Another Proof of the Total Positivity of the Discrete Spline Collocation Matrix

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We provide a different proof for Morken’s result on necessary and sufficient conditions for a minor of the discrete B-spline collocation matrix to be positive and supply intuition for those conditions.

1. Introduction

In [3, Theorem 6] Morken gives necessary and sufficient conditions for a minor of the discrete B-spline collocation matrix to be positive, correcting an error in an earlier theorem of Jia [2]. One of these conditions may not be intuitively obvious. In this note we attempt to supply such intuition, and we provide a different proof.

Recapping Morken’s notation, let \( k \) be a positive integer; let \( t = \{ t_i \}_{i=\infty}^{\infty} \) be a bi-infinite, nondecreasing sequence of real numbers (knots) with \( t_i < t_{i+k} \) for all \( i \); and let \( \tau \) be a bi-infinite subsequence of \( t \), \( \tau \subseteq t \). We study the discrete B-spline collocation matrix \( A_{\tau,t} \) with elements given by \( (A_{\tau,t})_{i,j} = \sigma_{j,k,t,\tau}(i) \). Here \( \sigma_{j,k,t,\tau}(i) \) are the coefficients in the expansion of the B-spline \( B_{j,k,t,\tau} \) on the coarse knot sequence \( \tau \) in terms of the B-splines on the fine knot sequence \( t \),

\[
B_{j,k,t} = \sum_i \sigma_{j,k,t,\tau}(i) B_i \kappa,t.
\]

Denote, further,

\[
m_{s}(x) = \max\{ q - p \mid t_q \leq x \text{ and } x \leq t_{p+1} \},
\]

\[
l_{s}(i) = \max\{ p \mid t_{i-p} = t_i \}
\]

\[
r_{s}(i) = \max\{ p \mid t_{i+p} = t_i \}
\]

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We are interested in the conditions under which a minor of $A_{i,t}$ has a strictly positive determinant, as formulated by Morken [3].

**Theorem 1.** Let $k \geq 1$ be given, let $t$ be a knot vector with $t_i < t_{i+1}$ for all $i$, and let $\tau$ be a subsequence of $t$. Let $i_1 < i_2 < \ldots < i_m$ and $j_1 < j_2 < \ldots < j_m$ be two increasing integer sequences. Then

$$\det A_{i,t}(i_1, \ldots, i_m, j_1, \ldots, j_m) \geq 0,$$

with strict positivity if and only if both of the following conditions are satisfied:

(i) $(A_{i,t})_{i_q,j_q} > 0$ for $q = 1, 2, \ldots, m$.

(ii) If for some $q$, the multiplicity of $i_q$ in $t$ is greater than the multiplicity of $j_q$ in $\tau$, that is $m_i(t_{i_q}) < m_{\tau}(t_{j_q})$, then

$$i_q - m_i(t_{i_q}) < j_q - m_{\tau}(t_{j_q}),$$

where

$$d_q = k - r(t_{i_q}),$$

$$f_q = \min\{l(t_{i_q}), m_{\tau}(t_{j_q}) - m_i(t_{i_q}) - 1\}.$$

To ease the ascertainment and use of condition (i) we employ the index mappings $\mu_L(j, \tau; t)$ and $\mu_R(j, \tau; t)$, introduced in [1]. By definition they are such that whenever $\tau_{b-1} \leq t_{a-1} < t_a = \tau_b$,

$$\begin{align*}
\mu_L(b + u; \tau; t) &= a + m_i(t_{i_q}) - m_i(t_{a}), \\
\mu_R(b + u; \tau; t) &= a + u,
\end{align*}$$

where $\tau_{b-1} \leq t_{a-1} < t_a = \tau_b$, $u = 0, \ldots, m(t_{a}) - 1$. Therefore $\mu_L(j; \tau; t)$ is the index of the $t$-knot corresponding to $\tau_j$, when multiple $\tau$-knots are viewed as aligned in order at the right end of the corresponding (multiple) $t$-knot. Note that $\mu_L(b + m_i(t_{i_q})) - 1 \geq \mu_L(b + m_{\tau}(t_{j_q}) - 1)$, and that both index mappings are strictly monotone. In these terms Jia [2, Lemma 5] can be rephrased as follow (see [1]).

**Lemma 1.2.** $(A_{i,t})_{i,j} > 0$ if and only if

$$\mu_L(j; \tau; t) = i \leq \mu_R(j + k; \tau; t) - k. \quad (1.1)$$

In the sequel we will therefore refer to condition (i) as the “interlacing conditions.”
For later use we record the following, easily proven, property of \( \mu \). If 
\[ \tau_j < \tau_{j+1}, \quad \text{or if } \tau_j = \tau_{j+1}, \quad \text{and } m_1(\tau_j) = m_2(\tau_j), \]
then
\[ \mu_2(j; \tau, t) \leq \mu_1(j + r; \tau, t) - r. \quad (1.2) \]
In particular, the assumption that \( t_i < t_{i+k} \) for all \( i \) implies that if \( \tau < t \) then
\[ \mu_2(j; \tau, t) \leq \mu_1(j + k - 1; \tau, t) - k + 1 \quad \text{for all } j. \quad (1.3) \]

Let us turn now to an examination of condition (ii). The intuition behind this condition and, indeed, our proof of the theorem, is based on the following observation of Jia [2].

**Lemma 1.3.** Suppose that \( \tau < \rho < t \). Then
\[ \det A_{\tau,\rho} \begin{pmatrix} i_1, ..., i_m \\ j_1, ..., j_m \end{pmatrix} > 0 \]
if and only if there exist \( \xi_1 < \cdots < \xi_m \) such that
\[ \det A_{\rho,\tau} \begin{pmatrix} \xi_1, ..., \xi_m \\ i_1, ..., i_m \end{pmatrix} \cdot \det A_{\rho,\tau} \begin{pmatrix} \xi_1, ..., \xi_m \\ j_1, ..., j_m \end{pmatrix} > 0. \]

In particular, given any intermediate knot sequence \( \rho \), it must be possible to pick a monotonically increasing integer sequence \( \xi_1 < \cdots < \xi_m \) such that the interlacing conditions are satisfied for \( t \) and \( \rho \), \( (A_{\rho,\tau})_{\xi_1 \xi_1} > 0 \) for \( q = 1, 2, ..., m \). Let us look at a case in which this is not possible. We will demonstrate this by showing that if the interlacing conditions do hold then the \( \xi \) sequence cannot be strictly monotonic.

Suppose there are indices \( i_p \) and \( i_q \) such that \( t_{i_p+k} = t_{i_q} = t_z \) with \( t_{z-1} < t_z \).

In fact, let us require slightly more: that \( z + m_z(t_z) \leq i_p + k \leq z + m_z(t_z) - 1 \) and that \( z \leq i_q \leq z + m_z(t_z) - 1 - m_z(t_z) \).

Consider now a sequence \( p \) which is the same as \( t \) except that the multiplicity of the knot \( t_z \) in \( p \) is \( m := i_p + k - z \) instead of \( m_z(t_z) \). Note that by assumption \( m_z(t_z) \leq m \leq m_z(t_z) - 1 \). If \( (A_{\rho,\tau})_{\xi_1 \xi_1} > 0 \) and \( (A_{\rho,\tau})_{\xi_1 \xi_1} > 0 \) for some \( \xi_p \) and \( \xi_q \), then it is easily seen (cf. the proof of Lemma 2.1) that necessarily
\[ \xi_p \geq i_p, \quad \xi_q \leq i_q - m_z(t_z) + \max(m, z + m_z(t_z) - 1 - i_q). \]

The sequence \( \xi \) will certainly fail to be strictly monotonic if \( \xi_p + q - p > \xi_q \), which by the above is assuredly true if
\[ i_p + q - p > i_q - m_z(t_z) + \max(i_p + k - z, z + m_z(t_z) - 1 - i_q). \]
The following lemma spells this condition out and shows that it is in fact equivalent to condition (ii); it is therefore, somewhat surprisingly, the only type of case that needs to be ruled out. Incidentally, the assumption \(0 \leq i_q - z \leq m_i(t_z) - 1 - m_i(t_z)\) is not stated explicitly because it is a consequence of the other conditions and the interlacing conditions for \(t\) and \(\tau\).

**Lemma 1.4.** Condition (ii) is violated for \(i_q\), with \(t_{q-1} < t_t = t_{i_q}\), if and only if there exists an \(i_p\) for which all of the following hold:

\[(a)\quad z + m_i(t_z) \leq i_q + k \leq z + m_i(t_z) - 1,\]

\[(b)\quad i_q + q - p \geq z,\]

\[(c)\quad q - p \geq d_q.\]

**Proof.** Suppose condition (ii) is violated, and set \(p = q - d_q\). Writing out the definition of \(f_q\) while noting that \(i_q - d_q = z + m_i(t_z) - 1 - k\) and \(i_q - l(i_q) = z\), we get

\[i_p \geq i_q - d_q - (m_i(t_z) - m_i(t_z) - 1) = z - k + m_i(t_z), \quad (1.4)\]

\[i_p \geq i_q - (q - p) - l(i_q) = z - (q - p), \quad (1.5)\]

proving (b) and half of (a). To prove the remaining half observe that \(i_{r-1} = i_r - 1\) for all \(r\) and so

\[i_p = i_q - d_q \leq i_q - d_q = z - k + m_i(t_z) - 1.\]

To prove the converse we note that if (a), (b), (c) hold for some \(\bar{p}\), then they must also hold for \(p = q - d_q \geq \bar{p}\). Namely, the left-hand side of (a) is immediate while the right-hand side follows from \(i_p = i_q - d_q \leq i_q - d_q\); and (b) follows from \(i_p - p \geq i_p - \bar{p}\). Therefore, the proof is already implicit in inequalities (1.4), (1.5). \(\square\)

2. Proof of the Theorem

The necessity of the interlacing conditions is fairly clear and proven in [2]. To establish the necessity of condition (ii), Lemma 2.1 exhibits a subsequence \(\tau \subset p \subset t\) such that if the condition is violated then

\[\det A_{p, t} (i_1, \ldots, i_m) = 0.\]
whatever the choice of $\{\xi_s\}$. It follows then from Lemma 1.3 that
\[
\det A_{ij} (i_j,\ldots, i_m) = 0.
\]
To prove the converse we proceed by induction on the difference in the number of knots in $t$ and $\tau$. If the difference is zero, $t = \tau$, it is easily seen from Lemma 1.2 that condition (i) implies $i_q = j_q$, $q = 1,\ldots, m$, and hence the determinant is positive. For the induction step we exhibit in Lemma 2.2, if conditions (i) and (ii) hold, a subsequence $\rho$, $\tau \subset \rho \subset t$, and a set of indices $\{\xi_s\}$ such that conditions (i) and (ii) hold again for $\rho$ and $t$ with respect to $\{\xi_s\}$ and $\{i_s\}$, and at the same time
\[
\det A_{ij} (i_{j_1},\ldots, i_{j_m}) > 0.
\]
Thus, another application of Lemma 1.3 completes the theorem.

**Lemma 2.1.** Suppose condition (ii) is violated for $i_q$. Then there exists a subsequence $\rho$, $\tau \subset \rho \subset t$, for which condition (i) can never hold, i.e., whatever the choice of $\xi_1 < \cdots < \xi_m$, there is an $i_s$ such that $(A_{ij})_{i_s\xi} = 0$. To be specific, $p$ coincides with $t$ everywhere except at $t_{iq}$ where it has a knot of multiplicity
\[
m_p(t_{iq}) = i_p + k - z,
\]
where $i_p$ is an index whose existence is ensured by Lemma 1.4.

**Proof.** Observe that Eq. (2.1) ensures $\tau \subset \rho \subset t$, by virtue of Lemma 1.4 (a). According to the definition of $\rho$,
\[
\mu_L(j; p, t) = \begin{cases} j, & \text{if } j \leq z - 1, \\ j + m_j(t_z) - m_p(t_z), & \text{if } j \geq z, \end{cases}
\]
\[
\mu_R(j; p, t) = \begin{cases} j, & \text{if } j \leq z + m_p(t_z) - 1, \\ j + m_j(t_z) - m_p(t_z), & \text{if } j \geq z + m_p(t_z). \end{cases}
\]

We will show that the interlacing conditions fail either at the index $p$ or at the index $q$. Suppose they do hold at $p$, so that we have to establish their failure at $q$. The interlacing condition at the index $p$,
\[
\mu_L(\xi_p; p, t) \leq i_p \leq \mu_R(\xi_p + k; p, t) - k,
\]
where $i_p$ is an index whose existence is ensured by Lemma 1.4.
forces \( \zeta_p = i_p \). This is so because the lefthand side implies, by (2.2), that 
\[ \zeta_p \leq i_p; \quad \text{and if} \quad \zeta_p \leq i_p \quad \text{then by the definition of} \quad m_y(t_z) \quad \text{in Eq. (2.1),} \]
\[ \zeta_p + k \leq i_p + k - 1 = m_y(t_z) + z - 1. \]
Hence, again by (2.2),
\[ \mu \phi(\zeta_p + k; \rho, t) - k = \zeta_p < i_p, \]
contradicting the right-hand side of (2.3).

Consider now \( i_q \). Since 
\[ \tilde{\xi}_q \geq \xi_p + q - p = i_p + q - p \geq z, \]
by Lemma 1.4 (b) it follows from (2.2) that
\[ \mu \phi(\tilde{\xi}_q; \rho, t) \geq i_q + q - p + m_y(t_z) - m_y(t_z). \]
Substituting the definition of \( m_y \), and then using Lemma 1.4(c),
\[ \mu \phi(\tilde{\xi}_q; \rho, t) \geq q - p + m_y(t_z) + z - k \geq d_q + m_y(t_z) + z - k = i_q + 1. \]
Hence the interlacing condition fails at the index \( q \).

**Lemma 2.2.** Suppose conditions (i) and (ii) hold. Let \( t_z \in \tau \) be the first knot not in \( \tau \), in the sense that \( m_y(t_z) < m_y(t_z) \), and set \( \rho := \tau \cup \{ t_z \} \), i.e.,
\[ \rho_j = \begin{cases} \tau_j, & \text{if} \quad \tau_j \leq t_z, \\ t_z, & \text{if} \quad \tau_{j-1} \leq t_z < \tau_j, \\ \tau_{j-1}, & \text{if} \quad \tau_{j-1} > t_z. \end{cases} \]

Then \( \xi_1 < \cdots < \xi_m \) can be chosen such that conditions (i) and (ii) hold for \( \rho \) and \( t \) with respect to \( \{ \xi_j \} \) and \( \{ i_j \} \), and
\[ \det A_{\rho, t}\left( \xi_1, \ldots, \xi_m \right) > 0. \quad (2.4) \]

**Proof.** Since \( \rho = \tau \cup \{ t_z \} \) we have, as pointed out by Jia [2], that inequality (2.4) holds if and only if the interlacing conditions are satisfied. It is easily seen, using Lemma 1.2, that this is the case if and only if
\[ \xi_s = \begin{cases} j_s, & \text{if} \quad j_s + k < y + m_y(t_z), \\ j_s + 1, & \text{if} \quad y + m_y(t_z) - k \leq j_s \leq y - 1, \\ j_s + 1, & \text{if} \quad j_s \geq y, \end{cases} \quad (2.5) \]
where \( y \) is such that \( \tau_{s-1} \leq t_z < \tau_s \). We have therefore to decide upon the value of \( \xi_s \) for those \( s \) for which \( y + m_y(t_z) - k \leq j_s \leq y - 1 \) and to prove that the resulting \( \xi \) sequence is strictly monotonic and that conditions (i) and (ii) hold again for \( \rho \) and \( t \) with respect to \( \{ \xi_j \} \) and \( \{ i_j \} \). Let us verify condition (ii) immediately since it does not depend at all on the definition of \( \xi \). Were condition (ii) to be violated, so that (a)–(c) of Lemma 1.4 hold for \( \rho \), then from \( m_y(t_z) > m_y(t_z) \) and \( z + m_y(t_z) \leq i_p + k \) it follows that condition (ii) is violated for \( \tau \), as well, a contradiction.
To complete the choice of $\zeta$ denote for brevity $\mu(f) = \mu(f; \tau, t)$ and $\bar{\mu}(f) = \mu(f; \rho, t)$. It is easily seen that

\[
\begin{align*}
\mu_L(f) &= \begin{cases} 
\mu_L(j), & \text{if } j < y, \\
z + m_{j}(t_{z}) - m_{j}(t_{z}), & \text{if } j = y, \\
\mu_L(j-1), & \text{if } j > y,
\end{cases} \\
\mu_R(f) &= \begin{cases} 
\mu_R(j), & \text{if } j < y + m_{j}(t_{z}) - 1, \\
z + m_{j}(t_{z}) - 1, & \text{if } j = y + m_{j}(t_{z}) - 1, \\
\mu_R(j-1), & \text{if } j > y + m_{j}(t_{z}) - 1.
\end{cases}
\end{align*}
\tag{2.6}
\]

Now for $s$ such that $y + m_{j}(t_{z}) - k \leq j_{s} \leq y - 1$ set

\[
\zeta_{s} = \begin{cases} 
j_{s}, & \text{if } i_{s} < \mu_{L}(j_{s} + 1), \\
j_{s} + 1, & \text{if } i_{s} + k > \mu_{R}(j_{s} + k), \\
\max(j_{s}, \zeta_{s-1} + 1), & \text{otherwise}.
\end{cases}
\tag{2.7}
\]

It is easily seen from the strict monotonicity of $\{j_{s}\}$ that with this definition, indeed, $j_{s} \leq \zeta_{s} \leq j_{s} + 1$ for all $s$.

Let us verify first that the interlacing conditions

\[
\mu_{L}(\zeta_{s}; \rho, t) \leq i_{s} \leq \mu_{R}(\zeta_{s} + k; \rho, t) - k, \quad r = 1, ..., m,
\tag{2.8}
\]

hold. When $j_{s} + k < y + m_{j}(t_{z})$ or $j_{s} \geq y$ it follows from (2.6) that

\[
\mu_{L}(\zeta_{s}) = \mu_{L}(j_{s}), \quad \bar{\mu}_{L}(\zeta_{s} + k) = \mu_{R}(j_{s} + k).
\]

Hence for these values of $\zeta_{s}$ inequality (2.8) is an immediate consequence of the corresponding interlacing conditions for $\tau$. On the other hand, for $s$ such that $y + m_{j}(t_{z}) - k \leq j_{s} \leq y - 1$ it follows from the interlacing conditions for $\tau$ and Eq. (2.6) that

\[
\mu_{L}(j_{s}) = \mu_{L}(j_{s} + 1) \leq i_{s} \leq \bar{\mu}_{L}(j_{s} + k) - k = \bar{\mu}_{R}(j_{s} + k + 1) - k.
\tag{2.9}
\]

Taking into account inequality (1.3),

\[
\bar{\mu}_{L}(j_{s} + 1) \leq \bar{\mu}_{R}(j_{s} + k) - k + 1,
\tag{2.10}
\]

we have that

- if $i_{s} < \mu_{L}(j_{s} + 1)$, so that $\zeta_{s} = j_{s}$, then $\bar{\mu}_{L}(\zeta_{s}) \leq i_{s}$ from inequality (2.9), and $i_{s} \leq \bar{\mu}_{R}(\zeta_{s} + k) - k$ from inequality (2.10);
- if $i_{s} + k > \bar{\mu}_{R}(j_{s} + k)$, so that $\zeta_{s} = j_{s} + 1$, then $\bar{\mu}_{L}(\zeta_{s}) \leq i_{s}$ from inequality (2.10), and $i_{s} \leq \bar{\mu}_{R}(\zeta_{s} + k) - k$ from inequality (2.9);
- if $\bar{\mu}_{L}(j_{s} + 1) \leq i_{s} \leq \bar{\mu}_{R}(j_{s} + k) - k$ then the interlacing condition for $\zeta_{s}$ holds whether $\zeta_{s}$ is defined as $j_{s}$ or as $j_{s} + 1$. 

This proves the interlacing conditions. Turning to the proof of the strict monotonicity of $\xi$, suppose to the contrary that there is a least $p$ and a $q$, $p < q$, such that $\xi_p - \xi_q < q - p$. Since $f_z \leq \xi_z \leq f_z + 1$ and $\{ f_z \}$ is strictly monotone, it must be the case that $f_q - f_p = q - p$ and that $\xi_p = f_p + 1$ and $\xi_q = f_q$. Hence it is seen from Eq. (2.5) that

$$y + m_z(t_z) - k \leq f_q - f_p \leq y - 1.$$  \hfill (2.11)

We obtain therefore from Eq. (2.7) that all of the following hold:

(1) $i_p + k > \mu f_p (f_p + k)$,
(2) $i_q < \mu f_q (f_q + 1)$,
(3) $j_q - f_p = q - p$.

To complete the proof we show that if (1), (2), and (3) hold then condition (ii) is violated in its formulation of Lemma 1.4.

It follows from inequality (2.11) and $\rho_z = \rho_{z+m_z(t_z)} = t_z$, that $\rho_{z+1} \leq t_z \leq \rho_{z+k}$. But $\rho_{z+1} < \rho_{z+2}$ is impossible because that, together with (1) and (2) and inequality (1.2), would imply

$$i_q + 1 \leq \mu f_q (j_q + 1) \leq \mu f_p (j_p + k) - (f_p + k - f_q - 1) \leq i_p + j_q - f_p.$$  

Upon substitution of (3) it is then seen that the $i$-indices cannot be strictly monotonic.

Thus, $\rho_{z+1} = t_z = \rho_{z+k}$. This implies

$$j_q + 1 = y, \quad j_p + k = y + m_z(t_z) - 1,$$  \hfill (2.12)

as follows: by (2.11) $j_q \leq y - 1$, but $j_q < y - 1$ would result in $\rho_{z+1} \leq \rho_{y-1} = \tau_{y-1} < t_z$; similarly, $j_p + k > y + m_z(t_z) - 1$ yields $\rho_{z+1} \geq \rho_{z+m_z(t_z)} = \tau_{z+m_z(t_z)} > t_z$, a contradiction.

From Eq. (2.12) it follows that

$$q - p = j_q - f_p = k - m_z(t_z),$$  \hfill (2.13)

and also, by (1), (2), and Eq. (2.6), that

$$i_q \leq \mu f_q (j_q + 1) - 1 = z + m_z(t_z) - m_z(t_z) - 1,$$  \hfill (2.14)

$$i_q + k \geq \mu f_q (j_q + k) + 1 = z + m_z(t_z) - 1.$$  \hfill (2.15)

In turn inequalities (2.13)–(2.15) imply

$$i_q + k \leq i_q - q + p + k \leq z + m_z(t_z) - 1,$$  \hfill (2.16)

$$i_q + q - p = i_p + k - m_z(t_z) \geq z,$$  \hfill (2.17)

$$q - p \geq k + i_q - z - m_z(t_z) + 1 = d_q.$$

\hfill (2.18)
Actually, for the last equality it still has to be shown that $t_i = t_z$. To this end, recall that $t_z$ is the first knot not in $\tau$, so that $\mu_i(j_q) = z - 1$. Therefore if $i_q < z$, then from the given $\mu_i(j_q) \leq l_q$, necessarily $i_q = z - 1 = j_q$. But then $l_q \leq i_q + p - q = j_q + p - q = j_p$, contradicting (1).

Since (2.16)–(2.18) establish the conditions of Lemma 1.4, we have shown that (1)–(3) can hold only if condition (ii) is violated.

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