admissible vertex of $(x; A')$. Hence, since neither $y_1$ nor $y_2$ is an admissible vertex of $(x; A')$ and $N(x) \cap V_5 = \{y_1, y_2\}$, we see that $\mathrm{Ad}(x; A') = \emptyset$, which contradicts the previous assertion. This contradiction proves Claim 3. □

By Claim 3, there is a $y_2$-opposite $xy_1$-fragment. Let $A$ be a minimal $y_2$-opposite $xy_1$-fragment and let $S = N(A)$. Since $A$ is $\{y_1, y_2\}$-free, we observe that $N(x) \cap A \cap V_5 = \emptyset$ and $|A| \geq 2$. Since $y_2 \in \bar{A}$ and $y_1y_2 \notin E(G)$, we also see that $|\bar{A}| \geq 2$.

Claim 4. $|\bar{A}| \geq 3$.

**Proof.** Assume $|\bar{A}| = 2$, say $\bar{A} = \{y_2, z\}$. Then, Claim 1 assures us that $|A| = |G| - |S| - |\bar{A}| \geq 3$. Let $B$ be an $xy_2$-fragment and let $T = N(B)$. Since $N(x) \cap \bar{A} \cap V_5 = \{y_2\}$, we observe $S - \{x\} \subset N(z)$, which implies that $B$ is an $(x, \bar{A})$-fit fragment.

We show $x$ is tractable with $\bar{A}$. Assume $x$ is not tractable with $\bar{A}$. Then, we observe that either $|S \cap B| = 1$ or $|S \cap \bar{B}| = 1$. Without loss of generality, we may assume $|S \cap B| = 1$, say $S \cap B = \{w\}$. Since $|S \cap B| < |\bar{A} \cap T|$, Lemma 2(2) assures us that $A \cap B = \emptyset$, which implies $B = S \cap B = \{w\}$ and $T = N(w)$. Hence $w \in V_5$ and $xw \in E(G)$, which implies $w = y_1$ since $N(x) \cap V_5 = \{y_1, y_2\}$. Then, since $T = N(w)$ and $y_2 \in T$, we see that $y_1y_2 \in E(G)$, which contradicts Proposition 1. This contradiction proves $|S \cap B| \geq 2$. By symmetry, we have $|S \cap \bar{B}| \geq 2$ and $|S \cap B| = |S \cap \bar{B}| = 2$. It is shown that $x$ is tractable with $\bar{A}$.

Since $x$ is tractable with $\bar{A}$, we observe that $S \cap T = \{x\}$. Then, without loss of generality, we may assume that $y_1 \in S \cap B$. Since $|\bar{A} \cap T| = 2$ and $S \cap T = \{x\}$, we observe that $|A \cap T| = 2$. Now we know that $|A \cap T| = 2$ and $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| = |(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| = 5$. Hence, since $y_1 \in S \cap B$, if $A \cap B \neq \emptyset$, then $A \cap B$ is a $y_2$-opposite $xy_1$-fragment, which contradicts the minimality of $A$. Thus $A \cap B = \emptyset$, which implies that $A \cap \bar{B} \neq \emptyset$ since $|A| \geq 3$ and $|A \cap T| = 2$. Let $v \in N(x) \cap (A \cap \bar{B})$. Then, since $y_1 \in S \cap B$ and $v \in A \cap \bar{B}$, we see that $vy_1 \notin E(G)$. Hence, since $N(x) \cap S \cap V_5 = \{y_1\}$ and $vy_1 \notin E(G)$, we observe that $\mathrm{Ad}(x, v; A) = \emptyset$. Since $|\bar{A}| = 2$, $|A| \geq 3$ and $\mathrm{Ad}(x, v; A) = \emptyset$, applying Lemma 3 with the role $y$ replaced by $v$, we see that there is a $xv$-fragment $A'$ such that $A' \subsetneq A$. Then, since $|\bar{A'}| \geq 2$, $A' \cap V_5 \neq \emptyset$, applying Lemma 6 with the role of $A$ replaced by $A'$, we see that $\mathrm{Ad}(x; A') \neq \emptyset$, which implies that $y_1$ is an admissible vertex of $(x; A')$ since $N(x) \cap N(A') \cap V_5 \subset \{y_1\}$. Hence, we observe that $A'$ is a $y_2$-opposite $xy_1$-fragment, which contradicts the minimality of $A$ since $A' \subsetneq A$. This contradiction proves Claim 4. □

Claim 5. $N(x) \cap N(y_1) \cap A \neq \emptyset$ *and* $|N(y_1) \cap A| \geq 2$.

**Proof.** Since $|\bar{A}| \geq 2$, $|A| \geq 2$ and $N(x) \cap A \cap V_5 = \emptyset$, applying Lemma 6, we see that there is an admissible vertex of $(x; A)$. Then, since $N(x) \cap S \cap V_5 = \{y_1\}$, we see that $y_1$ is an admissible vertex of $(x; A)$ and $|N(y_1) \cap A| \geq 2$, which implies the desired conclusion. □

Claim 6. $|N(y_1) \cap \bar{A}| = 1$.

**Proof.** Assume $|N(y_1) \cap \bar{A}| \geq 2$. Then, by Claim 5, we see that $|N(y_1) \cap \bar{A}| = |N(y_1) \cap A| = 2$, $N(y_1) \cap S = \{x\}$ and $N(x) \cap N(y_1) \cap A \neq \emptyset$.

Subclaim 6.1. $|A| = 2$.

**Proof.** Assume $|A| \geq 3$. Then, we know that $|\bar{A}| \geq 2$, $|A| \geq 3$, $|N(y_1) \cap A| = 2$, $N(y_1) \cap S = \{x\}$ and $N(x) \cap N(y_1) \cap A \neq \emptyset$. Then, applying Claim 2 with the role of $y$ replaced by $y_1$, we see that there is a fragment $A'$ with respect to $xy_1$ such that $A' \subsetneq A$, which contradicts the minimality of $A$. This contradiction proves that $|A| = 2$. □

Since $N(x) \cap A \cap V_5 = \emptyset$, we know that $A \cap V_{\geq 6} \neq \emptyset$. Furthermore, since $N(y_1) \cap S = \{x\}$ and $x \in V_{\geq 6}$, we observe that $\mathrm{Ad}(y_1; A) = \emptyset$. Since $|\bar{A}| \geq 2$, $|A| = 2$, $A \cap V_{\geq 6} \neq \emptyset$ and $\mathrm{Ad}(y_1; A) = \emptyset$, applying Lemma 5(2) with the role of $x$ replaced by $y_1$, we see that $N(y_1) \cap S = \emptyset$, which contradicts the fact that $N(y_1) \cap S = \{x\}$. This contradiction proves Claim 6. □

By Claim 6, we know that $|N(y_1) \cap \bar{A}| = 1$, say $N(y_1) \cap \bar{A} = \{z_2\}$. Since $|A| \geq 2$, $|\bar{A}| \geq 3$ and $|N(y_1) \cap \bar{A}| = 1$, applying Lemma 3 with the roles of $x$ and $A$ replaced by $y_1$ and $\bar{A}$, respectively, we see that $\mathrm{Ad}(y_1, \bar{A}) \neq \emptyset$. Since $|N(y_1) \cap A| \geq 2$ and $|N(y_1) \cap \bar{A}| = 1$, we have $|N(y_1) \cap S| \leq 2$. This together with the fact that $x \in V_{\geq 6}$ assures us that there is the only admissible vertex of $(y_1, \bar{A})$. Let $z_1$ be the admissible vertex of $(y_1, \bar{A})$. Then $z_1 \in V_5 \cap S \cap N(y_1)$, $z_1z_2 \in E(G)$ and $|N(z_1) \cap \bar{A}| \geq 2$. If $|N(z_1) \cap A| = 1$, say $N(z_1) \cap A = \{v\}$, then $A - \{v\}$ is a $y_2$-opposite $xy_1$-fragment, which contradicts the minimality of $A$. Hence we see that $|N(z_1) \cap A| \geq 2$. Since $|N(z_1)| = 5$, we know that $|N(z_1) \cap \bar{A}| = |N(z_1) \cap A| = 2$ and $N(z_1) \cap S = \{y_1\}$. Let $N(z_1) \cap \bar{A} = \{z_2, u_1\}$.

Claim 7. $z_2 \in V_5$ *and* $z_2u_1 \in E(G)$.

**Proof.** Since $N(y_1) \cap \bar{A} = \{z_2\}$, we observe that $y_1u_1 \notin E(G)$. Hence, since $N(z_1) \cap S = \{y_1\}$, we observe that $\mathrm{Ad}(z_1, u_1; \bar{A}) = \emptyset$. Since $|A| \geq 2$, $|\bar{A}| \geq 3$ and $\mathrm{Ad}(z_1, u_1; \bar{A}) = \emptyset$, applying Lemma 3 with the roles of $x, y$ and $A$ replaced by $z_1, u_1$ and $\bar{A}$, respectively, we see that there is a $z_1u_1$-fragment $A'$ such that $A' \subsetneq \bar{A}$. Then, since $N(z_1) \cap \bar{A} = \{u_1, z_2\}$ and $A' \subsetneq \bar{A}$ we observe that $N(z_1) \cap A' = \{z_2\}$. Since $y_1 \notin A$, $N(y_1) \cap \bar{A} = \{z_2\}$ and $A' \subsetneq \bar{A}$, we see that $y_1 \in N(A')$ and $N(y_1) \cap A' = \{z_2\}$. Hence we observe that $N(\{z_1, y_1\}) \cap A' = \{z_2\}$. Since $N(\{z_1, y_1\}) \cap A' = \{z_2\}$, applying Lemma 1 with the roles of $S$ and $A$ replaced by $\{z_1, y_1\}$ and $A'$, respectively, we see that $A' = \{z_2\}$, which implies $z_2 \in V_5$ and $z_2u_1 \in E(G)$ and Claim 7 is proved. □

By Claim 7, we know that $\{z_1, z_2\} \subset N(y_1) \cap V_5$. Now we know that $N(y_1) \cap S = \{x, z_1\}$ and $|N(y_1) \cap A| = 2$. Let $N(y_1) \cap A = \{z_3, z_4\}$. Since $N(x) \cap N(y_1) \cap A \neq \emptyset$, without loss of generality, we may assume that $z_4 \in N(x) \cap N(y_1) \cap A$.

To complete the proof of Proposition 2, we first consider the case that $|A| = 2$ and later we consider the case that $|A| \geq 3$.

At first suppose $|A| = 2$. Let $S = \{x, y_1, z_1, w_1, w_2\}$. In this case, since $|\bar{A}| \geq 2$, $|A| = 2$ and $A \cap V_{\geq 6} \neq \emptyset$, applying Lemma 5(3), we see that $S - \{x\} = \{y_1, z_1, w_1, w_2\} \subset V_5$. If $xz_3 \in E(G)$, then we see that $\{z_3, z_4\} \subset V_{\geq 6}$ and a configuration of the second kind arises. Otherwise, if $xz_3 \notin E(G)$, then we see that $\{z_1, z_2, z_3\} \subset N(y_1) \cap V_5$ and $\{x, z_4\} \subset N(y_1) \cap (V(G) - V_5)$. Hence a degenerated configuration of the first kind arises.

The remaining case is $|A| \geq 3$. Assume $|A| \geq 3$. We show a configuration of the first kind arises in this case. Recall that $N(y_1) \cap A = \{z_3, z_4\}$ and $z_4 \in N(x) \cap N(y_1) \cap A$.

Claim 8. $z_1z_3 \in E(G)$.

**Proof.** Assume $z_1z_3 \notin E(G)$. Then we observe that $\mathrm{Ad}(y_1, z_3; A) = \emptyset$ since $N(y_1 \cap S \cap V_5) = \{z_1\}$. Then, since $|\bar{A}| \geq 2$, $|A| \geq 3$ and $\mathrm{Ad}(y_1, z_3; A) = \emptyset$, applying Lemma 3 with the roles of $x$ and $y$replaced by $y_1$ and $z_3$, respectively, we see that there is a $y_1z_3$-fragment $A'$ such that $A' \subsetneq A$. Since $N(y_1) \cap A = \{z_3, z_4\}$ and $z_3 \in N(A')$, we observe that $z_4 \in A'$, which implies $x \in N(A')$ since $xz_4 \in E(G)$. Hence we see that $A'$ is a $y_2$-opposite $xy_1$-fragment, which contradicts the minimality of $A$. This contradiction proves Claim 8. □

By Claim 8, we have $z_3 \in N(z_1) \cap A$. Let $N(z_1) \cap A = \{z_3, u_2\}$.

Claim 9. $u_2 \neq z_4$.

**Proof.** Assume that $u_2 = z_4$. Then $N(\{y_1, z_1\}) \cap A = \{z_3, z_4\}$. Since $|A| \geq 3$, we observe that $A' = A - \{z_3, z_4\} \neq \emptyset$. Since $|N(A')| = |(S - \{y_1, z_1\}) \cup \{z_3, z_4\}| = 5$, $A'$ is an $xz_4$-fragment. Then, since $|\bar{A}'| \geq 2$ and $N(x) \cap A' \cap V_5 = \emptyset$, applying Lemma 6 with the role of $A$ replaced by $A'$, we see that $\mathrm{Ad}(x; A') \neq \emptyset$. On the other hand, since $A' \cap \{y_1, y_2\} = \emptyset$, we see that $\mathrm{Ad}(x; A') = \emptyset$, which contradicts the previous assertion. This contradiction proves Claim 9. □

Claim 10. $z_3 \in N(u_2) \cap V_5$.

**Proof.** Since $u_2 \neq z_4$, we observe that $u_2y_1 \notin E(G)$, which implies that $\mathrm{Ad}(z_1, u_2; A) = \emptyset$ since $N(z_1) \cap S = \{y_1\}$. Then, since $|\bar{A}| \geq 2$, $|A| \geq 3$ and $\mathrm{Ad}(z_1, u_2; A) = \emptyset$, applying Lemma 3 with the roles of $x$ and $y$ replaced by $z_1$ and $u_2$, respectively, we see that there is a $z_1u_2$-fragment $A'$ such that $A' \subsetneq A$. Since $N(z_1) \cap A = \{z_3, u_2\}$ and $u_2 \in N(A')$, we observe that $N(z_1) \cap A' = \{z_3\}$, which implies $y_1 \in N(A')$ since $y_1z_3 \in E(G)$. If $x \in N(A')$ then $A'$ is a $y_2$-opposite $xy_1$-fragment, which contradicts the minimality of $A$. Hence $x \notin N(A')$, which implies $z_4 \notin A'$ since $xz_4 \in E(G)$. Since $N(y_1) \cap A = \{z_3, z_4\}$, we see that $N(y_1) \cap A' = \{z_3\}$. Now we observe that $N(\{z_1, y_1\}) \cap A' = \{z_3\}$. Since $N(\{z_1, y_1\}) \cap A' = \{z_3\}$, applying Lemma 1 with the roles of $S$ and $A$ replaced by $\{z_1, y_1\}$ and $A'$, respectively, we see that $A' = \{z_3\}$. This implies $z_3 \in V_5$ and $z_3u_2 \in E(G)$. Now Claim 10 is proved. □

By Claims 8–10, we find a configuration of the first kind around $(x, y_1)$, and the proof of Proposition 2 is completed. □

## 5. Proof of Main Theorem

In this section we give a proof of Main Theorem.

We use a discharging method to prove Main Theorem.

Let $G$ be a contraction-critically 5-connected graph and let $x \in V(G)$. We put $\mathrm{ch}_0(x)$ unit of charge on $x$ before discharging process according to the following rule.

$$\mathrm{ch}_0(x) = \begin{cases} 0, & \text{if } x \in V_5 \\ 1 & \text{otherwise.} \end{cases}$$

In discharging process we move $\varphi(x, y)$ unit of charge from $x$ to $y$ by the following rule.

$$\varphi(x, y) = \begin{cases} \dfrac{1}{|N(x) \cap V_5|}, & \text{if } xy \in E_G(V(G) - V_5, V_5) \\ 0, & \text{otherwise.} \end{cases}$$

We denote $\mathrm{ch}(x)$ the amount of charge on $x \in V(G)$ after discharging process.

Then, since we put a unit of charge on each vertex of $V(G) - V_5$, we observe that $|G| - |V_5| = \sum_{x \in V(G)} \mathrm{ch}_0(x)$. Since the discharging process do not change the total amount of charge on $V(G)$, we see that $\sum_{x \in V(G)} \mathrm{ch}_0(x) = \sum_{x \in V(G)} \mathrm{ch}(x)$. According to the discharging rule, we know $\mathrm{ch}(x) = 0$ for each $x \in V(G) - V_5$. Hence, if $\mathrm{ch}(y) \leq 1$ for each $y \in V_5$, then $\sum_{y \in V(G)} \mathrm{ch}(y) \leq |V_5|$. Then $|V(G) - V_5| = \sum_{x \in V(G)} \mathrm{ch}_0(x) = \sum_{x \in V(G)} \mathrm{ch}(x) \leq |V_5|$, which implies that $|V_5| \geq \frac{1}{2}|G|$. Thus, it is enough to show that $\mathrm{ch}(y) \leq 1$ for each $y \in V_5$.

Let $X = \{x \in V(G) \mid \deg(x) \geq 6 \text{ and } |N(x) \cap V_5| = 2\}$. We divide $V_5$ into two sets $W$ and $W'$ as follows. $W = \{y \in V_5 \mid N(y) \cap X = \emptyset\}$ and $W' = V_5 - W$. Let $y \in V_5$. Let $\tilde{N}(y) = N(y) \cap (V(G) - V_5)$. Then, by Theorem E, we know that $|N(y) \cap V_5| \geq 2$, which implies that $|\tilde{N}(y)| \leq 3$. At first assume $y \in W$. Then, from each vertex of $\tilde{N}(y)$, $y$ receives at most $\frac{1}{3}$ unit of charge through the discharging process. Hence $\mathrm{ch}(y) \leq \frac{1}{3} \times |\tilde{N}(y)| \leq 1$ since $|\tilde{N}(y)| \leq 3$. Next assume $y \in W'$ and let $x \in N(y) \cap X$. Then, Proposition 2 assures us that there is either a configuration of the first kind or a configuration of the second kind around $(x, y)$. If there is a configuration of the first kind, then we observe that

$|\tilde{N}(y)| = 2$. Hence $\mathrm{ch}(y) \leq \frac{1}{2} \times |\tilde{N}(y)| \leq 1$. So assume there is a configuration of the second kind around $(x, y)$. In this case $|\tilde{N}(y)| = 3$. Let $\tilde{N}(y) = \{x, z_3, z_4\}$ as in Proposition 2. Then we see that $|N(z_3) \cap V_5| = |N(z_4) \cap V_5| = 4$, which implies that $\varphi(z_3, y) = \varphi(z_4, y) = \frac{1}{4}$. Hence, $\mathrm{ch}(y) = \varphi(x, y) + \varphi(z_3, y) + \varphi(z_4, y) = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$. Now it is shown that $\mathrm{ch}(y) \leq 1$ for each $y \in V_5$ and the proof of Main Theorem is completed. $\square$

## Acknowledgments

The authors would like to thank the referees for their detailed and valuable comments.

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