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Another Proof of the Total Positivity of the Discrete Spline Collocation Matrix

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supply intuition for those conditions. © 1996 Academic Press, Inc.

1. Introduction

In [3, Theorem 6] Morken gives necessary and sufficient conditions for a minor of the discrete B-spline collocation matrix to be positive, correcting an error in an earlier theorem of Jia [2]. One of these conditions may not be intuitively obvious. In this note we attempt to supply such intuition, and we provide a different proof.

Recapping Morken's notation, let k be a positive integer; let $t=\{t_i\}_{i=-\infty}^{\infty}$ be a bi-infinite, nondecreasing sequence of real numbers (knots) with $t_i<t_{i+k}$ for all i; and let τ be a bi-infinite subsequence of t, $\tau\subset t$. We study the discrete B-spline collocation matrix $A_{\tau,t}$ with elements given by $(A_{\tau,t})_{i,j}=\alpha_{j,k,t,\tau}(i)$. Here $\alpha_{j,k,t,\tau}(i)$ are the coefficients in the expansion of the B-spline $B_{j,k,\tau}$ on the coarse knot sequence τ in terms of the B-splines on the fine knot sequence t,

$$B_{j,k,\tau}=\sum_i \alpha_{j,k,t,\tau}(i)\, B_{i,k,t}.$$

Denote, further,

$$\begin{aligned} m_t(x) &= \max\{q-p \mid t_q\leqslant x \text{ and } x\leqslant t_{p+1}\},\\ l_t(i) &= \max\{p \mid t_{i-p}=t_i\},\\ r_t(i) &= \max\{p \mid t_{i+p}=t_i\}. \end{aligned}$$

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We are interested in the conditions under which a minor of $A_{\tau,t}$ has a strictly positive determinant, as formulated by Morken [3].

THEOREM 1.1. *Let $k \geqslant 1$ be given, let t be a knot vector with $t_i < t_{i+k}$ for all i, and let τ be a subsequence of t. Let $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_m$ be two increasing integer sequences. Then*

$$\det A_{\tau,t}\begin{pmatrix} i_1, ..., i_m \\ j_1, ..., j_m \end{pmatrix} \geqslant 0,$$

with strict positivity if and only if both of the following conditions are satisfied:

(i) $(A_{\tau,t})_{i_q, j_q} > 0$ *for* $q = 1, 2, ..., m$.

(ii) *If for some q, the multiplicity of t_{i_q} in t is greater than the multiplicity of t_{i_q} in τ, that is $m_\tau(t_{i_q}) < m_t(t_{i_q})$, then*

$$i_{q-d_q} < i_q - d_q - f_q,$$

where

$$\begin{aligned} d_q &= k - r_t(i_q), \\ f_q &= \min\{l_t(i_q), m_t(t_{i_q}) - m_\tau(t_{i_q}) - 1\}. \end{aligned}$$

To ease the ascertainment and use of condition (i) we employ the index mappings $\mu_L(j; \tau, t)$ and $\mu_R(j; \tau, t)$, introduced in [1]. By definition they are such that whenever $\tau_{b-1} \leqslant t_{a-1} < t_a = \tau_b$,

$$\left.\begin{aligned} \mu_L(b+u; \tau, t) &= a + m_t(t_a) - m_\tau(t_a) + u, \\ \mu_R(b+u; \tau, t) &= a + u, \end{aligned}\right\} \quad u = 0, ..., m_\tau(t_a) - 1.$$

Thus $\mu_L(j; \tau, t)$ is the index of the t-knot corresponding to τ_j, when multiple τ-knots are viewed as aligned in order at the right end of the corresponding (multiple) t-knot. Note that $\mu_R(b + m_\tau(t_a)) - 1 \geqslant \mu_L(b + m_\tau(t_a) - 1)$, and that both index mappings are strictly monotone. In these terms Jia [2, Lemma 5] can be rephrased as follow (see [1]).

LEMMA 1.2. $(A_{\tau,t})_{i,j} > 0$ *if and only if*

$$\mu_L(j; \tau, t) \leqslant i \leqslant \mu_R(j+k; \tau, t) - k. \tag{1.1}$$

In the sequel we will therefore refer to condition (i) as the "interlacing conditions."

For later use we record the following, easily proven, property of μ. If $\tau_j < \tau_{j+r}$, or if $\tau_j = \tau_{j+r}$ and $m_\tau(\tau_j) = m_t(\tau_j)$, then

$$\mu_L(j; \tau, t) \leqslant \mu_R(j+r; \tau, t) - r. \tag{1.2}$$

In particular, the assumption that $t_i < t_{i+k}$ for all i implies that if $\tau \subset t$ then

$$\mu_L(j; \tau, t) \leqslant \mu_R(j+k-1; \tau, t) - k + 1 \qquad \text{for all } j. \tag{1.3}$$

Let us turn now to an examination of condition (ii). The intuition behind this condition and, indeed, our proof of the theorem, is based on the following observation of Jia [2].

LEMMA 1.3. *Suppose that $\tau \subset \rho \subset t$. Then*

$$\det A_{\tau, t} \begin{pmatrix} i_1, ..., i_m \\ j_1, ..., j_m \end{pmatrix} > 0$$

if and only if there exist $\xi_1 < \cdots < \xi_m$ such that

$$\det A_{\rho, t} \begin{pmatrix} i_1, ..., i_m \\ \xi_1, ..., \xi_m \end{pmatrix} \cdot \det A_{\tau, \rho} \begin{pmatrix} \xi_1, ..., \xi_m \\ j_1, ..., j_m \end{pmatrix} > 0.$$

In particular, given any intermediate knot sequence ρ, it must be possible to pick a monotonically increasing integer sequence $\xi_1 < \cdots < \xi_m$ such that the interlacing conditions are satisfied for t and ρ, $(A_{\rho, t})_{i_q, \xi_q} > 0$ for $q = 1, 2, ..., m$. Let us look at a case in which this is *not* possible. We will demonstrate this by showing that if the interlacing conditions do hold then the ξ sequence cannot be strictly monotonic.

Suppose there are indices i_p and i_q such that $t_{i_p+k} = t_{i_q} = t_z$ with $t_{z-1} < t_z$. In fact, let us require slightly more: that $z + m_\tau(t_z) \leqslant i_p + k \leqslant z + m_t(t_z) - 1$ and that $z \leqslant i_q \leqslant z + m_t(t_z) - 1 - m_\tau(t_z)$. Consider now a sequence ρ which is the same as t except that the multiplicity of the knot t_z in ρ is $m := i_p + k - z$, instead of $m_t(t_z)$. Note that by assumption $m_\tau(t_z) \leqslant m \leqslant m_t(t_z) - 1$. If $(A_{\rho, t})_{i_p, \xi_p} > 0$ and $(A_{\rho, t})_{i_q, \xi_q} > 0$ for some ξ_p and ξ_q, then it is easily seen (cf. the proof of Lemma 2.1) that necessarily

$$\xi_p \geqslant i_p, \qquad \xi_q \leqslant i_q - m_t(t_z) + \max(m, z + m_t(t_z) - 1 - i_q).$$

The sequence ξ will certainly fail to be strictly monotonic if $\xi_p + q - p > \xi_q$, which by the above is assuredly true if

$$i_p + q - p > i_q - m_t(t_z) + \max(i_p + k - z, z + m_t(t_z) - 1 - i_q).$$

The following lemma spells this condition out and shows that it is in fact equivalent to condition (ii); it is therefore, somewhat surprisingly, the only type of case that needs to be ruled out. Incidentally, the assumption $0 \leqslant i_q - z \leqslant m_t(t_z) - 1 - m_\tau(t_z)$ is not stated explicitly because it is a consequence of the other conditions and the interlacing conditions for t and τ.

LEMMA 1.4. *Condition (ii) is violated for i_q, with $t_{z-1} < t_z = t_{i_q}$, if and only if there exists an i_p for which all of the following hold*:

(a) $z + m_\tau(t_z) \leqslant i_q + k \leqslant z + m_t(t_z) - 1$,

(b) $i_q + q - p \geqslant z$,

(c) $q - p \geqslant d_q$.

Proof. Suppose condition (ii) is violated, and set $p = q - d_q$. Writing out the definition of f_q while noting that $i_q - d_q = z + m_t(t_z) - 1 - k$ and $i_q - l_t(i_q) = z$, we get

$$i_p \geqslant i_q - d_q - (m_t(t_z) - m_\tau(t_z) - 1) = z - k + m_\tau(t_z), \tag{1.4}$$

$$i_p \geqslant i_q - (q - p) - l_t(i_q) = z - (q - p), \tag{1.5}$$

proving (b) and half of (a). To prove the remaining half observe that $i_{r-1} \leqslant i_r - 1$ for all r and so

$$i_p = i_{q - d_q} \leqslant i_q - d_q = z - k + m_t(t_z) - 1.$$

To prove the converse we note that if (a), (b), (c) hold for some $\bar{p}$, then they must also hold for $p = q - d_q \geqslant \bar{p}$. Namely, the left-hand side of (a) is immediate while the right-hand side follows from $i_p = i_{q-d_q} \leqslant i_q - d_q$; and (b) follows from $i_p - p \geqslant i_{\bar{p}} - \bar{p}$. Therefore, the proof is already implicit in inequalities (1.4), (1.5). ∎

2. PROOF OF THE THEOREM

The necessity of the interlacing conditions is fairly clear and proven in [2]. To establish the necessity of condition (ii), Lemma 2.1 exhibits a subsequence $\tau \subset \rho \subset t$ such that if the condition is violated then

$$\det A_{\rho, t}\begin{pmatrix} i_1, ..., i_m \\ \xi_1, ..., \xi_m \end{pmatrix} = 0,$$

whatever the choice of $\{\xi_r\}$. It follows then from Lemma 1.3 that

$$\det A_{\tau, t}\begin{pmatrix} i_j, \ldots, i_m \\ j_1, \ldots, j_m \end{pmatrix} = 0.$$

To prove the converse we proceed by induction on the difference in the number of knots in t and τ. If the difference is zero, $t = \tau$, it is easily seen from Lemma 1.2 that condition (i) implies $i_q = j_q$, $q = 1, \ldots, m$, and hence the determinant is positive. For the induction step we exhibit in Lemma 2.2, if conditions (i) and (ii) hold, a subsequence ρ, $\tau \subset \rho \subset t$, and a set of indices $\{\xi_r\}$ such that conditions (i) and (ii) hold again for ρ and t with respect to $\{\xi_r\}$ and $\{i_r\}$, and at the same time

$$\det A_{\tau, \rho}\begin{pmatrix} \xi_1, \ldots, \xi_m \\ j_1, \ldots, j_m \end{pmatrix} > 0.$$

Thus, another application of Lemma 1.3 completes the theorem.

LEMMA 2.1. *Suppose condition* (ii) *is violated for* i_q. *Then there exists a subsequence* ρ, $\tau \subset \rho \subset t$, *for which condition* (i) *can never hold, i.e., whatever the choice of* $\xi_1 < \cdots < \xi_m$, *there is an* i_s *such that* $(A_{\rho, t})_{i_s, \xi_s} = 0$. *To be specific,* ρ *coincides with* t *everywhere except at* t_{i_q} *where it has a knot of multiplicity*

$$m_\rho(t_{i_q}) = i_p + k - z, \tag{2.1}$$

where i_p *is an index whose existence is ensured by Lemma* 1.4.

Proof. Observe that Eq. (2.1) ensures $\tau \subset \rho \subset t$, by virtue of Lemma 1.4 (a). According to the definition of ρ,

$$\begin{aligned} \mu_L(j; \rho, t) &= \begin{cases} j, & \text{if } j \leqslant z-1, \\ j + m_t(t_z) - m_\rho(t_z), & \text{if } j \geqslant z, \end{cases} \\ \mu_R(j; \rho, t) &= \begin{cases} j, & \text{if } j \leqslant z + m_\rho(t_z) - 1, \\ j + m_t(t_z) - m_\rho(t_z), & \text{if } j \geqslant z + m_\rho(t_z). \end{cases} \end{aligned} \tag{2.2}$$

We will show that the interlacing conditions fail either at the index p or at the index q. Suppose they do hold at p, so that we have to establish their failure at q. The interlacing condition at the index p,

$$\mu_L(\xi_p; \rho, t) \leqslant i_p \leqslant \mu_R(\xi_p + k; \rho, t) - k, \tag{2.3}$$

forces $\xi_p = i_p$. This is so because the lefthand side implies, by (2.2), that $\xi_p \leqslant i_p$; and if $\xi_p < i_p$ then by the definition of $m_\rho(t_z)$ in Eq. (2.1), $\xi_p + k \leqslant i_p + k - 1 = m_\rho(t_z) + z - 1$. Hence, again by (2.2),

$$\mu_R(\xi_p + k; \rho, t) - k = \xi_p < i_p,$$

contradicting the right-hand side of (2.3).

Consider now i_q. Since $\xi_q \geqslant \xi_p + q - p = i_p + q - p \geqslant z$, by Lemma 1.4 (b) it follows from (2.2) that

$$\mu_L(\xi_q; \rho, t) \geqslant i_p + q - p + m_t(t_z) - m_\rho(t_z).$$

Substituting the definition of m_ρ, and then using Lemma 1.4(c),

$$\mu_L(\xi_q; \rho, t) \geqslant q - p + m_t(t_z) + z - k \geqslant d_q + m_t(t_z) + z - k = i_q + 1.$$

Hence the interlacing condition fails at the index q. ∎

LEMMA 2.2. *Suppose conditions* (i) *and* (ii) *hold. Let $t_z \in t$ be the first knot not in τ, in the sense that $m_\tau(t_z) < m_t(t_z)$, and set $\rho := \tau \cup \{t_z\}$, i.e.,*

$$\rho_j = \begin{cases} \tau_j, & \text{if } \tau_j \leqslant t_z, \\ t_z, & \text{if } \tau_{j-1} \leqslant t_z < \tau_j \\ \tau_{j-1}, & \text{if } \tau_{j-1} > t_z. \end{cases}$$

Then $\xi_1 < \cdots < \xi_m$ can be chosen such that conditions (i) *and* (ii) *hold for ρ and t with respect to $\{\xi_r\}$ and $\{i_r\}$, and*

$$\det A_{\tau,\rho}\begin{pmatrix} \xi_1, \ldots, \xi_m \\ j_1, \ldots, j_m \end{pmatrix} > 0. \tag{2.4}$$

Proof. Since $\rho = \tau \cup \{t_z\}$ we have, as pointed out by Jia [2], that inequality (2.4) holds if and only if the interlacing conditions are satisfied. It is easily seen, using Lemma 1.2, that this is the case if and only if

$$\xi_s = \begin{cases} j_s, & \text{if } j_s + k < y + m_\tau(t_z), \\ j_s \text{ or } j_s + 1, & \text{if } y + m_\tau(t_z) - k \leqslant j_s \leqslant y - 1, \\ j_s + 1, & \text{if } j_s \geqslant y, \end{cases} \tag{2.5}$$

where y is such that $\tau_{y-1} \leqslant t_{z-1} < t_z \leqslant \tau_y$. We have therefore to decide upon the value of ξ_s for those s for which $y + m_\tau(t_z) - k \leqslant j_s \leqslant y - 1$ and to prove that the resulting ξ sequence is strictly monotonic and that conditions (i) and (ii) hold again for ρ and t with respect to $\{\xi_r\}$ and $\{i_r\}$. Let us verify condition (ii) immediately since it does not depend at all on the definition of ξ. Were condition (ii) to be violated, so that (a)–(c) of Lemma 1.4 hold for ρ, then from $m_\rho(t_z) > m_\tau(t_z)$ and $z + m_\rho(t_z) \leqslant i_p + k$ it follows that condition (ii) is violated for τ as well, a contradiction.

To complete the choice of ξ denote for brevity $\mu(j)=\mu(j;\tau,t)$ and $\bar{\mu}(j)=\mu(j;\rho,t)$. It is easily seen that

$$\bar{\mu}_L(j)=\begin{cases}\mu_L(j), & \text{if } j<y,\\ z+m_t(t_z)-m_\rho(t_z), & \text{if } j=y,\\ \mu_L(j-1), & \text{if } j>y,\end{cases}$$
$$\bar{\mu}_R(j)=\begin{cases}\mu_R(j), & \text{if } j<y+m_\rho(t_z)-1,\\ z+m_\rho(t_z)-1, & \text{if } j=y+m_\rho(t_z)-1,\\ \mu_R(j-1), & \text{if } j>y+m_\rho(t_z)-1.\end{cases} \tag{2.6}$$

Now for s such that $y+m_\tau(t_z)-k\leqslant j_s\leqslant y-1$ set

$$\xi_s=\begin{cases}j_s, & \text{if } i_s<\bar{\mu}_L(j+1),\\ j_s+1, & \text{if } i_s+k>\bar{\mu}_R(j_s+k),\\ \max(j_s,\xi_{s-1}+1) & \text{otherwise.}\end{cases} \tag{2.7}$$

It is easily seen from the strict monotonicity of $\{j_s\}$ that with this definition, indeed, $j_s\leqslant\xi_s\leqslant j_s+1$ for all s.

Let us verify first that the interlacing conditions

$$\mu_L(\xi_r;\rho,t)\leqslant i_r\leqslant\mu_R(\xi_r+k;\rho,t)-k, \qquad r=1,\ldots,m, \tag{2.8}$$

hold. When $j_s+k<y+m_\tau(t_z)$ or $j_s\geqslant y$ it follows from (2.6) that

$$\bar{\mu}_L(\xi_s)=\mu_L(j_s), \qquad \bar{\mu}_R(\xi_s+k)=\mu_R(j_s+k).$$

Hence for these values of ξ_s inequality (2.8) is an immediate consequence of the corresponding interlacing conditions for τ. On the other hand, for s such that $y+m_\tau(t_z)-k\leqslant j_s\leqslant y-1$ it follows from the interlacing conditions for τ and Eq. (2.6) that

$$\bar{\mu}_L(j_s)=\mu_L(j_s)\leqslant i_s\leqslant\mu_R(j_s+k)-k=\bar{\mu}_R(j_s+k+1)-k. \tag{2.9}$$

Taking into account inequality (1.3),

$$\bar{\mu}_L(j_s+1)\leqslant\bar{\mu}_R(j_s+k)-k+1, \tag{2.10}$$

we have that

- if $i_s<\bar{\mu}_L(j_s+1)$, so that $\xi_s=j_s$, then $\bar{\mu}_L(\xi_s)\leqslant i_s$ from inequality (2.9), and $i_s\leqslant\bar{\mu}_R(\xi_s+k)-k$ from inequality (2.10);
- if $i_s+k>\bar{\mu}_R(j_s+k)$, so that $\xi_s=j_s+1$, then $\bar{\mu}_L(\xi_s)\leqslant i_s$ from inequality (2.10), and $i_s\leqslant\bar{\mu}_R(\xi_s+k)-k$ from inequality (2.9);
- if $\bar{\mu}_L(j_s+1)\leqslant i_s\leqslant\bar{\mu}_R(j_s+k)-k$ then the interlacing condition for ξ_s holds whether ξ_s is defined as j_s or as j_s+1.

This proves the interlacing conditions. Turning to the proof of the strict monotonicity of ξ, suppose to the contrary that there is a least p and a q, $p<q$, such that $\xi_q-\xi_p<q-p$. Since $j_s \leqslant \xi_s \leqslant j_s+1$ and $\{j_s\}$ is strictly monotone, it must be the case that $j_q-j_p=q-p$ and that $\xi_p=j_p+1$ and $\xi_q=j_q$. Hence it is seen from Eq. (2.5) that

$$y+m_\tau(t_z)-k \leqslant j_p<j_q \leqslant y-1. \tag{2.11}$$

We obtain therefore from Eq. (2.7) that all of the following hold:

(1) $i_p+k>\bar{\mu}_R(j_p+k)$,

(2) $i_q<\bar{\mu}_L(j_q+1)$,

(3) $j_q-j_p=q-p$.

To complete the proof we show that if (1), (2), and (3) hold then condition (ii) is violated in its formulation of Lemma 1.4.

It follows from inequality (2.11) and $\rho_y=\rho_{y+m_\tau(t_z)}=t_z$, that $\rho_{j_q+1} \leqslant t_z \leqslant \rho_{j_p+k}$. But $\rho_{j_q+1}<\rho_{j_p+k}$ is impossible because that, together with (1) and (2) and inequality (1.2), would imply

$$i_q+1 \leqslant \bar{\mu}_L(j_q+1) \leqslant \bar{\mu}_R(j_p+k)-(j_p+k-j_q-1) \leqslant i_p+j_q-j_p.$$

Upon substitution of (3) it is then seen that the i-indices cannot be strictly monotonic.

Thus, $\rho_{j_q+1}=t_z=\rho_{j_p+k}$. This implies

$$j_q+1=y, \qquad j_p+k=y+m_\rho(t_z)-1, \tag{2.12}$$

as follows: by (2.11) $j_q \leqslant y-1$, but $j_q<y-1$ would result in $\rho_{j_q+1} \leqslant \rho_{y-1}=\tau_{y-1}<t_z$; similarly, $j_p+k>y+m_\rho(t_z)-1$ yields $\rho_{j_q+1} \geqslant \rho_{y+m_\rho(t_z)}=\tau_{y+m_\tau(t_z)}>t_z$, a contradiction.

From Eq. (2.12) it follows that

$$q-p=j_q-j_p=k-m_\rho(t_z), \tag{2.13}$$

and also, by (1), (2), and Eq. (2.6), that

$$i_q \leqslant \bar{\mu}_L(j_q+1)-1=z+m_t(t_z)-m_\rho(t_z)-1, \tag{2.14}$$

$$i_p+k \geqslant \bar{\mu}_R(j_p+k)+1=z+m_\rho(t_z). \tag{2.15}$$

In turn inequalities (2.13)–(2.15) imply

$$i_p+k \leqslant i_q-q+p+k \leqslant z+m_t(t_z)-1, \tag{2.16}$$

$$i_p+q-p=i_p+k-m_\rho(t_z) \geqslant z, \tag{2.17}$$

$$q-p \geqslant k+i_q-z-m_t(t_z)+1=d_q. \tag{2.18}$$

Actually, for the last equality it still has to be shown that $t_{i_q} = t_z$. To this end, recall that t_z is the first knot not in τ, so that $\mu_L(j_q) = z - 1$. Therefore if $i_q < z$, then from the given $\mu_L(j_q) \leqslant i_q$, necessarily $i_q = z - 1 = j_q$. But then $i_p \leqslant i_q + p - q = j_q + p - q = j_p$, contradicting (1).

Since (2.16)–(2.18) establish the conditions of Lemma 1.4, we have shown that (1)–(3) can hold only if condition (ii) is violated. ∎

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