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Finding minimum hidden guard sets in polygons—tight approximability results ☆

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Abstract

We study the problem MINIMUM HIDDEN GUARD SET, which consists of positioning a minimum number of guards in a given polygon (or other structure such as a terrain) such that no two guards see each other and such that every point in the polygon is visible from at least one guard. By constructing a gap-creating reduction from 5-OCCURRENCE-3-SATISFIABILITY, we show that this problem cannot be approximated by any polynomial-time algorithm with an approximation ratio of $|I|^{1-\epsilon}$ for any $\epsilon > 0$, unless $NP = P$, where $|I|$ is the size of the input polygon. The result even holds for input polygons without holes, which separates the problem from other visibility problems such as guarding and hiding, where strong inapproximability results hold only for polygons with holes. We also show that a straight-forward approximation algorithm achieves an approximation ratio of $|I|$. These two results characterize the approximability threshold of MINIMUM HIDDEN GUARD SET exactly up to low-order terms.

Keywords: Hidden guard set; Inapproximability; Approximation algorithm; Visibility; Art gallery problems; Terrains; Polygons

1. Introduction

In the field of visibility problems, guarding and hiding are among the most prominent and most intensely studied problems. In guarding, we are given as input a simple polygon with or without holes and we need to find a minimum number of guard positions in the polygon such that every point in the interior of the polygon is visible from at least one guard. Two points in the polygon are visible from each other (or alternatively: see each other), if the straight line segment connecting the two points does not intersect the exterior of the polygon. In hiding, we need to find a maximum number of points in the given input polygon such that no two points see each other.

The combination of these two classic problems has been studied in the literature as well [12]. The problem is called MINIMUM HIDDEN GUARD SET and is formally defined as follows:

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Definition 1. The problem MINIMUM HIDDEN GUARD SET consists of finding a minimum set of guard positions in the interior of a given simple polygon such that no two guards see each other and such that every point in the interior of the polygon is visible from at least one guard.

We can define variations of this problem by allowing input polygons to contain holes or not or by letting the input be a 2.5 dimensional terrain. A 2.5 dimensional terrain is given as a triangulated set of vertices in the plane together with a height value for each vertex. The linear interpolation in between the vertices defines a bivariate continuous function, thus the name 2.5 dimensional terrain (see [9]). Extending our definition from polygons, we say that two points on or above a terrain are visible from (or see) each other if the straight line segment connecting the two points does not intersect the space below the terrain. In other variations, the guards are restricted to sit on vertices. Problems of this type arise in a variety of applications, most notably in telecommunications, where guards correspond to antennas in a network with a simple line-of-sight wave propagation model (see [3]).

While MINIMUM HIDDEN GUARD SET is *NP*-hard for input polygons with or without holes [12], no approximation algorithms or inapproximability results are known. For other visibility problems, such as guarding and hiding, the situation is different: MINIMUM VERTEX/POINT/EDGE GUARD are *NP*-hard [10] and cannot be approximated with an approximation ratio that is better than logarithmic in the number of polygon or terrain vertices for input polygons with holes or terrains [3]; these problems cannot be approximated with an arbitrarily small constant approximation ratio[1] for input polygons without holes [3]. The best approximation algorithms for these guarding problems achieve a logarithmic approximation ratio for MINIMUM VERTEX/EDGE GUARD for polygons [6] and terrains [5], which matches the logarithmic inapproximability result up to low-order terms in the case of input polygons with holes and terrains; the best approximation ratio known for MINIMUM POINT GUARD is $\Theta(|I|)$, where $|I|$ is the number of polygon or terrain vertices; intuitively, the lack of better approximation algorithms for MINIMUM POINT GUARD is due to the fact that the set of possible guard positions is infinite in this problem. The problem MAXIMUM HIDDEN SET cannot be approximated with an approximation ratio of $|I|^\epsilon$ for some $\epsilon > 0$ for input polygons with holes and it cannot be approximated with an arbitrarily small constant approximation ratio for polygons without holes [4]. The best approximation algorithms achieve approximation ratios of $\Theta(|I|)$ [4]. Thus, for both, hiding and guarding, the exact inapproximability threshold is still open for input polygons without holes. To get an overview of the multitude of results in visibility problems, consult [13] or [14].

In this article, we present an inapproximability result for MINIMUM HIDDEN GUARD SET: we show that no polynomial-time algorithm can guarantee an approximation ratio of $|I|^{1-\epsilon}$ for any $\epsilon > 0$, unless $NP = P$, where $|I|$ is the number of vertices in the input structure. The result holds for terrains, polygons with holes, and even polygons without holes as input structures. We obtain our result by constructing a gap-creating reduction from 5-OCCURRENCE-3-SATISFIABILITY, which is the *NP*-complete [2] satisfiability variation where each clause consists of at most three literals and each variable occurs at most five times as a literal. A gap-creating reduction is a straightforward variation of the concept of a gap-preserving reduction as defined in [1]. In our case, the gap-creating reduction maps the decision problem of 5-OCCURRENCE-3-SATISFIABILITY (i.e., the question whether a given formula can be satisfied or not) to a decision version of MINIMUM HIDDEN GUARD SET in which we are asked to decide whether a polygon can be guarded by either k or k' hidden guards where $k' > k$. The ratio k'/k between the two possible solutions of the resulting MINIMUM HIDDEN GUARD SET instance is a lower bound for the approximation ratio achievable by any polynomial-time algorithm and it is called a gap; thus, the term gap-creating reduction.

We also analyze an approximation algorithm for MINIMUM HIDDEN GUARD SET proposed in [12] and show that it achieves an approximation ratio of n, where n is the number of input vertices; in fact, our proof shows that any feasible solution achieves this bound. The algorithm works for polygons as input structures; the main idea can be easily adapted to work for terrains as well, resulting in the same approximation ratio. Thus, our results determine the approximability threshold of MINIMUM HIDDEN GUARD SET exactly up to low-order terms.

The remainder of this article is organized as follows: In Section 2, we present the construction of the reduction. We analyze the reduction and obtain our main result in Section 3. We analyze the approximation algorithm in Section 4. Section 5 contains extensions of our results to variations of MINIMUM HIDDEN GUARD SET. Concluding thoughts and directions for future research in the area of approximability of visibility problems are given in Section 6.

[1] In other words: there exists a constant $\delta > 0$ such that no polynomial-time approximation can guarantee an approximation ratio of $1 + \delta$.

2. Construction of the reduction

In this section, we show how to construct an instance of MINIMUM HIDDEN GUARD SET from a given instance of 5-OCCURRENCE-3-SATISFIABILITY in polynomial time. 5-OCCURRENCE-3-SATISFIABILITY is defined as follows:

Definition 2. Let I be a boolean formula given in conjunctive normal form, with each clause consisting of at most three literals and with each variable appearing in at most five clauses. The problem 5-OCCURRENCE-3-SATISFIABILITY consists of finding a truth assignment for the variables of I such that all clauses are satisfied.

We now show how to construct in polynomial time from a given instance I of 5-OCCURRENCE-3-SATISFIABILITY with n variables $x_1, \ldots, x_n$ and m clauses $c_1, \ldots, c_m$ an instance I' of MINIMUM HIDDEN GUARD SET, i.e., a simple polygon. An overview of the construction is given in Fig. 1. The main body of the constructed polygon is of rectangular shape. For each clause c_i, a *clause pattern* is constructed on the lower horizontal line of the rectangle, and for each variable x_i, we construct a *variable pattern* on the upper horizontal line as indicated in Fig. 1.

The construction will be such that a variable assignment that satisfies all clauses of I exists, if and only if the corresponding polygon I' has a hidden guard set with $\mathrm{O}(n)$ guards; otherwise, I' has a hidden guard set of size $\Omega(t)$, where parameter $t \gg n$ will be defined as part of the *rake-gadget* in the construction. The rake gadget, shown in Fig. 2, enables us to force a guard to a specific point R in the polygon.

Definition 3. The *rake gadget* consists of t polygon dents, which are small trapezoidal elements that point towards point R that is on the polygon boundary; points R' and R'' are auxiliary points that limit the area from which the dent bottom vertices can be seen; the rake gadget is embedded into the polygon such that no two guards can be hidden from each other inside the region enclosed by (R', R, R'', r, l).

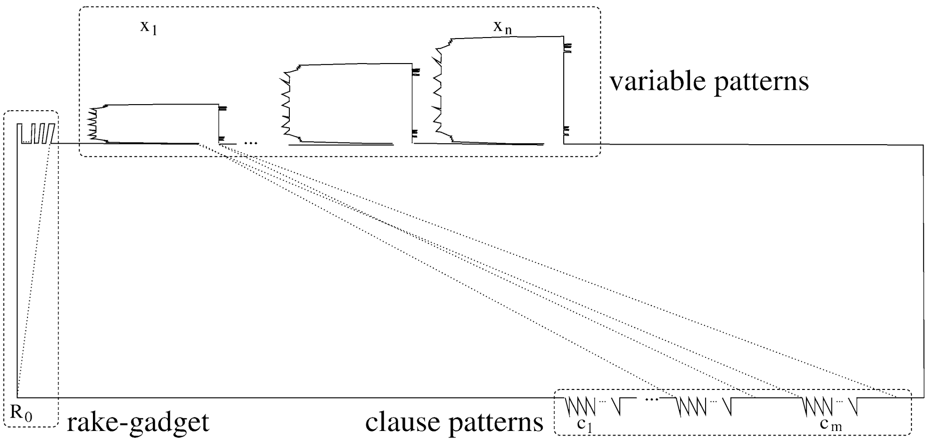

Fig. 1. Overview of construction.

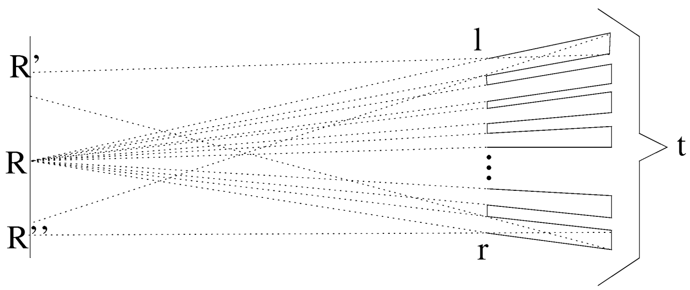

Fig. 2. Rake with t dents.

Lemma 1. *If the t dents of a rake are not covered by a single hidden guard at point R, then t hidden guards (namely one guard for each dent) are necessary to cover the dents.*

Proof. Clearly, any guard outside the region (R', R, R'', l, r) and outside the dents does not see a single dent completely. A guard in this region (but not at R) sees at most one dent completely, but only one such guard can exist as guards must be hidden from each other. Therefore, at least $t - 1$ guards must be hidden in the dents. □

In order to benefit from this property of a rake, we must place the rake in the polygon in such a way that the view from point R to the rake dents is not blocked by other polygon edges. As shown in Fig. 1, we place a rake at point R_0 in the lower left corner of the rectangle with the t dents at the top left corner.

A clause pattern, shown in Fig. 3, consists of t triangular-shaped spikes. Clause patterns are placed on the lower horizontal line of the rectangle. For each spike, the straight-line extension of the left (right) tight intersects with the upper left (right) corner of the large rectangle; thus the patterns are constructed in such a way that a guard on the upper horizontal line could see all spikes of all clause patterns. (This, however, will never happen, as we have already forced a guard to point R_0 to cover its rake. This guard would see any guard on the upper horizontal line.)

For each variable x_i, we construct a variable pattern, that is placed on top of the horizontal line of the rectangle. Each variable pattern opens the horizontal line for a unit distance. Each variable pattern has constant distance from its neighbors, and the right-most variable pattern (for variable x_n) is still to the left of the left-most clause pattern (for clause c_1), as indicated in Fig. 1. The variable patterns will differ in height, with the left-most variable pattern (for x_1) being the smallest and the right-most (for x_n) the tallest. Fig. 4 shows the variable pattern of variable x_i.

A variable pattern is a rectangular-shaped structure with a point F_i on top and a point T_i on the bottom. The construction is such that a guard sits at F_i, if the ith variable is set to false, and at T_i otherwise. Literals are represented

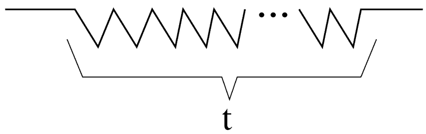

Fig. 3. Clause pattern consisting of t triangles.

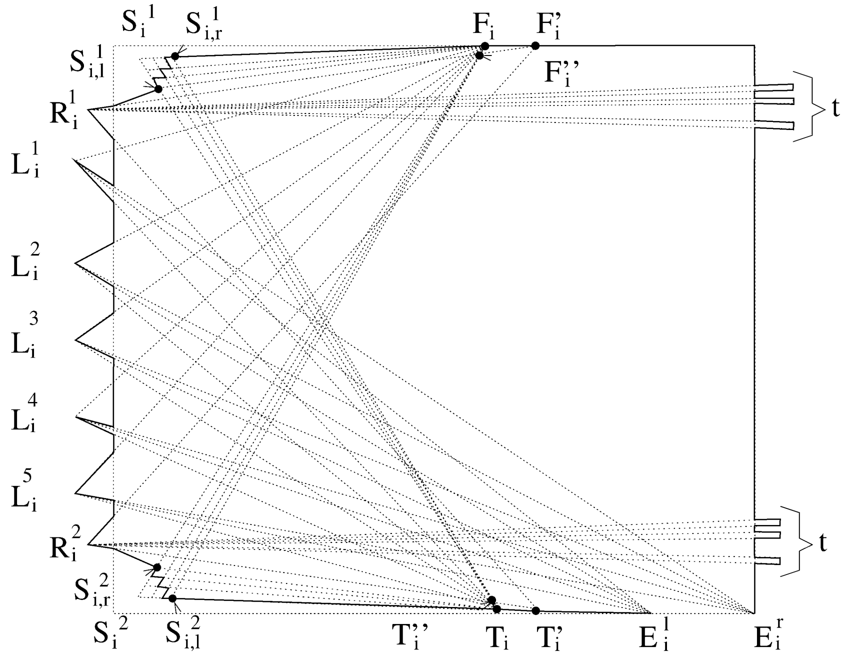

Fig. 4. Variable pattern with three positive and two negative literals.

by triangles with tips $L_i^1, \ldots, L_i^5$ for each of the five occurrences of the variable (some may be missing, if a variable occurs less than five times as literal). These triangles are constructed such that—for positive literals—they are completely visible from F_i, but not from T_i, and—for negative literals—they are completely visible from T_i, but not from F_i. A guard that sits at a point L_i^k, for any $k = 1, \ldots, 5$, can see through the exit of the variable pattern between points E_i^l and E_i^r. The construction is such that such a guard sees all spikes of the corresponding clause pattern (but no spikes of other clause patterns). This is shown schematically in Fig. 1.

In order to force a guard to sit at either F_i or T_i, we construct a rake point R_i^1 above L_i^1 and a rake point R_i^2 below L_i^5 with t dents each, all of which are on the right vertical line of the variable rectangle. Points R_i^1 and R_i^2 are at the tip of small triangles that point towards points F_i' and T_i', which lie a small distance to the right of F_i and T_i, respectively; points R_i^1 and R_i^2 are constructed such that they do not see any clause gadgets. In addition, we construct two areas S_i^1 and S_i^2 to the left of T_i and F_i, where we put t triangular spikes, each pointing exactly towards F_i and T_i. For simplicity, we have only drawn three triangular spikes in Fig. 4 instead of t. Area S_i^1 is the area of all these triangles at the top of the variable rectangle, area S_i^2 is the area of all these triangles at the bottom of the variable rectangle.

This completes our description of the constructed polygon that is an instance of MINIMUM HIDDEN GUARD SET. The polygon consists of a number of vertices that is polynomial in the size $|I|$ of the 5-OCCURRENCE-3-SATISFIABILITY instance I and in t. The coordinates of each vertex can be computed in time polynomial in $|I|$ and t, and they can be expressed by a polynomial (in $|I|$ and t) number of bits as they all can be modeled as intersection points of lines that are defined by points with polynomial coordinates. To see this, consider that an intersection point of two lines can be given in the input as the intersection point of those two lines and thus requires the sum of the bits that are needed to express each line; one can then define higher-order intersection points (i.e., points that are intersection points of lines that are in turn defined through intersection points of lines); it is fairly straight-forward to see in our construction that no point will achieve a nesting level of more than eight, i.e., a constant; thus each intersection point can be expressed with $\mathrm{O}(\log(nt)$ bits.[2] Therefore, the reduction is polynomial, if t is polynomial in $|I|$.

3. Analysis of the reduction

The following two lemmas describe the reduction as gap-creating and will allow us to prove our inapproximability result.

Lemma 2. *If the* 5-OCCURRENCE-3-SATISFIABILITY *instance I with n variables can be satisfied by a variable assignment, then the corresponding* MINIMUM HIDDEN GUARD SET *instance I' has a solution with at most* $8n + 1$ *guards.*

Proof. Given a satisfying assignment of the variables in I, we place guards in I' as follows: we set a guard at each rake point R_0 and R_i^1 and R_i^2, for $i = 1, \ldots, n$, which gives a total of $2n + 1$ hidden guards. For each variable x_i, we then place a guard at F_i or T_i depending on the truth value of the variable in the satisfying truth assignment; this yields additional n hidden guards. Finally, we place a guard at each literal L_i^k, if and only if the corresponding literal is true. This yields at most $5n$ hidden guards, as each variable occurs at most five times as a literal. Since the truth assignment satisfies all clauses, each clause pattern will be covered by at least one guard. The variable patterns and the main body rectangle are covered completely as well. Thus, the solution is feasible and consists of at most $8n + 1$ guards. □

Lemma 3. *If the* 5-OCCURRENCE-3-SATISFIABILITY *instance I with n variables cannot be satisfied by a variable assignment, then any solution of the corresponding* MINIMUM HIDDEN GUARD SET *instance I' has at least t guards.*

Proof. We prove the following equivalent formulation: If I' has a solution with strictly less than t guards, then I is satisfiable.

[2] For ease of presentation, we omit a detailed coordinate-level description of individual vertices.

Assume we have a solution for I' with less than t guards. Then, there must be a guard at each rake point R_0 and R_i^1 and R_i^2 for $i = 1, \dots, n$; this already restricts the possible positions for all other guards quite drastically, since they must be hidden from each other.

Observe in this solution, how the triangles of the areas S_i^1 and S_i^2 are covered. We need to introduce a few more points(see Fig. 4): point F_i'' is the intersection point of the line from R_i^1 to F_i' and from $S_{i,r}^2$ to F_i; point T_i'' is the intersection point of the line from R_i^2 to T_i' and from $S_{i,l}^1$ to T_i. Since we have guards at rake points R_i^1 and R_i^2, the guards for S_i^1 and S_i^2 can only lie in the 4-gons $(S_{i,l}^1, S_{i,r}^1, F_i, F_i')$ or $(S_{i,r}^2, S_{i,l}^2, T_i, T_i')$, but only a guard in the smaller triangle of either (F_i, F_i', F_i'') or (T_i, T_i', T_i'') can see both areas S_i^1 and S_i^2 If S_i^1 or S_i^2 is covered by a guard outside these triangles, then the other area can only be covered with t guards inside the S_i^1 or S_i^2 triangles. Therefore, there must be a guard in either one of the two triangles (F_i, F_i', F_i'') or (T_i, T_i', T_i'') in each variable pattern. We can move this guard to point F_i or T_i, respectively, without changing which literal triangles it sees.

Now, the only areas in the construction not yet covered are the literal triangles of those literals that are true and the spikes of the clause patterns. Assume for the sake of contradiction that one guard is hidden in a triangle of a clause pattern c_i. This guard sees the triangles of all literals that represent literals from the clause. This, however, implies that the remaining $t - 1$ triangles of the clause pattern c_i can only be covered by $t - 1$ additional guards in the clause pattern, thus resulting in t guards in total, contradicting our initial assumption. Therefore, all remaining guards must sit in the literal triangles in the variable patterns. W.l.o.g., we assume that there is a guard at each literal point L_i^k that is not yet covered by a guard at points F_i or T_i. If these guards do not collectively cover all clause patterns, then at least t guards are needed to cover the remaining clause patterns, which contradicts our assumption. □

Lemmas 2 and 3 immediately imply that we cannot approximate MINIMUM HIDDEN GUARD SET with an approximation ratio of $\frac{t}{8n+1}$ in polynomial time, because such an algorithm could be used to decide 5-OCCURRENCE-3-SATISFIABILITY. To get to an inapproximability result, we first observe that the number of vertices in I':

$$|I'| \leqslant (8t + 30)n + 2tm + 4t + 100 \leqslant 18tn + 30n + 4t + 100$$

by generously counting the constructed polygon vertices and using $m \leqslant 5n$. We now set

$$t = n^k$$

for an arbitrary but fixed $k > 1$. This implies $|I'| \leqslant n^{k+2}$ for large enough values of n, and thus

$$n \geqslant |I'|^{\frac{1}{k+2}}.$$

On the other hand, we cannot approximate MINIMUM HIDDEN GUARD SET with an approximation ratio of

$$\frac{t}{8n+1} \geqslant \frac{n^k}{n^2} = n^{k-2} \geqslant |I'|^{\frac{k-2}{k+2}} = |I'|^{1-\frac{4}{k+2}}.$$

Since k is an arbitrarily large constant, we have shown our main theorem:

Theorem 1. MINIMUM HIDDEN GUARD SET *on input polygons with or without holes cannot be approximated by any polynomial time approximation algorithm with an approximation ratio of* $|I|^{1-\epsilon}$ *for any* $\epsilon > 0$*, where* $|I|$ *is the number of vertices in the input polygon, unless* $NP = P$.

Theorem 1 does not address the issue of bit-level input size, but rather sticks to number of vertices. The theorem can be transformed such that it does take into account the bit-level size of the constructed polygon; however, such a transformation comes with a considerable amount of mathematical technicalities that we choose to omit.

Theorem 1 can be extended to terrains as input structures by using the following transformation from a polygon to a terrain (see [3]): Given a simple polygon, draw a bounding box around the polygon and then let all the area in the exterior of the polygon have height h (for some $h > 0$) and the interior height zero. This results in a terrain with vertical walls that we then triangulate. Any solution for MINIMUM HIDDEN GUARD SET on a polygon with k guards can be transformed into a solution of the corresponding terrain problem with $k + 1$ guards by placing the k guards from the polygon solution at the bottom (at height zero) of the terrain at their corresponding places and by placing one additional guard on a vertex of the bounding box of the terrain. The k guards cover all areas of the terrain that is at height zero as well as the walls. The additional guard covers all of the terrain that is at height h. On the other hand, any solution of the terrain problem can be transformed to a solution of the polygon problem with an equal or smaller number of guards by simply placing all guards from the terrain problem into their corresponding positions in the polygon (and possibly discarding any guards that are outside the polygon). Thus:

Theorem 2. MINIMUM HIDDEN GUARD SET *on terrains cannot be approximated by any polynomial time approximation algorithm with an approximation ratio of* $|I|^{1-\epsilon}$ *for any* $\epsilon > 0$*, where* $|I|$ *is the number of vertices in the input terrain, unless* $NP = P$.

4. An approximation algorithm

The following algorithm to find a feasible solution for MINIMUM HIDDEN GUARD SET was proposed in [12]: Iteratively add a guard to the solution by placing it in an area of the input polygon that is not yet covered by any other guard that is already in the solution. The idea of this algorithm can be applied to terrains as input structures in a straight-forward manner (i.e., iteratively place guards in a not yet covered area of the terrain). In terms of an approximation ratio for this algorithm, we have the following

Theorem 3. MINIMUM HIDDEN GUARD SET *can be approximated in polynomial time with an approximation ratio of* $|I|$*, where* $|I|$ *is the number of polygon or terrain vertices.*

Proof. If we have a polygon as input structure, any triangulation of the input polygon partitions the polygon into $|I|-2$ triangles. Now, fix any triangulation. If we have a terrain as input structure, a triangulation of the terrain vertices is already given as part of the input according to the definition of a terrain. Any guard that the approximation algorithm places (as described above) lies in at least one of the triangles of the triangulation and thus sees the corresponding triangle completely. Therefore, the solution will contain at most $|I| - 2$ guards. Since any solution must consist of at least one guard, the result follows. □

Note that this algorithm faces the typical issue regarding bit-level complexity of geometric algorithms; taking bit-complexity into account will increase the running time of the algorithm, but leave it polynomial (see [8] for details). An alternative way of interpreting the proof of Theorem 3 is to say that any feasible solution for a HIDDEN GUARD SET instance will consist of at most $n - 2$ hidden guards. Thus the task of any approximation algorithm for this problem turns into finding any feasible solution (rather than finding a good feasible solution) since any feasible solution is as good as an approximation algorithm can hope to get.

5. Extensions

A natural variation of our problem is MINIMUM HIDDEN VERTEX GUARD SET, where we have the additional requirement that all guards have to be placed at vertices (rather than anywhere in the terrain or in the interior of the polygon). Since we have always placed or moved guards to vertices throughout our construction, Theorem 1 would hold for MINIMUM HIDDEN VERTEX GUARD SET. However, MINIMUM HIDDEN VERTEX GUARD SET is a much harder problem in a sense, as it is *NP*-hard to even determine whether a feasible solution exists [12]. In view of this, an inapproximability result does not really make sense for MINIMUM HIDDEN VERTEX GUARD SET.

If we restrict the problem even more, namely to a variation, where the guards may only sit at vertices and they only need to cover the vertices rather than the whole polygon interior, we arrive at the problem MINIMUM INDEPENDENT DOMINATING SET for polygon or terrain visibility graphs. Also in this case, Theorem 1 holds, since we only placed or moved guards to vertices and since we always argued about covering particular vertices rather than whole areas in the interior. This observation adds the class of visibility graphs of polygons or terrains to the numerous graph classes for which MINIMUM INDEPENDENT DOMINATING SET cannot be approximated with a ratio of $|I|^{1-\epsilon}$ (see [2] and [7] for details). The approximation algorithm from Section 3 can be applied to this variation and achieves a matching ratio.

The complementary problem MAXIMUM HIDDEN GUARD SET, where we need to find a maximum number of hidden guards that cover a given polygon, is equivalent to MAXIMUM HIDDEN SET. Therefore, it cannot be approximated with an approximation ratio of $|I|^{\epsilon}$ for some $\epsilon > 0$ for input polygons with holes and it cannot be approximated to within an arbitrarily small constant approximation ratio for input polygons without holes [4]. The corresponding vertex-restricted variation cannot be approximated, as it is—again—*NP*-hard to even find a feasible solution [12].

6. Conclusion

We have presented an inapproximability result and an approximation algorithm for MINIMUM HIDDEN GUARD SET for polygons with or without holes or terrains as input structures. Our results are tight up to low-order terms. MINIMUM HIDDEN GUARD SET is a natural combination of the two classic visibility problems MINIMUM GUARDING and MAXIMUM HIDDEN SET. The approximability of these two classic problems has not yet been fully determined for input polygons without holes: For guarding, the best approximation algorithms achieve a logarithmic approximation ratio (for most variations), while it is only known that the problem cannot be approximated with an arbitrarily small constant approximation ratio. For hiding, we also know that it cannot be approximated with an arbitrarily small constant ratio, but the best approximation algorithms achieve a ratio of $\Theta(|I|)$. In view of this, it is remarkable that our approximability results for MINIMUM HIDDEN GUARD SET are tight even for input polygons without holes. Moreover, MINIMUM HIDDEN GUARD SET has the property that any feasible solution is within the best achievable approximation bounds; thus one could argue that this problem is in some sense, "as hard as it gets".

Closing the approximability gaps for guarding and hiding in polygons without holes is a key challenge for future research in the area of approximability of visibility problems. With this overall goal in mind, potential intermediate steps and research directions for closing the gaps for guarding are:

- Develop an approximation algorithm with constant ratio for certain restricted polygon classes (see [11] for an example).
- Give concrete lower bounds for the approximation ratio. Note that the known reductions showing the existence of a lower approximability bound [3] do not seem to give strong lower bounds (i.e., bounds significantly larger than one) as they reduce satisfiability variations.
- Propose a gap-preserving reduction showing a logarithmic lower bound on the approximation ratio that constructs a polygon with only a small number of holes (as compared with the reduction from [3]).
- Develop an approximation algorithm for MINIMUM POINT GUARD that achieves a logarithmic approximation ratio. Note that the approximation algorithm from [6] does not work for MINIMUM POINT GUARD, where the guards may be positioned anywhere in the interior of the polygon.

Potential intermediate steps for closing the gaps for hiding include:

- Develop an approximation algorithm with constant or logarithmic ratio for certain restricted polygon classes.
- Give concrete lower bounds for the approximation ratio. Again, the known reductions [4] do not seem to be suited for this.
- Propose a gap-preserving reduction showing a logarithmic or even stronger lower bound on the approximation ratio that constructs a polygon with only a small number of holes (as compared with the reduction from [4]).

A different line of future research might consist of investigating MINIMUM HIDDEN GUARD SET on restricted classes of polygons, such as orthogonal polygons, with the aim to find classes for which better approximation ratios are possible.

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