Principal frequency and existence of solutions for quasilinear elliptic obstacle problems

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Abstract

In this paper, we are interested in the first eigenvalue of \( p \)-Laplacian and the relation between the first eigenvalue and the existence (or nonexistence) of nontrivial (positive) solution for quasilinear elliptic obstacle problems. Utilizing the fact that obstacle problem have consanguineous relations with corresponding equation, we get a simple approach to study the properties of solutions of obstacle problems, such as existence and nonexistence, regularity and stability, etc. In this paper we are mainly concerned with the existence and nonexistence.

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1. Introduction

Let us recall some definitions and well-known results about quasilinear elliptic equations.

Definition. We say that \( \lambda \) is an eigenvalue, if there exists a continuous function \( u \in W^{1,p}_0(\Omega) \), \( u \not\equiv 0 \), such that

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \eta \, dx = \lambda \int_{\Omega} |u|^{p-2} u \eta \, dx
\]  

(1)

whenever \( \eta \in W^{1,p}_0(\Omega) \). The function \( u \) is called an eigenfunction.
The continuity of \( u \) is a standard elliptic regularity result. The first eigenvalue of the operator \( \text{div}(|\nabla u|^{p-2}\nabla u) \) is here defined as the least real number \( \lambda \) for which the equation

\[
\text{div} \left(|\nabla u|^{p-2}\nabla u + \lambda |u|^{p-2}u\right) = 0
\]

has a nontrivial solution \( u \) with zero boundary data in a given bounded domain in the \( n \)-dimensional Euclidean space. The first eigenvalue is the minimum of the Rayleigh quotient:

\[
\lambda_1 = \min_{u} \frac{\int \left| \nabla u \right|^p \, dx}{\int |u|^p \, dx},
\]

the infimum being taken among all \( u \in W^{1,p}_0(\Omega), 1 < p < \infty, u \neq 0 \). It is easily seen that this minimization problem is equivalent to Eq. (1) with \( \lambda = \lambda_1 \).

We say that \( \lambda_1 \) is the first eigenvalue or the principal frequency and the corresponding eigenfunction is called the first eigenfunction. We have a well-known result about solutions of Eq. (1), cf. [10]: there is a cone of solutions in each eigenvalue. In the linear case \( p = 2 \) one obtains “the principal frequency;” the associated first eigenfunction \( u \) describes the shape of a membrane when it vibrates emitting its gravest tone. (We shall often use the term principal frequency for the nonlinear case as well.)

The first eigenvalue of the operator \( \text{div}(|\nabla u|^{p-2}\nabla u) \) has a distinct feature that it is simple in any bounded domain \( \Omega \) in \( \mathbb{R}^n \), i.e., all the associated first eigenfunctions are merely constant multiples of each other, cf. [9]. This phenomenon is a reflection of distinguishing feature of the first eigenfunctions: they never change signs in \( \Omega \). Higher eigenvalues are not simple even in the case \( \Delta u + \lambda u = 0 \).

We want to mention an interesting result that the first eigenvalue in bounded domain \( \Omega \) is the reciprocal of the best constant \( c(\Omega) \), where \( c(\Omega) \) is a positive constant only depend on \( \Omega \) such that the Poincare’s inequality

\[
\int_{\Omega} |u|^p \, dx \leq c(\Omega) \int_{\Omega} \left| \nabla u \right|^p \, dx
\]

valid for every \( u \in W^{1,p}_0(\Omega) \).

In the literature, much work has been devoted to study the existence of (positive) solutions for linear, quasilinear obstacle problems, see [1–8]. However, it has not been analyzed (for quasilinear elliptic obstacle problem) what occurs when the parameter \( \lambda \) interacts with the principal frequency of the \( p \)-Laplacian. In the present work, we obtain several results on the existence of weak solutions under assumptions that relate \( \lambda \) and the principal frequency \( \lambda_1 \) of the \( p \)-Laplacian.

Our paper is organized in the following way. In Section 2 we study the typical case, that is the corresponding energy functional is

\[
F(u) = \frac{1}{p} \int_{\Omega} \left| \nabla u \right|^p \, dx - \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx,
\]

(4)
where \( p = q \). In Section 3, we consider critical case, that is the higher growth power of \( u \) is a critical exponent \( p^* \):

\[
p^* = \begin{cases} 
\frac{Np}{N-p}, & \text{if } p < N, \\
\infty, & \text{if } p \geq N.
\end{cases}
\]

For the different cases \( p = q \) and \( p \neq q \), the functional \( F(u) \) has different properties, such as described by the following

**Lemma** [10, Proposition 2.3, p. 1395]. Assume that \( p, q \) verifies \( 1 < q < p^* \). Then:

1. If \( p \neq q \) the functional \( F \) defined by (4) verifies the following Palais–Smale condition: Let \( \{u_j\} \subset W^{1,p}_0(\Omega) \) verifies \( |F(u_j)| \leq c \) and \( F'(u_j) \to 0 \) in \( W^{-1,p'}(\Omega) \), \( 1/p + 1/p' = 1 \). Then, there exists a subsequence \( u_j \to u \) in \( W^{1,p}_0(\Omega) \).

2. If \( p = q \) the following local Palais–Smale condition is verified: Let \( M_\alpha = \{ u \in W^{1,p}_0(\Omega): \frac{1}{p} \int_\Omega |\nabla u|^p \, dx = \alpha \} \) and \( \{u_j\} \subset M_\alpha \) be such that:

   (i) \( b(u_j) = \frac{1}{p} \int_\Omega |u_j|^p \, dx \geq \beta > 0 \) for all \( j \in \mathbb{N} \);

   (ii) \( D(u_j) = |u_j|^{p-2}u_j - \frac{1}{ap}(-\Delta pu_j) \to 0 \) in \( W^{-1,p'}(\Omega) \), \( 1/p + 1/p' = 1 \).

Then, there exists a convergent subsequence \( u_j \to u \in M_\alpha \) and, moreover, \( -\Delta pu = \lambda |u|^{p-2}u \). (In other words, \( u \) is a critical point of \( F \) defined by (4), \( \Delta pu = \text{div} |\nabla u|^{p-2}|\nabla u| \).)

This difference make us particularly interested in the typical case, that is \( q = p \).

2. Typical problem

In this section, we are devoted to the problem

\[
u \in \mathcal{N}: \quad \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla (v-u) \, dx \geq \lambda \int_\Omega |u|^{p-2}u(v-u) \, dx \quad \forall v \in \mathcal{N}, \tag{5}
\]

where \( \mathcal{N} = \{ v \in W^{1,p}_0(\Omega), v \geq \psi \text{ a.e. in } \Omega \} \), \( \psi \in W^{1,p}(\Omega) \) is a given obstacle function.

**Definition.** We call \( u \) is a weak solution of the quasilinear elliptic obstacle problem (5), if (5) holds for all \( v \in \mathcal{N} \).

The energy functional of (5) is defined by

\[
I(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \lambda \frac{1}{p} \int_\Omega |u|^p \, dx. \tag{6}
\]

In linear case \( p = 2 \), we have the following nonexistence result.
Theorem 2.1. There is no positive solution of (5) when $p = 2, \lambda > \lambda_1$. ($\lambda_1$ is the first eigenvalue of $-\Delta$ with zero Dirichlet condition on $\Omega$.)

Proof. Let $\phi_1$ be the eigenfunction of $-\Delta$ corresponding to $\lambda_1$ with $\phi_1 > 0$ on $\Omega$. Suppose $u$ is a positive solution of

$$
\int_{\Omega} \nabla u \cdot \nabla (v - u) \, dx \geq \lambda \int_{\Omega} u(v - u) \, dx.
$$

We have

$$
-\int (\Delta u)\phi_1 = \lambda_1 \int u\phi_1 = -\int \Delta u(u + \phi_1 - u) + \int \nabla u \cdot \nabla [(u + \phi_1) - u] \geq \lambda \int u\phi_1
$$

and thus $\lambda \leq \lambda_1$, a contradiction to our hypothesis $\lambda > \lambda_1$. \qed

We know that calculus of variations is a typical method to solve an important class of partial differential equations or variational inequalities. The following two theorems are known assertions, we record them with proof for convenience and completeness.

Theorem 2.2. The minimum of the functional $I(u), u \in \mathfrak{B}$, is a solution of the obstacle problem, that is

$$
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (v - u) \, dx \geq \lambda \int_{\Omega} |u|^{p-2} u(v - u) \, dx \text{ for all } v \in \mathfrak{B}.
$$

Proof. We assume that $I(u) = \min_{v \in \mathfrak{B}} I(u)$.

Fix any element $v \in \mathfrak{B}$. Then for each $0 \leq \tau \leq 1$

$$
u + \tau (v - u) = (1 - \tau)u + \tau v \in \mathfrak{B}
$$
since $\mathfrak{B}$ is convex. Thus if we set

$$i(\tau) := I\left[u + \tau (v - u)\right]
$$

we see that $i(0) \leq i(\tau)$ for all $0 \leq \tau \leq 1$. Hence

$$i'(0) \geq 0. \quad (7)
$$

Now, if $0 < \tau \leq 1$,

$$
\lim_{\tau \to 0^+} \frac{i(\tau) - i(0)}{\tau} = \lim_{\tau \to 0^+} \frac{1}{\tau} \left\{ \int_{\Omega} \left[ \frac{|\nabla u|^{p-2} |\nabla u|^{p} - |\nabla u|^{p} - \lambda |u + \tau (v - u)|^{p} - |u|^{p}}{p} \right] \, dx \right\}
\int_{\Omega} \left[ |\nabla u|^{p-2} \nabla u \cdot \nabla (v - u) - \lambda |u|^{p-2} u(v - u) \right] \, dx. \quad (8)
$$

Thus (7) implies...
\[ i'(0) = \int_{\Omega} \left[ |\nabla u|^{p-2} \nabla u \cdot \nabla (v-u) - \lambda |u|^{p-2} u (v-u) \right] dx \geq 0, \quad \forall v \in \mathfrak{R}, \]

and the theorem follows. □

**Definition.** By a critical point \( u \) of the functional \( I(u) \) in \( \mathfrak{R} \) we mean that
\[ \langle I'(u), v-u \rangle \geq 0, \quad \forall v \in \mathfrak{R}. \]

By this theorem we can conclude that the critical point of the functional \( I(u) \) in \( \mathfrak{R} \) happens to be a weak solution for obstacle problem (5).

**Theorem 2.3.** Assume \( u \in W^{1,p}_0(\Omega) \) satisfies
\[ I(u) = \min_{w \in W^{1,p}_0(\Omega)} I(w). \]

Then there exists at least one positive weak solution for equation
\[ -\text{div} |\nabla u|^{p-2} \nabla u = \lambda |u|^{p-2} u, \quad \lambda > 0, \quad x \in \Omega. \tag{9} \]

**Proof.** We see that if \( I(u) = \min_{w \in W^{1,p}_0(\Omega)} I(w) \), so does \( |u| \); it may be assumed without loss of generality that \( u \geq 0 \) a.e. on \( \Omega \).

We take care about differentiating under the integrals. Fix any \( v \in W^{1,p}_0(\Omega) \) and set
\[ i(\tau) := I(u + \tau v) \quad (\tau \in \mathbb{R}). \]

It is clear that
\[ \frac{1}{p} |\nabla u|^p = \frac{\lambda}{p} |u|^p \leq C (|\nabla u|^p + |u|^p). \tag{10} \]

In view of (10) we see that \( i(\tau) \) is finite for all \( \tau \).

Let \( \tau \neq 0 \) and write the difference quotient
\[
\frac{i(\tau) - i(0)}{\tau} = \frac{1}{\tau} \left\{ \int_{\Omega} \left[ \frac{|\nabla u + \tau \nabla v|^p}{p} - \frac{|\nabla u|^p}{p} - \frac{\lambda |u + \tau v|^p}{p} - \frac{|u|^p}{p} \right] dx \right\}
\]
\[ = \int_{\Omega} I^\tau(x) dx, \tag{11} \]

where
\[ I^\tau(x) = \frac{1}{\tau} \left[ \frac{|\nabla u + \tau \nabla v|^p}{p} - \frac{|\nabla u|^p}{p} - \frac{\lambda |u + \tau v|^p}{p} - \frac{|u|^p}{p} \right] \]
for a.e. \( x \in \Omega \). Clearly
\[ I^\tau(x) \to |\nabla u|^{p-2} \nabla u \cdot \nabla v - \lambda |u|^{p-2} u \quad \text{a.e.} \tag{12} \]
as \( \tau \to 0 \). Furthermore
\[ I^\tau(x) = \frac{1}{\tau} \int_0^\tau \left[ \nabla u + s \nabla v \right]^{p-2} \left( \nabla u + s \nabla v \right) \cdot \nabla v - \lambda |u + sv|^{p-2} (u + sv) v \, ds. \]

Next recall from Young’s inequality:
\[ ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1. \]

Then since \( u, v \in W^{1,p}_0(\Omega) \), Young’s inequality implies after some elementary calculations that
\[ \left| I^\tau(x) \right| \leq C \left( |\nabla u|^p + |\nabla v|^p + |u|^p + |v|^p \right) \in L^1(\Omega) \]
for each \( \tau \neq 0 \). Consequently we may invoke the dominated convergence theorem to conclude from (11), (12) that \( i'(0) \) exists and equals
\[ \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v - \lambda |u|^{p-2} uv \, dx. \]

But then, since \( i(\cdot) \) has a minimum for \( \tau = 0 \), we know \( i'(0) = 0 \); thus \( u \) is a positive solution of (9), \( u > 0 \) on \( \Omega \) is of course a consequence of Harnack inequality [12, Theorem 1.1].

By comparing the last two theorems, we see that if \( I(u_1) = \min_{w \in W^{1,p}_0(\Omega)} I(w), u_1 > 0 \) on \( \Omega \), and \( u_1 \) happens to be an admissible function in \( \mathcal{R} (u_1 \in \mathcal{R} \text{ follows from } 0 \in \mathcal{R}) \), we can obtain \( u_1 = \min_{w \in \mathcal{R}} I(w) \), by Theorem 2.2, \( u_1 \) is precisely a positive solution of the obstacle problem (5). Therefore, we have the following

Corollary. Suppose \( I(u_1) = \min_{w \in W^{1,p}_0(\Omega)} I(w) \) (so does \( |u_1| \)), \( 0 \in \mathcal{R} \); then there exists at least one positive solution for obstacle problem (5).

The following theorem express a simple approach to investigate the properties of solutions for a class of obstacle problem, not only the existence or nonexistence, but also regularity, stability, etc.

**Theorem 2.4.** If \( 0 \in \mathcal{R} \), Eq. (9) has one positive solution, then obstacle problem (5) has at least one positive solution.

**Proof.** Suppose \( u_1 \) is a positive solution of Eq. (9); this means that
\[ \int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \phi \, dx = \lambda \int_\Omega |u_1|^{p-2} u_1 \phi \, dx \quad (13) \]
for all \( \phi \in W^{1,p}_0(\Omega) \).
Fix any \( v \in \mathbb{R} \), \( v-u_1 \in W^{1,p}_0(\Omega) \), let \( \phi = v-u_1 \). From (13), we see that
\[
\int_{\Omega} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla (v-u_1) \, dx = \lambda \int_{\Omega} |u_1|^{p-2} u_1 (v-u_1) \, dx \\
\geq \lambda \int_{\Omega} |u_1|^{p-2} u_1 (v-u_1) \, dx
\]
for all \( v \in \mathbb{R} \). Hence \( u_1 \) is a solution of obstacle problem (5). \( \square \)

The following lemma is a known result, see [7], we list it with simple proof.

**Lemma.** If \( \lambda < \lambda_1 \) (\( \lambda_1 \) is the first eigenvalue of the operator \( \text{div}(|\nabla u|^{p-2} \nabla u) \)), there is no nontrivial solution for Eq. (13).

**Proof.** Suppose to the contrary that \( u \) is a nontrivial solution for (13) when \( \lambda < \lambda_1 \). By (13), applied with \( \phi \) replaced by \( u \), we have that
\[
\int_{\Omega} |\nabla u|^p \, dx = \lambda \int_{\Omega} |u|^p \, dx < \lambda_1 \int_{\Omega} |u|^p \, dx
\]
we see that
\[
\lambda_1 > \frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\Omega} |u|^p \, dx}.
\]
But this is a direct contradiction to the definition of \( \lambda_1 \), assertion is proved. \( \square \)

This lemma illustrates that the eigenvalues of the operator \( \text{div}(|\nabla u|^{p-2} \nabla u) \) are bounded below for a positive constant \( \lambda_1 \), \( \lambda_1 \) is the lowest eigenvalue.

The following definition is used in our next theorem, we give it for reader’s convenience.

**Definition** [6, Definition 2, p. 779]. A function \( u \in \mathbb{R} \) is called a \( W \)-supersolution of (5) if
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (v-u) \, dx \geq \lambda \int_{\Omega} |u|^{p-2} u (v-u) \, dx \quad \forall v \in u \lor \mathbb{R},
\]
where \( u \lor \mathbb{R} = \{ \max[u, v] : v \in \mathbb{R} \} \).

**Theorem 2.5.** If \( \lambda < \lambda_1 \) (\( \lambda_1 \) is the first eigenvalue of the operator \( \text{div}(|\nabla u|^{p-2} \nabla u) \)), \( 0 \in \mathbb{R} \), then there is no nontrivial solution for the obstacle problem (5), and the positive first eigenfunctions are \( W \)-supersolutions of obstacle problem (5); if \( \lambda \geq \lambda_1 \), then the obstacle problem (5) may have infinitely many nontrivial solutions.

**Proof.** When \( \lambda \leq \lambda_1 \), suppose to the contrary that \( u \) is a nontrivial solution for (5). Since \( 0 \in \mathbb{R} \), \( 0 \) is an admissible function for obstacle problem (5). Let \( v = 0 \); from (5) we see that
\[-\int_\Omega |\nabla u|^p \, dx \geq -\lambda \int_\Omega |u|^p \, dx,\]
\[\lambda \int_\Omega |u|^p \, dx \geq \int_\Omega |\nabla u|^p \, dx.\]

We deduce from the last inequality that
\[\lambda \geq \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}.\]

Since \(\lambda < \lambda_1\), we see that
\[\lambda_1 > \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx};\]
it contradicts with the definition of \(\lambda_1\), hence there is no nontrivial solution for obstacle problem (5).

Suppose \(u_1\) is a positive first eigenfunction corresponding to \(\lambda_1\); by the definition of first eigenfunction we have that
\[\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \eta \, dx = \lambda_1 \int_\Omega |u_1|^{p-2} u_1 \eta \, dx, \quad \forall \eta \in W^{1,p}_0(\Omega). \tag{14}\]

For \(\forall v \in u_1 \cap \mathbb{R}\), we see that \(v - u_1 \in W^{1,p}_0(\Omega)\). Let \(\eta = v - u_1\); we have that
\[\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla (v - u_1) \, dx = \lambda_1 \int_\Omega |u_1|^{p-2} u_1 (v - u_1) \, dx\]
\[\geq \lambda \int_\Omega |u_1|^{p-2} u_1 (v - u_1) \, dx.\]

By the definition of \(W\)-supersolution of (5), we see that \(u_1\) is a \(W\)-supersolution of (5).

Our last assertion follows from that if \(\lambda \geq \lambda_1\), there exists infinitely many nontrivial solutions for Eq. (1), cf. [10]. Suppose \(\{u_n\} (n = 1, 2, \ldots)\) are nontrivial solutions for Eq. (1), \(u_n \in \mathbb{R}\). By the definition of weak solution for quasilinear elliptic equation and obstacle problem (similar to the proof of Theorem 2.4), we can conclude that \(u_n, n = 1, 2, \ldots\), are solutions for obstacle problem (5). The theorem is proved. \(\square\)

We next employ the above theorem to give a new definition for principal frequency \(\lambda_1\).

**Definition.** We call the principal frequency \(\lambda_1\) of the operator \(\text{div}(|\nabla u|^{p-2} \nabla u)\) a sharp value of the existence or nonexistence of weak solution for obstacle problem (5).

3. Critical case

In this section we begin with the hypothesis that \(1 < p < N\). The problem now is to prove that
are nontrivial solutions where \( p^* = Np/(N - p) \).

First, let us recall some existence assertion for weak solutions of corresponding equations:

\[
\begin{aligned}
- \text{div}(|\nabla u|^{p-2}\nabla u) &= \lambda |u|^{p-2}u + |u|^{p^* - 2}u \\
\text{in } \Omega, \\
u &\geq 0 \text{ in } \Omega, \\
u|_{\partial \Omega} &= 0.
\end{aligned}
\]

If \( 0 < \lambda < \lambda_1 \) and \( p^2 < N \), then (16) has a nontrivial solution. For \( \lambda > \lambda_1 \), the conjecture is that the only solution is the trivial one. The case \( N \in (p, p^2) \) is much more difficult, see [10]. In the linear case, \( p = 2 \), paper [11] contains the following beautiful result: Let \( \lambda_1 \) be the first eigenvalue of the Laplacian in the unit ball \( B \subset \mathbb{R}^3 \); then (16) has solution in \( B \) if and only if \( \frac{4}{3}\lambda_1 < \lambda < \lambda_1 \).

Our main result of this section is the following two theorems.

**Theorem 3.1.** Suppose \( B \) is a unit ball \( B \subset \mathbb{R}^3 \), \( \frac{1}{4}\lambda_1 < \lambda < \lambda_1 \), \( 0 \in \mathbb{R} \); then obstacle problem (15) on \( B \) with \( p = 2 \) has at least one positive solution.

**Proof.** Suppose \( u_0 \) is a weak solution of (16) with \( p = 2 \), \( \Omega = B \); \( u_0 \in \mathbb{R} \) follows from \( 0 \in \mathbb{R} \). When \( p = 2 \), by the definition of weak solution of equation, we have

\[
\int_{\Omega} \nabla u_0 \cdot \nabla \phi \, dx = \int_{\Omega} u_0^2 \phi \, dx + \lambda \int_{\Omega} u_0 \phi \, dx
\]

for all \( \phi \in W^{1,2}_0(B) \).

\[
\forall v \in \mathbb{R} = \{ v \in W^{1,2}_0(B), v \geq \psi \text{ a.e. on } B \}, v - u_0 \in W^{1,2}_0(B), \text{ let } \phi = v - u_0, \text{ we have}
\]

\[
\int_{\Omega} \nabla u_0 \cdot \nabla (v - u_0) \, dx = \int_{\Omega} u_0^2 (v - u_0) \, dx + \lambda \int_{\Omega} u_0 (v - u_0) \, dx \\
\geq \int_{\Omega} u_0^2 (v - u_0) \, dx + \lambda \int_{\Omega} u_0 (v - u_0) \, dx.
\]

Applying the definition of weak solution for obstacle problem, we see that \( u_0 \) is a positive solution for problem

\[
u \in \mathbb{R}, \int_{\Omega} \nabla u \cdot \nabla (v - u) \, dx \geq \int_{\Omega} |u|^{p} u (v - u) \, dx + \lambda \int_{\Omega} u (v - u) \, dx \text{ } \forall v \in \mathbb{R}.
\]

The theorem follows. \( \square \)
Theorem 3.2. Assume $0 < \lambda < \lambda_1$ and $p^2 < N$, $0 \in \mathbb{R}$; then (15) has at least one positive solution.

Proof. From the conditions of the theorem, we can assume that $u_0$ is a nontrivial solution of problem (16). By the definition of weak solution for equation, we can conclude that $u_0$ is also a positive solution for obstacle problem (15). The theorem is proved. □

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References