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Triangular Norm-Based Measures and Their Markov Kernel Representation

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We approach the problem whether left-continuous triangular norm-based valuations (called T-measures or T-probability measures) defined on triangular norm-based tribes of the unit cube can be disintegrated by Markov kernels. We prove that each T-measure based on a "fundamental" triangular norm (these triangular norms T, together with their corresponding triangular conorms S, satisfy the functional equation $T(x, y) + S(x, y) = x + y$) can be uniquely represented as a sum of a "disintegrable" T-measure and a "hard core" which is either identically zero or which is monotonically irreducible (i.e., cannot be disintegrated). © 1991 Academic Press, Inc.

INTRODUCTION

The concept of a *triangular norm* is due to Menger [28], and it was studied from algebraic and topological points of view in fields like *Probabilistic Metric Spaces* (Wald [39], Schweizer and Sklar [34–37]), *Multivalued Logic* (Rose and Rosser [30], Hamacher [17]), and *Semigroups* (Climescu [11], Schweizer and Sklar [36], Paalman-de Miranda [29], Ling [27], Kimberling [19]).

Frank [15] has shown that the class of continuous triangular norms T, which together with their corresponding triangular conorms S satisfy the equation $T(x, y) + S(x, y) = x + y$, consists of the ordinal sums of sequences of "fundamental" triangular norms and conorms.

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Triangular norm-based measures (*T-measures*) appear under various names, and in specific analytical forms, in fields ranging from *Mathematical Statistics* (Dvoretzki, Wald, and Wolfowitz [13], Aczel and Alsina [2]), to *Capacity Theory* (Frank [15]), *Probability and Measure Theory* (Schmidt [31], Klement *et al.* [25, 26], Klement [22–24], Butnariu [5, 6]), *Pattern Recognition* (Sugeno [38]), *Game Theory* (Aumann and Shapley [4], Aubin [3], Butnariu [7, 9, 10]), etc. In this paper we study the triangular norm-based measures in their proximal context, namely T-measures defined on subsets of the unit cube $[0,1]^X$, which are triangular norm-based tribes (*T-tribes*). The main purpose is to find out whether, or under which conditions, T-measures can be represented as integrals of specific Markov kernels. We concentrate on fundamental triangular norm-based T-measures mainly because this class of T-measures is of interest in most of the applications mentioned above. One may also note that there are classes of nonfundamental triangular norms, on which no nontrivial T-measure is based (cf. Klement [24]).

We first deal with T-tribes, the main result being Theorem 2.1 showing that any fundamental triangular norm based T-tribe $\mathscr{T}$ consists of functions, which are measurable with respect to the intrinsic σ-algebra $\mathscr{T}^{\vee}$ corresponding to $\mathscr{T}$ (i.e., with respect to the σ-algebra of those sets whose characteristic functions belong to $\mathscr{T}$). In this context, we give a characterization of the generated T-tribes introduced by Klement [22]—see Theorem 2.1 and Remark 2.3. Theorem 2.1 allows the deduction (see Section 5) that on a fundamental triangular norm based tribe $\mathscr{T}$ any function $\mathbf{m}$ of the form

$$\mathbf{m}(A) = \int_{\{A>0\}} (g + h \cdot A)\, d\mathbf{p} \qquad (*)$$

is a well-defined monotone T-measure, provided g, h are nonnegative $\mathscr{T}^{\vee}$-measurable functions, and $\mathbf{p}$ is a probability measure on $\mathscr{T}^{\vee}$. The question is whether any monotone T-measure on $\mathscr{T}$ is of the form $(*)$. In fact, this is equivalent to the question whether any T-measure is disintegrable by a Markov kernel, and it is not essentially new. It arises implicitly in many works dealing with T-measures, and it was already known that for fundamental triangular norm-based measures on generated T-tribes $\mathscr{T}$ the answer is affirmative (cf. Klement [24]). Also, it was previously known that, even if $\mathscr{T}$ is nongenerated, T_∞-based measures are necessarily of the form $(*)$ (cf. Butnariu [10]—see also Theorem 4.1). The main result of the paper is Theorem 5.3 showing that, in general, each finite monotone fundamental triangular norm-based measure can be uniquely decomposed into a sum of a T-measure of the form $(*)$ and a monotonically irreducible T-measure $\mathbf{m}^*$ (that is a T-measure which is either identically zero, or such that there is no T-measure of the form $(*)$ differing monotonically from $\mathbf{m}^*$).

The relevance of our results may be seen under several aspects. First, we describe analytically a large class of T-tribes, which are in fact abstractions of the concept of a Boolean ring (see Schmidt [32]), and we characterize fundamental triangular norm-based measures defined on general T-tribes. These are among the generalizations of ordinary probability measures naturally involved in problems of *Pattern Recognition* and *Plausibility Theory* (cf. Sugeno [38], Höhle and Klement [18]), *Automata Theory* (Eilenberg [14]), *Capacity Theory* (Frank [15]), *Mathematical Economics* (Aczel and Alsina [2]), and *Game Theory* (Butnariu [10]). On the other hand, one may look at our results from a probabilistic point of view. In such a context, Theorems 3.5 and 4.1 say that fundamental triangular norm-based T-measures, which are defined on generated T-tribes and T_∞-measures on arbitrary T_∞-tribes, are "totally disintegrable" (i.e., they can be written as integrals of Markov kernels). Theorem 5.3 implies that, in general, fundamental triangular norm-based measures are disintegrable up to a *hard core* which is essentially irreducible. These facts open a way to a proof that on a significant space of coalitional games (known in the literature as pM) a maximally monotone *multivalued value operator* exists. On the other hand, Theorem 4.1 allows formulation of an alternative interpretation of the concept of Lebesgue integral; i.e., it shows that a Lebesgue integral on the set X is precisely a T_∞-measure on a T_∞-tribe in the unit cube $[0, 1]^X$.

Finally, we must point out that our representation theorems for triangular norm-based measures are valid for monotone T-measures only. The question whether they are true for nonmonotone T-measures, too, is equivalent to whether for triangular norm-based measures there exist Jordan decompositions (by monotone T-measures). It follows from a result of Schmidt [31] that T-measures on T-tribes can be written as differences of monotone T-countable additive functions, but this does not mean automatically that for T-measures Jordan decompositions exist (except in the case of T_∞-measures, where Schmidt's results apply according to Example 3.2 (iii) and Remark 4.2 (iii)).

1. Triangular Norms, T-Clans and T-Tribes

A *triangular norm* (*t-norm* for short) is a two-place function T: $[0, 1] \times [0, 1] \to [0, 1]$ which is commutative, associative, monotone in each component and satisfies the boundary condition $T(x, 1) = x$. A t-norm T is called *strict* if it is continuous und satisfies $T(x, y) < T(x, z)$ whenever $y < z$. It is called *Archimedian* if it satisfies $T(x, x) < x$ for all $x \in]0, 1[$. The corresponding *t-conorm* of T is the function $S: [0, 1] \times [0, 1] \to [0, 1]$ defined by $S(x, y) = 1 - T(1 - x, 1 - y)$.

As an example, a most important family of t-norms $\{T_s\}_{s \in [0, \infty]}$ (cf. [15]), which we call *fundamental t-norms*, is given by

$$
\begin{aligned}
T_s(x, y) &= \min(x, y) && \text{if } s = 0,\\
&= x \cdot y && \text{if } s = 1,\\
&= \max(0, x + y - 1) && \text{if } s = \infty,\\
&= \log_s\left[1 + \frac{(s^x - 1) \cdot (s^y - 1)}{s - 1}\right] && \text{if } s \in]0, \infty[\backslash\{1\}.
\end{aligned}
$$

Their corresponding t-conorms are

$$
\begin{aligned}
S_s(x, y) &= \max(x, y) && \text{if } s = 0,\\
&= x + y - x \cdot y && \text{if } s = 1,\\
&= \min(1, x + y) && \text{if } s = \infty,\\
&= 1 - \log_s\left[1 + \frac{(s^{1-x} - 1) \cdot (s^{1-y} - 1)}{s - 1}\right] && \text{if } s \in]0, \infty[\backslash\{1\}.
\end{aligned}
$$

This is a "continuous" family of t-norms in the sense that $\lim_{s \to t} T_s = T_t$. Moreover, each pair (T_s, S_s) satisfies the functional equation

$$
T(x, y) + S(x, y) = x + y. \tag{1}
$$

T_0 is not Archimedean (and hence not strict), T_∞ is Archimedean but not strict, and each T_s with $s \in]0, \infty[$ is strict (and hence Archimedean).

Consider a countable set J, a family $\{]a_j, b_j[\}_{j \in J}$ of mutually disjoint open subintervals of $[0, 1]$, and a family of t-norms $\{T_j\}_{j \in J}$. Then the function $T: [0, 1] \times [0, 1] \to [0, 1]$ defined by

$$
\begin{aligned}
T(x, y) &= a_j + (b_j - a_j) \cdot T_j\left(\frac{x - a_j}{b_j - a_j}, \frac{y - a_j}{b_j - a_j}\right) && \text{if } x, y \in]a_j, b_j[\text{ for some } j \text{ in } J,\\
&= \min(x, y) && \text{otherwise,}
\end{aligned}
$$

is a t-norm called *ordinal sum of the t-norms* $\{T_j\}_{j \in J}$ *over the intervals* $\{]a_j, b_j[\}_{j \in J}$ (see [37]). Ordinal sums of t-conorms can be defined dually. Frank [15] proved that the only pairs (T, S) of continuous t-norms and corresponding t-conorms solving Eq. (1) are the fundamental t-norms and the ordinal sums of fundamental t-norms T_s $(s > 0)$ together with their corresponding t-conorms S_s $(s > 0)$.

The function

$$W(x, y)=\min(x, y) \quad \text{if} \quad \max(x, y)=1,$$
$$=0 \quad \text{otherwise,}$$

is a t-norm, its corresponding t-conorm is

$$V(x, y)=\max(x, y) \quad \text{if} \quad \min(x, y)=0,$$
$$=1 \quad \text{otherwise.}$$

W is not Archimedean (and hence not strict) and not continuous. It is the "smallest" t-norm, and the fundamental t-norm T_0 is the "largest" t-norm, i.e., for any t-norm T we have

$$W \leqslant T \leqslant T_0.$$

Given a t-norm T and its corresponding t-conorm S their associativity allows to extend them to n-ary operations $\mathbf{T}_{i=1}^{n}: [0, 1]^n \to [0, 1]$ and $\mathbf{S}_{i=1}^{n}: [0, 1]^n \to [0, 1]$. In what follows we write $\mathbf{T}_{i=1}^{n} x_i$ and $\mathbf{S}_{i=1}^{n} x_i$ instead of $\mathbf{T}_{i=1}^{n} (x_1, ..., x_n)$ and $\mathbf{S}_{i=1}^{n} (x_1, ..., x_n)$, respectively. For any sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, 1]$ the sequence $\{\mathbf{T}_{i=1}^{n} x_i\}_{n \in \mathbb{N}}$ is nonincreasing; therefore its limit $\mathbf{T}_{n=1}^{\infty} x_n = \lim_{n \to \infty} \mathbf{T}_{i=1}^{n} x_i$ always exists. By duality, the sequence $\{\mathbf{S}_{i=1}^{n} x_i\}_{n \in \mathbb{N}}$ is nondecreasing, its limit, denoted $\mathbf{S}_{n=1}^{\infty} x_n$, exists, and we have $\mathbf{S}_{n=1}^{\infty} x_n = 1 - \mathbf{T}_{n=1}^{\infty} (1 - x_n)$.

1.1 PROPOSITION. *Let T be a continuous Archimedean t-norm and let $\{x_n\}_{n \in \mathbb{N}}$ be a constant sequence in $[0, 1[$. Then we have*

$$\mathop{\mathbf{T}}_{n=1}^{\infty} x_n = 0.$$

Proof. Assume that $x_n = a \neq 0$ for each $n \in \mathbb{N}$. Let us consider the continuous function h from X to $[0, 1]$ defined by $h(x) = T(x, x)$. Putting $h^1 = h$ and $h^{n+1} = h \circ h^n$, we have, for each $x \in]0, 1[$, $h(x) < x$ and $h^{n+1}(x) \leqslant h^n(x)$. Then for $b = \lim_{n \to \infty} h^n(a)$ we obtain the equality $h(b) = h(\lim_{n \to \infty} h^n(a)) = \lim_{n \to \infty} h^{n+1}(a) = b$ which implies $b = 0$. Since the sequence $\{h^n(a)\}_{n \in \mathbb{N}}$ is a subsequence of the convergent sequence $\{\mathbf{T}_{i=1}^{n} x_i\}_{n \in \mathbb{N}}$, our result follows. ∎

1.2 PROPOSITION. (i) *If T is a t-norm which is either fundamental or the ordinal sum of a family of fundamental t-norms, then $T_\infty \leqslant T \leqslant T_0$.*

(ii) *If $0 \leqslant s < 1 < t \leqslant \infty$, and T_s and T_t are the corresponding fundamental t-norms, then $T_t \leqslant T_1 \leqslant T_s$.*

Proof. (i) Let T be a t-norm which is either fundamental or the ordinal sum of a family of fundamental t-norms. If T is fundamental, then $T_\infty \leqslant T$ because T and its corresponding t-conorm S satisfy (1). If T itself is not fundamental but an ordinal sum of a family of fundamental t-norms $\{T_j\}_{j\in J}$ over a family of subintervals $\{]a_j, b_j[\}_{j\in J}$ of $[0, 1]$, then for any $j\in J$ and for any $x, y\in]a_j, b_j[$ we have

$$\begin{aligned} T(x, y) &\geqslant a_j + (b_j - a_j)\, T_\infty\left(\frac{x-a_j}{b_j-a_j}, \frac{y-a_j}{b_j-a_j}\right) \\ &= \max[x+y-b_j, a_j] \\ &\geqslant T_\infty(x, y), \end{aligned}$$

and this, together with $T \leqslant T_0$ completes the proof of (i).

(ii) Consider two fundamental t-norms T_s and T_t with $0<s<1<t<\infty$. If $x\in\{0, 1\}$ or $y\in\{0, 1\}$, then $T_t(x, y) = T_1(x, y) = T_s(x, y)$. It remains to show that if $x, y\in]0, 1[$ then $T_t(x, y)\leqslant T_1(x, y)\leqslant T_s(x, y)$. The first inequality is equivalent to

$$\log_t\left[1+\frac{(t^x-1)\cdot(t^y-1)}{t-1}\right]\leqslant x\cdot y.$$

This inequality is equivalent to

$$\frac{t^x-1}{t^{xy}-1}\leqslant\frac{t-1}{t^y-1} \qquad (x, y\in]0, 1[, t>1)$$

and, substituting $v=t^x$, to

$$\frac{v-1}{v^y-1}\leqslant\frac{t-1}{t^y-1} \qquad (1<v<t \text{ and } y\in]0, 1[).$$

Thus, it is sufficient to show that the function $f_y(v) = (v-1)/(v^y-1)$ is nondecreasing in the interval $]1, t[$ for any fixed y in $]0, 1[$. Computing the derivative we get

$$f_y'(v) = y\cdot\frac{\dfrac{v^y-1}{y-0}-\dfrac{v^y-v^{y-1}}{y-(y-1)}}{(v^y-1)^2} \qquad (v>1, y\in]0, 1[).$$

The function $y\to v^y$ is convex on $\mathbb{R}$ for any fixed $v>0$. Therefore, the denominator of f_y' is nonnegative in $]1, \infty[$, and f_y is nondecreasing. For the second inequality fix $s\in]0, 1[$. In fact, it is sufficient to prove that the function f_y attains its minimal value in the interval $[s, 1[$ at the point $v=s$ for any $y\in]0, 1[$. Since, for $v\in]s, 1[$, f_y' is as above and the denominator of the derivative is still nonnegative for $v\in [s, 1[$, it follows that f_y is nondecreasing on $[s, 1[$ for any y fixed in $]0, 1[$. By consequence, the minimal value of f_y on $[s, 1[$ is attained at $v = s$. ∎

A function $A: X \to [0, 1]$ has been called a *fuzzy subset* of the ordinary set X (Zadeh [41]). This generalizes the concept of a (Cantorian) subset A of X which can be identified with its characteristic function $A: X \to \{0, 1\}$ defined by $A(x) = 1$ if $x \in A$, and $A(x) = 0$ if $x \notin A$. If A is a fuzzy subset of X, then the value $A(x)$ is interpreted as the degree of membership of the point x in A. The collection of all fuzzy subsets of X is denoted $[0, 1]^X$, as usual.

Let T be a t-norm and S be its corresponding t-conorm. We extend T and S to $[0, 1]^X$ pointwise, i.e., $(A \mathbf{T} B)(x) = T(A(x), B(x))$ and $(A \mathbf{S} B)(x) = S(A(x), B(x))$. These operations can be considered as "intersection" and "union" of fuzzy subsets, respectively. Also, finite (countable) "intersections" $\mathbf{T}_{i=1}^{n} A_i$ ($\mathbf{T}_{n=1}^{\infty} A_i$) and "unions" $\mathbf{S}_{i=1}^{n} A_i$ ($\mathbf{S}_{n=1}^{\infty} A_i$) of fuzzy subsets are defined in the straightforward way. They satisfy the De Morgan laws

$$\left(\mathop{\mathbf{T}}_{n=1}^{\infty} A_n\right)' = \mathop{\mathbf{S}}_{n=1}^{\infty} A_n' \quad \text{and} \quad \left(\mathop{\mathbf{S}}_{n=1}^{\infty} A_n\right)' = \mathop{\mathbf{T}}_{n=1}^{\infty} A_n',$$

where the "complement" A' is defined by $A'(x) = 1 - A(x)$. Restricted to ordinary sets (i.e., characteristic functions), these operations coincide with intersection, union, and complement, respectively, regardless which t-norm and t-conorm is considered. The class $[0, 1]^X$ of the fuzzy subsets of X together with the operations $\mathbf{T}$ and $\mathbf{S}$ form a partially ordered commutative semigroup having $\varnothing$ as smallest (and as null) element and X as largest (and as unit) element. However, $[0, 1]^X$ provided with the operations $\mathbf{T}$, $\mathbf{S}$, and the complement "$'$" is not a Boolean algebra. It is not even a lattice, except in the case $T = T_0$ and $S = S_0$. In general, $\mathbf{T}$ and $\mathbf{S}$ are not distributive with respect to each other, $A \mathbf{T} A'$ may be different from $\varnothing$ and $A \mathbf{S} A'$ may be different from X.

Let T be a t-norm. A subfamily $\mathscr{C}$ of $[0, 1]^X$ containing $\varnothing$ and being closed under the operation $\mathbf{T}$ and under complementation will be called a *T-clan*. Obviously, by the duality of $\mathbf{T}$ and $\mathbf{S}$, the closedness with respect to $\mathbf{T}$ can be replaced by the closedness with respect to $\mathbf{S}$ in the definition above.

1.3 EXAMPLE. (i) Since we identify ordinary subsets of X with their characteristic functions, any algebra of subsets of X is a T-clan with respect to any t-norm T.

(ii) For any $n \in \mathbb{N}$ the family $\mathscr{C}_n(X) = \{0, 1/n, ..., (n-1)/n, 1\}^X$ is a T-clan for $T = T_0$, and also for $T = T_\infty$, but not with respect to any other fundamental t-norm.

(iii) If the t-norm T is continuous (measurable), and if X is a topological (measurable) space, then the family of all continuous (measurable) fuzzy subsets of X is a T-clan.

A T-clan $\mathscr{T}$ which is also closed under countable "intersections," i.e., which satisfies

$$\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{T} \Rightarrow \mathop{\mathbf{T}}_{n=1}^{\infty} A_n \in \mathscr{T},$$

is called a *T-tribe*. A pair $(X, \mathscr{T})$, where X is a set and $\mathscr{T}$ is a T-tribe, is called a *T-measurable space*.

1.4 EXAMPLE. (i) Obviously, not any T-clan is a T-tribe. For instance, the family of all constant functions on X with values in $\mathbb{Q} \cap [0, 1]$ is a T_∞-clan but not a T_∞-tribe.

(ii) Any σ-algebra of subsets of X is a T-tribe with respect to any t-norm T.

(iii) Given a σ-algebra $\mathscr{A}$ of subsets of X, the family $\mathscr{A}^\vee$ of all $\mathscr{A}$-measurable fuzzy subsets of X is a T-tribe with respect to any Borel-measurable t-norm T.

(iv) Given a T-tribe $\mathscr{T}$, the family $\mathscr{T}^\vee$ of all characteristic functions contained in $\mathscr{T}$ is a σ-algebra, and hence a T-tribe.

(v) (Klement [20]) The family $\mathscr{T}$ consisting of all fuzzy subsets of $X = [0, 1]$ which are either constant or have all their values in the interval $[\frac{1}{3}, \frac{2}{3}]$ is a T-tribe in the case $T = W$ and for $T = T_0$, but it is not a T-tribe for $T = T_t$ with $t \in]0, \infty]$. It is interesting to note that there is no σ-algebra of sets $\mathscr{A}$ such that $\mathscr{T} = \mathscr{A}^\wedge$.

(vi) (Klement [22]) If in Example (v) we additionally require all elements of $\mathscr{T}$ to be continuous, then $\mathscr{T}$ is a W-tribe but not a T_0-tribe.

(vii) Consider a nonempty subset Y of X such that $Y \neq X$. The family $\mathscr{T}$ of fuzzy subsets of X which are constant on Y and assume only values 0 and 1 outside of Y, is a T-tribe with respect to any t-norm T, but it does not contain any constant fuzzy subset (except $\varnothing$ and X).

1.5 THEOREM. *If $s \in]0, \infty[$ and T_s is the corresponding fundamental t-norm, then any T_s-tribe $\mathscr{T}$ is a T_∞-tribe. Moreover, any T_∞-tribe is a T_0-tribe.*

Proof. We fix an arbitrary $s \in]0, \infty[$. The proof is carried out in several steps.

(a) First we prove that if $A, B \in \mathscr{T}$ then there exists a $C \in \mathscr{T}$ such that $A\, \mathbf{S}_\infty\, B = A\, \mathbf{S}_s\, C$. Let A and B be any two fuzzy subsets in the T_s-tribe

$\mathscr{T}$. Define a double sequence as follows: $A_1 = A$, $B_1 = B$, $A_{n+1} = A_n \mathbf{S}_s B_n$, and $B_{n+1} = A_n \mathbf{T}_s B_n$. The sequence $\{A_n\}_{n \in \mathbb{N}}$ is nondecreasing, the sequence $\{B_n\}_{n \in \mathbb{N}}$ is nonincreasing, and both sequences are contained in $\mathscr{T}$. Since the pair (T_s, S_s) satisfies (1), by induction we get for all $n \in \mathbb{N}$

$$A_n + B_n = A + B. \tag{2}$$

Claim 1. For each $a \in [0, 1[$ there exists a number $c \in [0, 1[$ such that for all b in $[0, a]$ we have

$$T_s(a, T_s(a, b)) \leqslant c \cdot T_s(a, b). \tag{3}$$

Indeed, from Proposition 1.2 we have that $T_s \leqslant T_1$ for $s \geqslant 1$; and this implies that if $s \geqslant 1$ we can choose $c = a$. If $s < 1$, consider $c = (s^a - 1)/(s-1)$. It is clear that $c < 1$. Then for each $b \in [0, a[$ the inequality (3) is equivalent to

$$\ln[1 + c^2(s^b - 1)] \geqslant c \cdot \ln[1 + c(s^b - 1)].$$

The expansion of the logarithms in power series leads to

$$\sum_{i=1}^{\infty} (-1)^{i-1} \cdot \frac{c^{2i}(s^b - 1)^i}{i} \geqslant \sum_{i=1}^{\infty} (-1)^{i-1} \cdot \frac{c^{i+1}(s^b - 1)^i}{i}$$

which is equivalent to $c^{i-1} \leqslant 1$. Since this inequality holds for all $i \in \mathbb{N}$, it follows that (3) is valid for all $s \in]0, \infty[$.

Claim 2. We have

$$A \mathbf{S}_\infty B = \mathop{\mathbf{S}_s}_{n=1}^{\infty} C_n, \tag{4}$$

where $C_1 = A_1$ and $C_{n+1} = B_n$, $(n \in \mathbb{N})$. In order to prove that, we fix an $x \in X$ and put $\alpha = (A \mathbf{S}_\infty B)(x)$. If $\alpha < 1$, then $A_n(x) \leqslant \alpha < 1$ for all $n \in \mathbb{N}$ because of (2). Let c be a number in $[0, \alpha[$ such that (3) is satisfied. Then, by the monotonicity of the t-norm T_s we get

$$\begin{aligned} B_n(x) &\leqslant c^{n-2} \cdot \alpha && \text{if} \quad s < 1 \\ &\leqslant \alpha^n && \text{if} \quad s \geqslant 1 \end{aligned}$$

for all $n \geqslant 2$. Since α and c are both in $[0, 1[$, it follows that

$$\lim_{n \to \infty} B_n(x) = 0$$

and, because of (2),

$$\lim_{n \to \infty} A_n(x) = (A \mathbf{S}_\infty B)(x), \tag{5}$$

which is exactly (4) by the definition of the double sequence. Now, assume that $\alpha = 1$. In this case (5) also holds, since assuming the contrary we get

$$\lim_{n \to \infty} A_n(x) < 1,$$

and this means that there exists a number d in $[0, 1[$ such that $A_n(x) \leqslant d < 1$ for all $n \in \mathbb{N}$. But using analogous arguments as above, with d instead of α, we deduce that

$$\lim_{n \to \infty} A_n(x) \geqslant 1$$

contradicting our assumption. Hence (4) is always true. Putting $C = \mathbf{S}_{s\,n=2}^{\infty} C_n$, the proof of part (a) is complete. This also shows that the T_s-tribe $\mathscr{T}$ is a T_∞-clan.

(b) Let $\{D_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathscr{T}$. There exists a sequence $\{E_n\}_{n \in \mathbb{N}}$ in $\mathscr{T}$ such that for each $n \in \mathbb{N}$ we have

$$\mathop{\mathbf{S}_\infty}_{j=1}^{n} D_j = \mathop{\mathbf{S}_s}_{k=1}^{n} E_k. \tag{6}$$

For $n = 2$ this follows from part (a). Suppose we have proved (6) for $n \in \mathbb{N}$. Then we get, again using part (a),

$$\begin{aligned}\mathop{\mathbf{S}_\infty}_{j=1}^{n+1} D_j &= \left(\mathop{\mathbf{S}_\infty}_{j=1}^{n} D_j\right) \mathbf{S}_\infty D_{n+1} = \left(\mathop{\mathbf{S}_s}_{j=1}^{n} E_j\right) \mathbf{S}_\infty D_{n+1} \\ &= \left(\mathop{\mathbf{S}_s}_{j=1}^{n} E_j\right) \mathbf{S}_s E_{n+1} = \mathop{\mathbf{S}_s}_{j=1}^{n+1} E_j.\end{aligned}$$

Now, because of (6), we obtain

$$\mathop{\mathbf{S}_\infty}_{n=1}^{\infty} D_n = \lim_{n \to \infty} \mathop{\mathbf{S}_\infty}_{j=1}^{n} D_j = \lim_{n \to \infty} \mathop{\mathbf{S}_s}_{j=1}^{n} E_j = \mathop{\mathbf{S}_s}_{n=1}^{\infty} E_n,$$

the latter fuzzy subset being an element of $\mathscr{T}$. This shows that $\mathscr{T}$ is a T_∞-tribe.

(c) In order to show that a T_∞-tribe $\mathscr{T}$ is a T_0-clan it suffices to observe that for any two fuzzy subsets A and B one has $A\,\mathbf{T}_0\,B = A\,\mathbf{T}_\infty\,(B\,\mathbf{S}_\infty\,A')$. Actually, $\mathscr{T}$ is even a T_0-tribe: If $\{A_n\}_{n \in \mathbb{N}}$ is an increasing sequence in $\mathscr{T}$ put $B_n = A_n\,\mathbf{T}_\infty\,A'_{n-1}$ for each $n \in \mathbb{N}$ with $A_0 = \varnothing$ and observe that

$$A_n = \mathop{\mathbf{S}_\infty}_{i=1}^{n} B_i \qquad (n \in \mathbb{N}).$$

Hence,

$$\mathbf{S}_0{}_{n=1}^{\infty} A_i = \lim_{n\to\infty} A_n = \mathbf{S}_\infty{}_{n=1}^{\infty} B_n \in \mathscr{T}. \qquad \blacksquare$$

Theorem 1.5 shows that T-tribes based on fundamental t-norms T are implicitly σ-complete lattices with respect to the pointwise order and that they are closed sets with respect to the weak topology on the unit cube $[0, 1]^X$. Moreover, viewed as T_∞-tribes, the fundamental norm based T_s-tribes with $s \in]0, \infty]$ are implicitly "clans" in the sense of Wyler [40], where the subtraction is defined by $A \ominus B = A \, \mathbf{T}_\infty \, B'$ (see also Schmidt [22]).

2. Representation of T-Tribes, Disjointness

We already observed that any T-tribe $\mathscr{T}$ on X includes a σ-algebra $\mathscr{T}^\vee$ of subsets of X, and that the family $(\mathscr{T}^\vee)^\wedge$ of all $\mathscr{T}^\vee$-measurable functions $X \to [0, 1]$ is a T-tribe (if T is Borel-measurable (see Example 1.4)). Now we study the precise relationship between $\mathscr{T}$ and $(\mathscr{T}^\vee)^\wedge$. In particular, we are interested to know under which conditions they coincide. In this case, the T-tribe $\mathscr{T}$ is called *generated.* For a nongenerated T-tribe see Example 1.4 (v).

2.1 Theorem. *For any fundamental t-norm T_s with $s > 0$, and for each T_s-tribe $\mathscr{T}$ we have $\mathscr{T} \subseteq (\mathscr{T}^\vee)^\wedge$*

Proof. For any $A \in \mathscr{T}$ and for any $a \in [0, 1]$ we denote $A_a = \{x \in X; A(x) \geqslant a\}$, and we must show that $A_a \in \mathscr{T}^\vee$. For $a = 0$ this is trivial. Assume $a = 1$. Then, because of Proposition 1.1, for any fuzzy subset B we have

$$B(x) > 0 \Leftrightarrow \mathbf{S}_s{}_{n=1}^{\infty} B_n(x) = 1 \qquad \text{(where } B_n = B \text{ for each } n \in \mathbb{N}\text{)}.$$

Putting $B = A'$ this yields $A_1 \in \mathscr{T}^\vee$. Now choose $a \in]0, 1[$. Then there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ of positive rational numbers increasing to a, and we have

$$A_a = \bigcap_{n=1}^{\infty} A_{a_n}.$$

Thus, it suffices to show that $A_a \in \mathscr{T}^\vee$ for any $a \in]0, 1[$ which can be represented in the form

$$a = \sum_{i=1}^{k} \frac{a(i)}{2^i} \quad \text{with} \quad a(i) \in \{0, 1\} \quad \text{for} \quad 1 \leqslant i \leqslant k, \quad \text{and} \quad k \in \mathbb{N}. \tag{7}$$

We proceed by induction upon the positive integer k involved in (7). If $k=1$ than $a(1)=1$ and $a=\frac{1}{2}$. Thus we have $A_a=\{x\in X; (A\,\mathbf{S}_\infty\, A)(x)=1\}=(A\,\mathbf{S}_\infty\, A)_1$. But $A\,\mathbf{S}_\infty\, A\in\mathcal{T}$ because of Theorem 1.5. Therefore, $A_a=(A\,\mathbf{S}_\infty\, A)_1$ belongs to $\mathcal{T}^\vee$. Let us assume that for every A in $\mathcal{T}$ and for every $a\in\,]0,1[$ of the form (7) with $k\leqslant n$ we have $A_a\in\mathcal{T}^\vee$. Suppose that

$$a=\sum_{i=1}^{n+1}\frac{a(i)}{2^i}\qquad\text{with}\quad a(i)\in\{0,1\}$$

$$\text{for}\quad 1\leqslant i\leqslant n+1\quad\text{and}\quad a(n+1)\neq 0. \tag{8}$$

Then $a=b/2$, where $b=\sum_{i=1}^{n+1}(a(i)/2^{i-1})$. If $b\in[0,1]$, then $A_a=(A\,\mathbf{T}_\infty\, A)_b\in\mathcal{T}^\vee$ by the inductive assumption. If $b\notin[0,1]$, then $a>\frac{1}{2}$, and it can be written as

$$a=(c+1)/2\qquad\text{with}\quad c=\sum_{i=1}^{n}\frac{a(i+1)}{2^i}\in\,]0,1].$$

Thus we get $A_a=(A'\,\mathbf{T}_\infty\, A')'_c\in\mathcal{T}^\vee$ by Theorem 1.5 and the inductive assumption. ∎

In general a T_s-tribe $\mathcal{T}$ is not generated (i.e., $\mathcal{T}$ may be different from $(\mathcal{T}^\vee)^\wedge$) even if T_s is a fundamental t-norm with $s>0$ (see Example 1.4 (v)). However, we have the following result:

2.2 Theorem. *For any fundamental t-norm T_s with $s>0$, a T_s-tribe on X is generated if and only if it contains all the constant fuzzy subsets of X.*

Proof. Necessity is obvious. Conversely, assume that the T_s-tribe $\mathcal{T}$ contains all the constant fuzzy subsets of X. We must show that each A in $(\mathcal{T}^\vee)^\wedge$ is contained in $\mathcal{T}$ (cf. Theorem 2.1). Define the sequence of fuzzy subsets

$$U_n=\mathop{\mathbf{S}_\infty}_{k=1}^{2^{n-1}}(A_k^{[n]}\,\mathbf{T}_\infty\, V_k^{[n]}),$$

where

$$\begin{aligned} V_k^{[n]}&=\{x\in X; (2k-1)/2^n\leqslant A(x)<2k/2^n\} &&\text{if}\quad k<2^{n-1},\\ &=\{x\in X; (2^n-1)/2^n\leqslant A(x)\} &&\text{otherwise}\end{aligned}$$

and $A_k^{[n]}$ is the constant fuzzy subset $A_k^{[n]}(x)=a_k^{[n]}$, with the number $a_k^{[n]}$ chosen such that $S_s((2k-2)/2^n, a_k^{[n]})=(2k-1)/2^n$. Note that this choice of $a_k^{[n]}$ is possible since the function $S_s((2k-2)/2^n,\cdot)\colon[0,1]\to[(2k-2)/2^n,1]$ is a surjection. Because of the $\mathcal{T}^\vee$-measurability of A each

$V_k^{[n]}$ is contained in $\mathcal{T}^{\vee}$, and hence in $\mathcal{T}$. Since $\mathcal{T}$ contains the constant fuzzy subsets $A_k^{[n]}$, and since it is a T_∞-clan (cf. Theorem 1.5), it follows that $U_n \in \mathcal{T}$. Now it is a matter of computation to check that

$$\mathop{\mathbf{S}_s}_{i=1}^{n} U_i = \sum_{k=1}^{2^n} \frac{k-1}{2^n} \cdot W_k^{[n]}, \tag{9}$$

where

$$\begin{aligned} W_k^{[n]} &= \{x \in X; (k-1)/2^n \leqslant A(x) < k/2^n\} && \text{if} \quad k < 2^n, \\ &= \{x \in X; (2^n - 1)/2^n \leqslant A(x)\} && \text{otherwise.} \end{aligned}$$

Since the functions on the right-hand side of (9) are convergent to A, we get $A = \mathop{\mathbf{S}_\infty}_{n=1}^{\infty} U_n$, showing that $A \in \mathcal{T}$. ∎

2.3 *Remark.* (i) A T_s-tribe $\mathcal{T}$ may be not generated even if T_s is a fundamental t-norm with $s > 0$. See, for instance, Example 1.4 (vii).

(ii) By virtue of Theorem 1.5, in Theorem 2.2 the condition "$\mathcal{T}$ contains all the constant fuzzy subsets of X" can be replaced by the condition "$\mathcal{T}$ contains a sequence $\{A_n\}_{n \in \mathbb{N}}$ of constant fuzzy subsets of X with $A_n(x) = 1/z^n$ ($n \in \mathbb{N}$), where $z \geqslant 2$ is an integer."

In order to introduce the important concept of disjointness of fuzzy subsets with respect to t-norms, let X be a nonempty set, T a t-norm and S its corresponding t-conorm. A *finite family* of fuzzy subsets $A_1, A_2, ..., A_n$ of X is said to be *T-disjoint* if

$$\left(\mathop{\mathbf{S}}_{j \neq k} A_j \right) \mathbf{T}\, A_k = \varnothing \qquad (1 \leqslant k \leqslant n). \tag{10}$$

An *infinite sequence* $\{A_j\}_{j \in \mathbb{N}}$ of fuzzy subsets of X is called *T-disjoint* if for any $n \in \mathbb{N}$, $n \geqslant 2$, the finite family $A_1, ..., A_n$ is T-disjoint.

2.4 *Remark.* (i) Any subfamily $\{A_i\}_{i \in I}$ of a countable T-disjoint family $\{A_i\}_{i \in J}$ is also T-disjoint. Obviously, it suffices to prove this for a finite set $J = \{1, 2, ..., n\}$ and a subset $I = \{i_1, ..., i_k\} \subseteq J$. Indeed, for any $i_r \in I$ we have

$$\left(\mathop{\mathbf{S}}_{\substack{h=1 \\ h \neq r}}^{k} A_{i_h} \right) \mathbf{T}\, A_{i_r} \leqslant \left(\mathop{\mathbf{S}}_{\substack{h=1 \\ h \neq r}}^{n} A_{i_h} \right) \mathbf{T}\, A_{i_r} = \varnothing.$$

(ii) The definition of T-disjointness does not depend on the order in which the fuzzy subsets $\{A_j\}_{j \in \mathbb{N}}$ are numbered, i.e., if π is a permutation of $\mathbb{N}$ and $\{A_j\}_{j \in \mathbb{N}}$ is T-disjoint so is $\{A_{\pi(j)}\}_{j \in \mathbb{N}}$.

Different t-norms may lead to different concepts of "disjointness." However, for some classes of t-norms the corresponding "disjointness" concepts do not depend on the choice of the t-norm in that class.

2.5 EXAMPLE. Let $\{A_j\}_{j \in \mathbb{N}}$ be a countable family of fuzzy subsets.

(i) If all A_j are (characteristic functions of) ordinary sets then T-disjointness is equivalent with pairwise disjointness with respect to any t-norm.

(ii) T-disjointness implies pairwise T-disjointness according to 2.4 (i), but the converse is not generally true: if we take $A_i = \frac{1}{2}$ for $i = 1, 2, 3$, then A_1, A_2, A_3 are pairwise T_∞-disjoint, but they are not T_∞-disjoint.

(iii) For $s \in [0, \infty[$ we get: $\{A_j\}_{j \in \mathbb{N}}$ is T_s-disjoint if and only if each x is "contained" in at most one A_k (that is if and only if $A_k(x) > 0$ for at most one k).

(iv) W-disjointness of $\{A_j\}_{j \in \mathbb{N}}$ means that for each $x \in X$ exactly one of the following conditions holds:

(1) There is at most one $k \in \mathbb{N}$ such that $A_k(x) = 1$.

(2) There are at most two indices $k, l \in \mathbb{N}$ such that $0 < A_k(x), A_l(x) < 1$.

2.6 PROPOSITION. *Let T be a t-norm and S be its corresponding t-conorm such that* (1) *holds. Then for any $n \geqslant 2$ the following conditions are equivalent*:

(i) *$A_1, ..., A_n$ are T-disjoint.*

(ii) *For any $k = 2, ..., n$: $(\mathbf{S}_{i=1}^{k-1} A_i)\, \mathbf{T}\, A_k = \varnothing$.*

(iii) *For any $k = 2, ..., n$: $\mathbf{S}_{i=1}^{k} A_i = \sum_{i=1}^{k} A_i$.*

(iv) *For each set $I \subseteq \{1, 2, ..., n\}$ containing at least $n-1$ elements*: $\mathbf{S}_{i \in I} A_i = \sum_{i \in I} A_i$.

Proof. (i) $\Rightarrow$ (ii) is an immediate consequence of Remark 2.4 (i).

(ii) $\Rightarrow$ (iii). Using (1) we have

$$\left(\mathop{\mathbf{S}}_{i=1}^{k-1} A_i\right) \mathbf{T}\, A_k + \left(\mathop{\mathbf{S}}_{i=1}^{k-1} A_i\right) \mathbf{S}\, A_k = \mathop{\mathbf{S}}_{i=1}^{k-1} A_i + A_k, \tag{11}$$

which implies

$$\mathop{\mathbf{S}}_{i=1}^{k} A_i = \mathop{\mathbf{S}}_{i=1}^{k-1} A_i + A_k.$$

Repeating this $(k-1)$ times gives the desired result.

(iii) ⇒ (iv). If $n=2$ or $I=\{1, 2, ..., n\}$ or $I=\{1, 2, ..., n-1\}$ nothing is to prove. Otherwise, observe that for $2 \leqslant k \leqslant n$ we have (11), which implies

$$\left(\mathbf{S}_{i=1}^{k-1} A_i\right) \mathbf{T}\, A_k = \varnothing.$$

Now, for $k=2, ..., n+1$ and $j \leqslant k-1$ define $I_{k,j}=\{1, ..., k-1\} \backslash \{j\}$. Then (11) together with the monotonicity of S and T implies

$$\left(\mathbf{S}_{i \in I_{k,j}} A_i\right) \mathbf{T}\, A_k = \varnothing. \tag{12}$$

Since from (1) we have

$$\left(\mathbf{S}_{i \in I_{k,j}} A_i\right) \mathbf{T}\, A_k + \left(\mathbf{S}_{i \in I_{k,j}} A_i\right) \mathbf{S}\, A_k = \mathbf{S}_{i \in I_{k,j}} A_i + A_k$$

and because of (12) we get

$$\mathbf{S}_{i \in I_{k+1,j}} A_i = \mathbf{S}_{i \in I_{k,j}} A_i + A_k. \tag{13}$$

Now, put $k=n$ and $j \leqslant n-1$. If $j=n-1$ we obtain the desired result from (13). If $j<n-1$, compute $\mathbf{S}_{i \in I_{k,j}} A_i$ using (13), and insert it in (13) again. Continue until $j=k-1$, and this gives again the desired result.

(iv) ⇒ (i). For $1 \leqslant k \leqslant n$ put $I_k=\{1, 2, ..., n\} \backslash \{k\}$. Then because of (1) we have

$$\left(\mathbf{S}_{i \in I_k} A_i\right) \mathbf{T}\, A_k + \left(\mathbf{S}_{i \in I_k} A_i\right) \mathbf{S}\, A_k = \mathbf{S}_{i \in I_k} A_i + A_k,$$

which immediately implies T-disjointness. ∎

2.7 Corollary. *Let T be a t-norm and S its corresponding t-conorm such that* (1) *holds. Then the following assertions are equivalent*:

(i) *The family $\{A_n\}_{n \in \mathbb{N}}$ is T-disjoint.*

(ii) *For any $k \geqslant 2$ we have*: $(\mathbf{S}_{i=1}^{k-1} A_i)\, \mathbf{T}\, A_k = \varnothing$.

(iii) *For any $k \geqslant 2$ we have*: $\mathbf{S}_{i=1}^{k} A_i = \sum_{i=1}^{k} A_i$.

(iv) *For each finite subset I of $\mathbb{N}$ we have*: $\mathbf{S}_{i \in I} A_i = \sum_{i \in I} A_i$.

2.8 *Remark.* Let $\{A_j\}_{j \in J}$ be a countable family of fuzzy subsets.

(i) $\{A_j\}_{j \in J}$ is T_∞-disjoint if and only if $\sum_{i \in J} A_i \leqslant 1$.

(ii) From Proposition 2.6 and Corollary 2.7 we know that if T and its corresponding t-conorm S satisfy (1) and if $\{A_j\}_{j \in J}$ is T-disjoint, then

$\sum_{i\in J} A_i \leqslant 1$. However, the converse is not generally true (see Example 2.5(iii)).

(iii) The requirement that T and S satisfy (1) cannot be dropped in Proposition 2.6 and in Corollary 2.7. If, for instance, we take $S=V$ and $T=W$, then the conditions (i), (ii), and (iii) are no longer equivalent.

3. T-Measures and a First Representation Theorem

Throughout this paragraph let X be a nonempty set, T a t-norm, and S its corresponding t-conorm. For a T-clan $\mathscr{T} \subseteq [0, 1]^X$ we consider functions $\mathbf{m}: \mathscr{T} \to [-\infty, +\infty]$ which assume at most one of the values $-\infty$ and $+\infty$. A function $\mathbf{m}: \mathscr{T} \to [-\infty, +\infty]$ is called a *T-valuation* (*on* $\mathscr{T}$) if it satisfies the following conditions:

$$\mathbf{m}(\varnothing)=0 \tag{14}$$

$$A, B \in \mathscr{T} \Rightarrow \mathbf{m}(A \mathbin{\mathrm{T}} B)+\mathbf{m}(A \mathbin{\mathrm{S}} B)=\mathbf{m}(A)+\mathbf{m}(B). \tag{15}$$

A function $\mathbf{m}: \mathscr{T} \to [-\infty, +\infty]$ is said to be T-additive if it satisfies (14) and

$$(A, B \in \mathscr{T} \text{ and } A \mathbin{\mathrm{T}} B=\varnothing) \Rightarrow \mathbf{m}(A \mathbin{\mathrm{S}} B)=\mathbf{m}(A)+\mathbf{m}(B). \tag{16}$$

3.1 *Remark.* (i) If $\mathbf{m}: \mathscr{T} \to [-\infty, +\infty]$ is a T-valuation on the T-clan $\mathscr{T}$ then $\mathbf{m}$ is also T-additive, the converse not being generally true since, for instance, if $\mathscr{T}$ consists of all the constant functions in $[0, 1]^X$ and if $s \in [0, +\infty[$, then, because of the absence of any nontrivial T_s-disjoint elements in the T_s-clan $\mathscr{T}$, each function $\mathbf{m}: \mathscr{T} \to [-\infty, +\infty]$ which satisfies (14) is T_s-additive without necessarily being a T_s-valuation. This shows that our T-valuations are particular *additive functions* in the sense of Schmidt [31, p. 558] and that, consequently, if they are finite, they can be represented as differences of monotone T-additive functions (cf. Schmidt [31, Theorem 2.2]). However, this is not sufficient to conclude directly that T-valuations always have Jordan decompositions.

(ii) If $\mathscr{T}$ is a T_∞-clan and if $\mathbf{m}$ is a finite T_∞-additive function on $\mathscr{T}$, then $\mathbf{m}$ is also a T_∞-valuation.

(iii) If $\mathscr{T}$ is a T-clan consisting of characteristic functions only, then the finite T-additive functions are Q-valuations for any t-norm Q.

A function $\mathbf{m}$ from a T-clan $\mathscr{T}$ to $[-\infty, +\infty]$ is called a *T-measure* if it is a T-valuation and if the following *left-continuity* is satisfied

$$(\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{T},\ A_n \uparrow A \text{ and } A \in \mathscr{T}) \Rightarrow \lim_{n\to\infty} \mathbf{m}(A_n)=\mathbf{m}(A). \tag{17}$$

A function $\mathbf{m}$ from a T-tribe $\mathscr{T}$ to $[-\infty, +\infty]$ is said to be *T-countably additive* if it satisfies (14) and if

$$\mathbf{m}\left(\mathop{\mathbf{S}}_{i=1}^{\infty} A_n\right) = \sum_{i=1}^{\infty} \mathbf{m}(A_n) \tag{18}$$

for any T-disjoint sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{T}$.

3.2 EXAMPLE. (i) T-countably additive functions are T-additive.

(ii) T-measures on T-tribes are T-countably additive: Take a T-disjoint sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathscr{T}$, and define $B_n = \mathbf{S}_{i=1}^{n} A_i$; then $\{B_n\}_{n\in\mathbb{N}} \subseteq \mathscr{T}$ and $B_n \uparrow (\mathbf{S}_{n=1}^{\infty} A_n)$ and

$$\mathbf{m}\left(\mathop{\mathbf{S}}_{n=1}^{\infty} A_n\right) = \mathbf{m}(\lim_{n\to\infty} B_n) = \lim_{n\to\infty} \mathbf{m}(B_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{m}(A_i) = \sum_{n=1}^{\infty} \mathbf{m}(A_n).$$

(iii) T_∞-countably additive functions on T_∞-tribes are necessarily T_∞-measures (cf. Butnariu [5]), but, in general, for arbitrary t-norms T and T-tribes $\mathscr{T}$ the T-countably additivity does not imply the left-continuity (17). For example, if $T = T_s$ with $s \in [0, \infty[$ and $\mathscr{T} = [0, 1]^X$, then for any fixed $x_0 \in X$ the function $\mathbf{m}$ from $\mathscr{T}$ to $[-\infty, +\infty]$ defined by $\mathbf{m}(A) = 1$ if $A(x_0) = 1$, and $\mathbf{m}(A) = 0$ if $A(x_0) < 1$ is T_s-countably additive, but it is not a T_s-measure since it does not satisfy (17).

(iv) If $\mathscr{T}$ is a T-tribe which consists of characteristic functions only, then the family of T-countable additivity functions coincides with the family of T-measures for any t-norm T, since in this case all t-norms on $\mathscr{T}$ coincide with T_∞.

(v) One can strengthen in some way the left-continuity (17) replacing it by

$$(\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{T},\ A_n \uparrow A \text{ and } A \in \mathscr{T}) \Rightarrow \lim_{n\to\infty} \mathbf{m}(A_n) = \mathbf{m}\left(\mathop{\mathbf{S}}_{n=1}^{\infty} A_n\right). \tag{19}$$

(vi) For T-tribes consisting of characteristic functions only conditions (17) and (19) are equivalent.

(vii) If $\mathbf{m}: \mathscr{T} \to [-\infty, +\infty]$ is *monotone* in the sense that

$$(A, B \in \mathscr{T} \text{ and } A \leqslant B) \Rightarrow \mathbf{m}(A) \leqslant \mathbf{m}(B), \tag{20}$$

then (19) implies (17) for any t-norm T and for any T-tribe $\mathscr{T}$, since for $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{T}$, $A_n \uparrow A$ and $A \in \mathscr{T}$ we have

$$A_n \leqslant A = \mathop{\mathbf{S}_0}_{n=1}^{\infty} A_n \leqslant \mathop{\mathbf{S}}_{n=1}^{\infty} A_n,$$

which implies $\lim_{n\to\infty} \mathbf{m}(A_n) = \mathbf{m}(A) \leqslant \mathbf{m}(\mathbf{S}_{n=1}^{\infty} A_n)$.

3.3 PROPOSITION. *If $\mathscr{T}$ is both a T-clan and a T_0-clan, then each T-valuation is a T_0-valuation.*

Proof. For any A and B in $\mathscr{T}$ we have

$$\begin{aligned} &\mathbf{m}(A\,\mathbf{T}_0\,B) + \mathbf{m}(A\,\mathbf{S}_0\,B) \\ &\quad = \mathbf{m}((A\,\mathbf{T}_0\,B)\,\mathbf{T}\,(A\,\mathbf{S}_0\,B)) + \mathbf{m}((A\,\mathbf{T}_0\,B)\,\mathbf{S}\,(A\,\mathbf{S}_0\,B)) \\ &\quad = \mathbf{m}(A\,\mathbf{T}\,B) + \mathbf{m}(A\,\mathbf{S}\,B) = \mathbf{m}(A) + \mathbf{m}(B). \quad \blacksquare \end{aligned}$$

3.4 *Remark.* (i) Proposition 3.3 shows that if $\mathbf{m}$ is a T-measure on $\mathscr{T}$ ($\mathscr{T}$ being both a T- and a T_0-clan), then it is also a T_0-measure.

(ii) If $\mathbf{m}$ is a T_s-measure on a T_s-tribe $\mathscr{T}$ with $s \in\,]0, \infty]$, then it is also a T_0-measure (cf. Theorem 1.5).

(iii) The converse of Proposition 3.3 does not generally hold: Let $\mathscr{T}$ be the family of all Borel-measurable fuzzy subsets on $X = [0, 1]$. Then the function $\mathbf{m}: \mathscr{T} \to [-\infty, +\infty]$ defined by

$$\mathbf{m}(A) = \int_{\{A>0\}} (1 + A(x))\, dx$$

is a T_0-valuation (even a T_0-measure) but not a T_∞-valuation.

The T_0-measures play a fundamental role in the following. In order to give an integral representation for them let $(X, \mathscr{A})$ be a measurable space, $\mathscr{B}_0$ be the family of all Borel subsets of $[0, 1]$ and $\mathscr{B}_1 = \mathscr{B}_0 \cap [0, 1[$. A function $K: X \times \mathscr{B}_1 \to \mathbb{R}$ is called an *$\mathscr{A}$-Markov kernel* if it satisfies the following conditions:

(a) For each $x \in X$, the function $K(x, \cdot): \mathscr{B}_1 \to \mathbb{R}$ is a probability measure on $\mathscr{B}_1$;

(b) For each $B \in \mathscr{B}_1$, the function $K(\cdot, B): X \to \mathbb{R}$ is measurable.

It was observed above (see Example 1.4 (iii)) that, if $(X, \mathscr{A})$ is a measurable space then the family $\mathscr{A}^\wedge$ of all $\mathscr{A}$-measurable functions from X to $[0, 1]$ is a T_0-tribe. The following result shows that T_0-measures on $\mathscr{A}^\wedge$ can be represented as integrals of Markov kernels.

3.5 THEOREM (First Representation Theorem, Klement [21]). *If T_s is a fundamental t-norm with $s \in [0, \infty]$, if $\mathscr{T}$ is a generated T_s-tribe and if* $\mathbf{m}$ *is a finite monotone T_s-measure on $\mathscr{T}$, then there exists a unique measure $\check{\mathbf{m}}$ on $\mathscr{T}^{\vee}$ and an $\check{\mathbf{m}}$-a.e. uniquely determined $\mathscr{T}^{\vee}$-Markov kernel $K: X \times \mathscr{B}_1 \to \mathbb{R}$ such that*

$$\mathbf{m}(A) = \int_X K(x, [0, A(x)[)\, d\check{\mathbf{m}}(x) \qquad (A \in \mathscr{A}). \tag{21}$$

Proof. Immediate if one combines Proposition 3.3, Remark 3.4 (ii), and the representation theorem in Klement [21, Section 6]. ∎

4. INTEGRAL REPRESENTATION OF $\mathbf{T}_\infty$-MEASURES

The First Representation Theorem shows that monotone finite measures based on fundamental t-norms T and defined on generated T-tribes can be represented as integrals of Markov kernels. It is clear that this holds for $T = T_\infty$, too. However, in this particular case the condition that $\mathscr{T}$ must be generated can be dropped. This is a consequence of the results of [8] showing that for finite T_∞-measures on T_∞-tribes nonnegativity implies *continuity* in the sense of

$$(\{A_n\}_{n \in \mathbb{N}} \subseteq \mathscr{T} \text{ and } \lim_{n \to \infty} A_n = A) \Rightarrow \lim_{n \to \infty} \mathbf{m}(A_n) = \mathbf{m}(A), \tag{22}$$

and that nonnegativity is equivalent to monotonicity. The following Representation Theorem of T_∞-measures is essentially Theorem 2.6 (c) of Butnariu [10]. We present it here with an alternative proof.

4.1 THEOREM. *If $\mathscr{T}$ is a T_∞-tribe and if* $\mathbf{m}$ *is a finite nonnegative T_∞-measure on $\mathscr{T}$ then there exists a unique measure $\check{\mathbf{m}}$ on $\mathscr{T}^{\vee}$, namely the restriction of* $\mathbf{m}$ *to $\mathscr{T}^{\vee}$, such that for any A in $\mathscr{T}$*

$$\mathbf{m}(A) = \int_X A(x)\, d\check{\mathbf{m}}(x). \tag{23}$$

Proof. It is clear that if (23) holds then $\check{\mathbf{m}}$ must be the restriction of $\mathbf{m}$ to $\mathscr{T}^{\vee}$.

Claim 1. If $A \in \mathscr{T}$, $\alpha, \beta \in [0, 1]$ and $\alpha < \beta$ then the set

$$A_{\alpha,\beta} = \{x \in X; \alpha < A(x) \leqslant \beta\}$$

belongs to $\mathscr{T}^{\vee}$, $A \cdot A_{\alpha,\beta}$ belongs to $\mathscr{T}$, and

$$\alpha \cdot \mathbf{m}(A_{\alpha,\beta}) \leqslant \mathbf{m}(A \cdot A_{\alpha,\beta}). \tag{24}$$

The first assertion follows from Theorem 2.1. The second results from the fact that for any $M \in \mathscr{T}^{\vee}$ we have $A \cdot M = A \, \mathbf{T}_\infty \, M \in \mathscr{T}$. Now, in order to prove (24) it is sufficient to show that it holds for any A in $\mathscr{T}$ and for any α, β in $[0, 1]$, where $\alpha < \beta$ and $\alpha = a$ with a of the form (8). Indeed, if (24) is true in this case, then for any $0 < \alpha < \beta \leqslant 1$ we can find a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ which is nonnegative, nondecreasing, and convergent to α, and such that each α_n is of the form (8). Using the continuity of $\mathbf{m}$ we get

$$\alpha \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) = \lim_{n \to \infty} \alpha_n \cdot \check{\mathbf{m}}(A_{\alpha_n,\beta}) \leqslant \lim_{n \to \infty} \mathbf{m}(A \cdot A_{\alpha_n,\beta}) = \mathbf{m}(A \cdot A_{\alpha,\beta}),$$

since in our setting we have $A_{\alpha_n,\beta} \downarrow A_{\alpha,\beta}$. Let us assume $\alpha = a$, where a is of the form (8). If $a = 0$ there is nothing to prove. Suppose $a > 0$. In this situation we proceed by induction upon the number k involved in (8). If $k = 1$, then $\alpha = 0$ or $\alpha = \frac{1}{2}$. In the first case (24) clearly holds. In the second case we have $A_{\alpha,\beta} = (A \cdot A_{\alpha,\beta}) \, \mathbf{S}_\infty \, (A \cdot A_{\alpha,\beta})$, and this implies

$$\begin{aligned}
\check{\mathbf{m}}(A_{\alpha,\beta}) &= \mathbf{m}(A_{\alpha,\beta}) \\
&= \mathbf{m}(A \cdot A_{\alpha,\beta}) + \mathbf{m}(A \cdot A_{\alpha,\beta}) - \mathbf{m}[(A \cdot A_{\alpha,\beta}) \, \mathbf{T}_\infty \, (A \cdot A_{\alpha,\beta})] \\
&\leqslant 2 \cdot \mathbf{m}(A \cdot A_{\alpha,\beta})
\end{aligned}$$

which is exactly (24) with $\alpha = \frac{1}{2}$. Suppose that (24) holds for all $A \in \mathscr{T}$ and for all α, β in $[0, 1]$ with $\alpha < \beta$ and $\alpha = a$, where a is of the form (8) with $k \leqslant m$. Consider

$$\alpha = \sum_{i=1}^{m+1} \frac{a(i)}{2^i} \qquad \text{with} \quad a(i) \in \{0, 1\}, \quad (1 \leqslant i \leqslant m+1), \quad \text{and} \quad a(m+1) \neq 0.$$

Then we get $\alpha = \theta/2$ with

$$\theta = \sum_{i=1}^{m+1} \frac{a(i)}{2^{i-1}}. \tag{25}$$

Case 1. Assume $\theta < 1$ and $\beta < \frac{1}{2}$. Then $A_{\alpha,\beta} = (A \, \mathbf{S}_\infty \, A)_{\theta, 2\beta}$. Using the inductive assumption for the set $(A \, \mathbf{S}_\infty \, A)_{\theta, 2\beta}$ (this is possible since the sum in (25) has at most m nonzero terms in our case) we get $\theta \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) \leqslant \mathbf{m}[(A \, \mathbf{S}_\infty \, A) \cdot A_{\alpha,\beta}]$ and, observing that $(A \, \mathbf{S}_\infty \, A) \cdot A_{\alpha,\beta} \leqslant (A \cdot A_{\alpha,\beta}) \, \mathbf{S}_\infty \, (A \cdot A_{\alpha,\beta})$, we obtain

$$\begin{aligned}
\theta \cdot \check{\mathbf{m}}(A_{\alpha,\beta}) &\leqslant \mathbf{m}[(A \, \mathbf{S}_\infty \, A) \cdot A_{\alpha,\beta}] \leqslant \mathbf{m}[(A \cdot A_{\alpha,\beta}) \, \mathbf{S}_\infty \, (A \cdot A_{\alpha,\beta})] \\
&= 2 \cdot \mathbf{m}(A \cdot A_{\alpha,\beta}) - \mathbf{m}[(A \cdot A_{\alpha,\beta}) \, \mathbf{T}_\infty \, (A \cdot A_{\alpha,\beta})] \leqslant 2 \cdot \mathbf{m}(A \cdot A_{\alpha,\beta})
\end{aligned}$$

by the monotonicity and additivity of $\mathbf{m}$ (see [8]). This implies (24) in this specific case.

Case 2. Assume $\theta<1$ and $\beta=\frac{1}{2}$. Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be an increasing sequence in $]\alpha,\beta[$ which converges to β. By Case 1 we have $\alpha\cdot\check{\mathbf{m}}(A_{\alpha,\gamma_n})\leqslant \mathbf{m}(A\cdot A_{\alpha,\gamma_n})$ for each $n\in\mathbb{N}$. Since $\mathbf{m}$ is continuous (cf. [8]) and because of

$$A_{\alpha,\gamma_n}\uparrow \mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}$$

we get

$$\alpha\cdot\check{\mathbf{m}}\left(\mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}\right)\leqslant \mathbf{m}\left[A\cdot\left(\mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}\right)\right],$$

where

$$\left(\mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}\right)\cup\{x\in X;\, A(x)=\beta\}=A_{\alpha,\beta}.$$

Since the sets forming the union are disjoint one may write

$$\begin{aligned}\alpha\cdot\check{\mathbf{m}}(A_{\alpha,\beta})&=\alpha\cdot\check{\mathbf{m}}\left(\mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}\right)+\alpha\cdot\check{\mathbf{m}}(\{A=\beta\})\\ &\leqslant \mathbf{m}\left[A\left(\mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}\right)\right]+\alpha\cdot\check{\mathbf{m}}(\{A=\beta\}).\end{aligned}\tag{26}$$

Since for $\beta=\frac{1}{2}$ the fuzzy subsets B_1 and B_2 defined by

$$\begin{aligned}B_1(x)=B_2(x)&=0 \quad\text{if}\quad A(x)\neq\beta\\ &=\beta \quad\text{if}\quad A(x)=\beta\end{aligned}$$

are T_∞-disjoint and elements of $\mathscr{T}$, we have $\mathbf{m}(B_1\,\mathbf{S}_\infty\,B_2)=2\cdot\mathbf{m}(\beta\cdot\{A=\beta\})$ (cf. Remark 3.1 (i)) and $B_1\,\mathbf{S}_\infty\,B_2=\{A=\beta\}$. Hence,

$$\mathbf{m}(\beta\cdot\{A=\beta\})=\beta\cdot\check{\mathbf{m}}(\{A=\beta\})\geqslant\alpha\cdot\mathbf{m}(\{A=\beta\}).$$

Combining this with (26) and using the additivity of $\mathbf{m}$, we deduce

$$\begin{aligned}\alpha\cdot\mathbf{m}(A_{\alpha,\beta})&\leqslant\mathbf{m}\left[A\cdot\left(\mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}\right)\right]+\mathbf{m}(\beta\cdot\{A=\beta\})\\ &=\mathbf{m}\left[A\cdot\left(\{A=\beta\}\cup\left(\mathop{\mathbf{S}_0}_{n=1}^{\infty} A_{\alpha,\gamma_n}\right)\right)\right]=\mathbf{m}(A\cdot A_{\alpha,\beta}).\end{aligned}$$

This proves (24) in this case.

Case 3. Assume $\theta<1$ and $\beta>\frac{1}{2}$. Then $A\cdot A_{\alpha,\beta}=(A\cdot A_{\alpha,1/2})$ $\mathbf{S}_\infty\,(A\cdot A_{1/2,\beta})$, where the fuzzy subsets on the right-hand side are T_∞-disjoint. Hence, according to Remark 3.1 (i) we have $\mathbf{m}(A\cdot A_{\alpha,\beta})=\mathbf{m}(A\cdot A_{\alpha,1/2})+\mathbf{m}(A\cdot A_{1/2,\beta})$. The first term on the right-hand side falls under the circumstances of Case 2, and the second one falls under the circumstances of the inductive assumption. Hence

$$\begin{aligned}\check{\mathbf{m}}(A_{\alpha,\beta})&\geqslant\alpha\cdot\check{\mathbf{m}}(A_{1/2,\beta})+\tfrac{1}{2}\cdot\check{\mathbf{m}}(A_{1/2,\beta})\\&\geqslant\alpha\cdot[\check{\mathbf{m}}(A_{\alpha,1/2})+\check{\mathbf{m}}(A_{1/2,\beta})]=\alpha\cdot\check{\mathbf{m}}(A_{\alpha,\beta}),\end{aligned}$$

showing that (24) holds in this case too.

Case 4. Assume finally $\theta\geqslant 1$. In this case α can be written as

$$\alpha=(\varepsilon+1)/2,\qquad\text{where}\quad\varepsilon=\sum_{i=1}^{m}\frac{a(i)}{2^i}\in[0,1].$$

We also have $A_{\alpha,\beta}=[(A'\,\mathbf{S}_\infty\,A')']_{\varepsilon,2\delta}$ with $\delta=\beta-\frac{1}{2}$. Using the inductive assumption for the set $[(A'\,\mathbf{S}_\infty\,A')']_{\varepsilon,2\delta}$, we obtain

$$\begin{aligned}\alpha\cdot\check{\mathbf{m}}(A_{\alpha,\beta})&=\frac{1}{2}\cdot\check{\mathbf{m}}(A_{\alpha,\beta})+\frac{\varepsilon}{2}\cdot\check{\mathbf{m}}(A_{\alpha,\beta})\\&=\frac{1}{2}\cdot\check{\mathbf{m}}(A_{\alpha,\beta})+\frac{\varepsilon}{2}\cdot\check{\mathbf{m}}([(A'\,\mathbf{S}_\infty\,A')']_{\varepsilon,2\delta})\\&\leqslant\frac{1}{2}\cdot\check{\mathbf{m}}(A_{\alpha,\beta})+\frac{1}{2}\cdot\check{\mathbf{m}}([(A'\,\mathbf{S}_\infty\,A')']_{\varepsilon,2\delta}).\end{aligned}\tag{27}$$

Observe that $A_{\alpha,\beta}[(A'\,\mathbf{S}_\infty\,A')']=D\,\mathbf{T}_\infty\,C'$ with $D=A_{\alpha,\beta}$ and $C=A_{\alpha,\beta}(A'\,\mathbf{S}_\infty\,A')$. According to Remark 3.1 (i) we have that

$$(E,F\in\mathscr{T}\text{ and }E\geqslant F)\Rightarrow\mathbf{m}(E\,\mathbf{T}_\infty\,F')=\mathbf{m}(E)-\mathbf{m}(F).\tag{28}$$

Since we clearly have $D\geqslant C$, (27) combined with (28) gives

$$\alpha\cdot\check{\mathbf{m}}(A_{\alpha,\beta})\leqslant\tfrac{1}{2}\cdot\check{\mathbf{m}}(A_{\alpha,\beta})+\tfrac{1}{2}\cdot[\mathbf{m}(D)-\mathbf{m}(C)]=\check{\mathbf{m}}(A_{\alpha,\beta})-\tfrac{1}{2}\cdot\mathbf{m}(C).\tag{29}$$

Now, taking into account that $\theta\geqslant 1$ we deduce $C=(A'\cdot A_{\alpha,\beta})\,\mathbf{S}_\infty\,(A'\cdot A_{\alpha,\beta})$ and $(A'\cdot A_{\alpha,\beta})\,\mathbf{T}_\infty\,(A'\cdot A_{\alpha,\beta})=\varnothing$. Thus $\mathbf{m}(C)=2\cdot\mathbf{m}(A'\cdot A_{\alpha,\beta})$ by the additivity of $\mathbf{m}$. Since $A'\cdot A_{\alpha,\beta}=A_{\alpha,\beta}\,\mathbf{T}_\infty\,(A\cdot A_{\alpha,\beta})'$, we get $\mathbf{m}(C)=2\cdot[\mathbf{m}(A_{\alpha,\beta})-\mathbf{m}(A\cdot A_{\alpha,\beta})]$ because of (28). Substituting this in (29) we obtain (24), and Claim 1 is completely proved.

Claim 2. If $A\in\mathscr{T}$, if $0\leqslant\alpha<\beta\leqslant 1$, and if we put $\bar{A}_{\alpha,\beta}=\{x\in X;\alpha\leqslant A(x)<\beta\}$, then $\bar{A}_{\alpha,\beta}\in\mathscr{T}^{\vee}$, $A\cdot\bar{A}_{\alpha,\beta}\in\mathscr{T}$, and

$$\mathbf{m}(A\cdot\bar{A}_{\alpha,\beta})\leqslant\beta\cdot\check{\mathbf{m}}(\bar{A}_{\alpha,\beta}).\tag{30}$$

The first two assertions are obvious. To prove (30) observe that

$$A_{\alpha,\beta} = \bar{A}'_{\beta',\alpha'}, \qquad \text{where} \quad \alpha' = 1-\alpha \qquad \text{and} \qquad \beta' = 1-\beta. \tag{31}$$

Then

$$\begin{aligned}
\mathbf{m}[(A \cdot A_{\alpha,\beta})'] &= \check{\mathbf{m}}(X) - \mathbf{m}(A \cdot \bar{A}_{\alpha,\beta}) = \check{\mathbf{m}}(\bar{A}_{\alpha,\beta}) + \check{\mathbf{m}}([\bar{A}_{\alpha,\beta}]') - \mathbf{m}(A \cdot \bar{A}_{\alpha,\beta}) \\
&= \mathbf{m}(A' \cdot \bar{A}_{\alpha,\beta}) + \check{\mathbf{m}}([\bar{A}_{\alpha,\beta}]') = \mathbf{m}(A' \cdot A'_{\beta',\alpha'}) + \check{\mathbf{m}}([\bar{A}_{\alpha,\beta}]') \\
&\geqslant \beta' \cdot \check{\mathbf{m}}(X) + \check{\mathbf{m}}([\bar{A}_{\alpha,\beta}]') = \check{\mathbf{m}}(X) - \beta \cdot \check{\mathbf{m}}(\bar{A}_{\alpha,\beta}),
\end{aligned}$$

where the inequality and the last equality are consequences of (31) and Claim 1, respectively. This implies (30), and Claim 2 is proved.

In order to complete the proof of our theorem let A be in $\mathscr{T}$. Denote

$$\begin{aligned}
G_{n,i} &= \{x \in X;\, A(x) = 0\} && \text{if } \ i = 0, \\
&= A_{(i-1)/2^n,\, i/2^n} && \text{if } \ 1 \leqslant i \leqslant 2^n,
\end{aligned}$$

and

$$\begin{aligned}
H_{n,i} &= \bar{A}_{i/2^n,\, (i+1)/2^n} && \text{if } \ 1 \leqslant i < 2^n, \\
&= \{x \in X;\, A(x) = 1\} && \text{if } \ i = 2^n.
\end{aligned}$$

From Claims 1 and 2 it follows that the step functions

$$s_n = \sum_{i=1}^{m} \frac{i-1}{m} \cdot G_{n,i} \qquad \text{and} \qquad t_n = \sum_{i=0}^{m-1} \frac{i+1}{m} \cdot H_{n,i} + H_{n,m},$$

where $m = 2^n$ are $\mathscr{T}^{\vee}$-measurable. It is clear that $s_n \uparrow A$ and $t_n \downarrow A$. Taking into account (24) and (30), we deduce

$$\int_X s_n \, d\check{\mathbf{m}} = \sum_{i=1}^{m} \frac{i-1}{m} \cdot \check{\mathbf{m}}(G_{n,i}) \leqslant \sum_{i=0}^{m-1} \mathbf{m}(A \cdot G_{n,i}) = \mathbf{m}(A)$$

and

$$\int_X t_n \, d\check{\mathbf{m}} \geqslant \sum_{i=1}^{m-1} \frac{i+1}{m} \cdot \check{\mathbf{m}}(H_{n,i}) \geqslant \sum_{i=0}^{m} \mathbf{m}(A \cdot H_{n,i}) = \mathbf{m}(A).$$

Taking the limit $n \to \infty$ in these relations we obtain (23), therefore completing the proof of the theorem. ∎

4.2 *Remark.* (i) Comparing the results of [8] with the First Representation Theorem, one can easily see that on a generated T-tribe, with T being a fundamental t-norm, the T_∞-measures are exactly those

T-measures for which the corresponding Markov kernel K in the representation (21) is given by

$$K(x, [\alpha, \beta[) = \beta - \alpha \qquad (x \in X).$$

(ii) For fundamental t-norms T_s with $s \in]0, \infty[$ and generated T_s-tribes, one can also specify the form of the Markov kernel involved in (21). To be precise, it was shown in [24] that if T_s is a fundamental t-norm with $s \in]0, \infty[$, and if the T_s-tribe $\mathscr{T}$ is generated, then for any monotone finite T_s-measure $\mathbf{m}$ on $\mathscr{T}$ there exists a unique measure $\check{\mathbf{m}}$ on $\mathscr{T}^{\vee}$, namely the restriction of $\mathbf{m}$ to $\mathscr{T}^{\vee}$, and an $\check{\mathbf{m}}$–a.e. uniquely determined $\mathscr{T}^{\vee}$-measurable function $f: X \to [0, 1]$ such that for all $A \in \mathscr{T}$

$$\mathbf{m}(A) = \int_{\{A > 0\}} [f + (1 - f) \cdot A] \, d\check{\mathbf{m}}. \tag{32}$$

(iii) Theorems 4.1 and 1.5 imply that T_∞-measures on T_s-tribes with $s \in]0, \infty[$ are also T_s-measures. However, not each T_s-measure is necessarily a T_∞-measure, even on generated T_s-tribes. It was shown in [21] that it is necessary and sufficient for a monotone finite T_s-measure $\mathbf{m}$ with $s \in]0, \infty[$ to be a T_∞-measure, that the following condition be satisfied:

$$(\{A_n\}_{n \in \mathbb{N}} \subseteq \mathscr{T} \text{ and } A_n \downarrow \varnothing) \Rightarrow \lim_{n \to \infty} \mathbf{m}(A_n) = 0. \tag{33}$$

5. Decompositions of T-Measures

According to (32), finite monotone measures $\mathbf{m}$, based on fundamental t-norms T_s with $s \in]0, \infty]$, on generated tribes differ from T_∞-measures (i.e., from integrals, according to Theorem 4.2) by functions of the form $A \to \int_{\{A > 0\}} f \, d\check{\mathbf{m}}$, which are also monotone finite T_s-measures. The question is now how much a T_s-measure, defined on a nongenerated T_s-tribe, differs from a T_∞-measure (i.e., an integral).

5.1 Proposition. *Let T_s be a fundamental t-norm with $s \in [0, \infty]$. If $\mathscr{T}$ is a T_s-tribe and if $\mathbf{m}$ is a finite monotone T_s-measure on $\mathscr{T}$, then there exists a unique pair $(\mathbf{m}_\infty, \mathbf{m}_s)$ of functions from $\mathscr{T}$ to $\mathbb{R}_+$ such that:*

(a) $\mathbf{m}_\infty$ *is a T_∞-measure on $\mathscr{T}$;*

(b) $\mathbf{m}_s$ *is a T_s-measure on $\mathscr{T}$;*

(c) $\mathbf{m} = \mathbf{m}_\infty + \mathbf{m}_s$;

(d) $\mathbf{m}_\infty$ *is "maximal" in the sense that if $\mathbf{m}': \mathscr{T} \to \mathbb{R}_+$ is another T_∞-measure such that $\mathbf{m} - \mathbf{m}'$ is monotone, then $\mathbf{m}' \leqslant \mathbf{m}_\infty$.*

Moreover, the functions $\mathbf{m}_\infty$ *and* $\mathbf{m}_s$ *have the property that there exists a unique measure* $\check{\mathbf{m}}$ *on* $\mathscr{T}^\vee$, *namely the restriction of* $\mathbf{m}$ *to* $\mathscr{T}^\vee$, *and an* $\check{\mathbf{m}}$-*a.e. unique* $\mathscr{T}^\vee$-*measurable function* $f: X \to [0, 1]$ *such that for all* $A \in \mathscr{T}$

$$\mathbf{m}_\infty(A) = \int_X (1 - f) \cdot A \, d\check{\mathbf{m}}, \tag{34}$$

and for all $M \in \mathscr{T}^\vee$

$$\mathbf{m}_s(M) = \int_M f \, d\check{\mathbf{m}}. \tag{35}$$

Proof. Denote by $\mathscr{M}_\infty$ the family of all T_∞-measures $\mathbf{p}: \mathscr{T} \to \mathbb{R}_+$ such that $\mathbf{m} - \mathbf{p}$ is monotone. The family $\mathscr{M}_\infty$ is nonempty since it contains the zero T_∞-measure on $\mathscr{T}$. The family $\mathscr{M}_\infty$ is provided with the partial order

$$\mathbf{p} \leqslant \mathbf{p}' \Leftrightarrow (\forall A \in \mathscr{T}: \mathbf{p}(A) \leqslant \mathbf{p}'(A)). \tag{36}$$

If $\{\mathbf{p}_\alpha\}_{\alpha \in J}$ is a chain in $\mathscr{M}_\infty$, then the function $\mathbf{p}: \mathscr{T} \to [0, \infty]$ defined by

$$\mathbf{p}(A) = \sup_{\alpha \in J} \mathbf{p}_\alpha(A) \qquad (A \in \mathscr{T}) \tag{37}$$

is a T_∞-valuation. Indeed, $\mathbf{p}(\varnothing) = 0$ clearly holds, and for any A and B in $\mathscr{T}$ we have

$$\begin{aligned}\mathbf{p}(A \, \mathbf{S}_\infty \, B) + \mathbf{p}(A \, \mathbf{T}_\infty \, B) &= \sup_{\alpha \in J} \mathbf{p}_\alpha(A \, \mathbf{S}_\infty \, B) + \sup_{\alpha \in J} \mathbf{p}_\alpha(A \, \mathbf{T}_\infty \, B) \\ &= \sup_{\alpha \in J} [\mathbf{p}_\alpha(A \, \mathbf{S}_\infty \, B) + \mathbf{p}_\alpha(A \, \mathbf{T}_\infty \, B)] \\ &= \sup_{\alpha \in J} [\mathbf{p}_\alpha(A) + \mathbf{p}_\alpha(B)] = \mathbf{p}(A) + \mathbf{p}(B),\end{aligned}$$

where the second and the last equality hold because of the monotonicity of $\{\mathbf{p}_\alpha(C)\}_{\alpha \in J}$ for every C in $\mathscr{T}$. It is clear that $\mathbf{p}$ is monotone. Hence, $0 \leqslant \mathbf{p}(A) \leqslant \mathbf{m}(A) \leqslant \mathbf{m}(X)$ for all A in $\mathscr{T}$, implying that $\mathbf{p}$ is also finite. If $\{A_n\}_{n \in \mathbb{N}}$ is a nondecreasing sequence in $\mathscr{T}$, then

$$\begin{aligned}\lim_{n \to \infty} \mathbf{p}(A_n) &= \sup_{n \in \mathbb{N}} \mathbf{p}(A_n) = \sup_{n \in \mathbb{N}} (\sup_{\alpha \in J} \mathbf{p}_\alpha(A_n)) \\ &= \sup_{\alpha \in J} (\sup_{n \in \mathbb{N}} \mathbf{p}_\alpha(A_n)) = \sup_{\alpha \in J} (\lim_{n \to \infty} \mathbf{p}_\alpha(A_n)) = \mathbf{p}(\lim_{n \to \infty} A_n),\end{aligned}$$

showing that $\mathbf{p}$ is left-continuous. Hence, $\mathbf{p}$ is a finite nonnegative T_∞-measure on $\mathscr{T}$. It is easy to see that $\mathbf{m} - \mathbf{p}$ is also monotone, since

$\mathbf{m} - \mathbf{p}_\alpha$ is monotone for each $\alpha \in J$. Thus we have $\mathbf{p} \in \mathcal{M}_\infty$. In other words, each nondecreasing chain in $\mathcal{M}_\infty$ has an upper bound in $\mathcal{M}_\infty$ and, by Zorn's lemma, $\mathcal{M}_\infty$ has a maximal element denoted $\mathbf{m}_\infty$. Since $\mathbf{m}_\infty$ is a T_∞-measure it is a T_s-measure, too, and so is the difference $\mathbf{m}_s = \mathbf{m} - \mathbf{m}_\infty$. By the definition of $\mathbf{m}_\infty$, $\mathbf{m}_s$ is monotone and finite, and for the pair $(\mathbf{m}_\infty, \mathbf{m}_s)$ the conditions (a), (b), and (c) are satisfied. By the maximality of $\mathbf{m}_\infty$ in $\mathcal{M}_\infty$ the condition (d) is also satisfied; (a) and (d) imply uniqueness. It remains to show that there exists an f such that (34) and (35) hold. To this end observe that $\mathcal{T}$ is a T_∞-tribe (cf. Theorem 1.5) and that $\mathbf{m}_\infty$ is a T_∞-measure on $\mathcal{T}$. Hence, according to Theorem 4.1, $\mathbf{m}_\infty$ can be written as

$$\mathbf{m}_\infty = \int_X A \, d\check{\mathbf{m}}_\infty \qquad (A \in \mathcal{T}), \tag{38}$$

where $\check{\mathbf{m}}_\infty$ is the restriction of $\mathbf{m}_\infty$ to $\mathcal{T}^\vee$. Since $\mathbf{m}_\infty \leqslant \mathbf{m}$, it follows that $\check{\mathbf{m}}_\infty$ is absolutely continuous with respect to the restriction $\check{\mathbf{m}}$ of $\mathbf{m}$ to $\mathcal{T}^\vee$, and the Radon–Nikodym derivative $d\check{\mathbf{m}}_\infty / d\check{\mathbf{m}}$ is $\check{\mathbf{m}}$-a.e. equal to a function g mapping X into $[0, 1]$. Putting $f = 1 - g$, then f is a $\mathcal{T}^\vee$-measurable function with values in $[0, 1]$, and (34) is satisfied due to (38). Now, taking into account the definition of $\mathbf{m}_s$, we can write

$$\mathbf{m}_s(M) = \check{\mathbf{m}}(M) - \check{\mathbf{m}}_\infty(M) = \check{\mathbf{m}}(M) - \int_M (1 - f) \, d\check{\mathbf{m}} = \int_M f \, d\check{\mathbf{m}}$$

for all $M \in \mathcal{T}^\vee$, and therefore (35) also holds. It also follows that if (34) and (35) hold then $\check{\mathbf{m}}$ must be equal to the restriction of $\mathbf{m}$ to $\mathcal{T}^\vee$. ▌

5.2 *Remark.* (i) The component $\mathbf{m}_s$ of a T_s-measure in Proposition 8.2 is a "pure" T_s-measure in the sense that it has a zero T_∞-component: $(\mathbf{m}_s)_\infty = 0$. In fact, assuming the contrary would contradict the maximality of $\mathbf{m}_\infty$.

(ii) If in Proposition 5.1 one assumes the T_s-tribe $\mathcal{T}$ to be generated, then (34) holds for any M in $\mathcal{T}$ (and not only for M in the σ-algebra $\mathcal{T}^\vee$). Indeed, this follows comparing (32) with (34) and (35), and keeping in mind that the function f in (32) must be $\check{\mathbf{m}}$-a.e. unique.

Consider a T_s-tribe $\mathcal{T}$, where T_s is a fundamental t-norm with $s \in \,]0, \infty[$. If $\check{\mathbf{p}}$ is any measure on $\mathcal{T}^\vee$, and if g and h are any $\mathcal{T}^\vee$-measurable functions from X to $[0, \infty]$, then the function $\mathbf{m}: \mathcal{T} \to [0, +\infty]$ defined by

$$\mathbf{m}(A) = \int_{\{A > 0\}} (g + h \cdot A) \, d\check{\mathbf{p}} \tag{39}$$

is a monotone T_s-measure on $\mathscr{T}$. Monotonicity and $\mathbf{m}(\varnothing)=0$ are obvious; the left continuity of $\mathbf{m}$ follows from the Lebesgue monotone convergence theorem, taking into account that for a nondecreasing sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathscr{T}$ whose pointwise limit is A, one has $\bigcup_{n=1}^{\infty}\{A_n>0\}=\{A>0\}$; the T_s-additivity of $\mathbf{m}$ is shown as

$$\begin{aligned}
\mathbf{m}(A\,\mathbf{T}_s\,B)+\mathbf{m}(A\,\mathbf{S}_s\,B) &= \int_{\{A>0\}\cap\{B>0\}}[g+h\cdot(A\,\mathbf{T}_s\,B)]\,d\check{\mathbf{p}} \\
&\quad+\int_{\{A>0\}\cup\{B>0\}}[g+h\cdot(A\,\mathbf{S}_s\,B)]\,d\check{\mathbf{p}} \\
&=\int_{\{A>0\}\cap\{B>0\}}[2\cdot g+h\cdot(A\,\mathbf{T}_s\,B+A\,\mathbf{S}_s\,B)]\,d\check{\mathbf{p}} \\
&\quad+\int_{\{A>0\}\cap\{B=0\}}[g+h(A\,\mathbf{S}_s\,B)]\,d\check{\mathbf{p}} \\
&\quad+\int_{\{A=0\}\cap\{B>0\}}[g+h(A\,\mathbf{S}_s\,B)]\,d\check{\mathbf{p}} \\
&=\int_{\{A>0\}\cap\{B>0\}}[2\cdot g+h\cdot(A+B)]\,d\check{\mathbf{p}} \\
&\quad+\int_{\{A>0\}\cap\{B=0\}}(g+h\cdot A)\,d\check{\mathbf{p}} \\
&\quad+\int_{\{A=0\}\cap\{B>0\}}(g+h\cdot B)\,d\check{\mathbf{p}} \\
&=\int_{\{A>0\}}(g+h\cdot A)\,d\check{\mathbf{p}}+\int_{\{B>0\}}(g+h\cdot B)\,d\check{\mathbf{p}} \\
&=\mathbf{m}(A)+\mathbf{m}(B).
\end{aligned}$$

A T_s-measure $\mathbf{m}$ on $\mathscr{T}$, which can be represented in the form (39) by some nonnegative measure $\check{\mathbf{p}}$ on $\mathscr{T}^{\vee}$ and by some pair (g,h) of nonnegative $\mathscr{T}^{\vee}$-measurable functions on X, is said to be *generated* (*by* $\check{\mathbf{p}}$, g *and* h).

It follows from [24] (see Remark 4.2) that if $\mathscr{T}$ is generated then all finite monotone T_s-measures with $s\in\,]0,\infty[$ on $\mathscr{T}$ are generated. From Theorem 4.2 we already know that the T_∞-measures on $\mathscr{T}$ are generated, even when $\mathscr{T}$ is not generated. Thus, it is natural to ask whether, in general, T_s-measures on T_s-tribes are always generated. In order to answer this question, we define a T_s-measure $\mathbf{m}$ on the T_s-tribe $\mathscr{T}$ to be *monotonically irreducible*, if it is monotone and if there is no nonidentically zero generated T_s-measure $\mathbf{q}$ on $\mathscr{T}$ such that $\mathbf{m}-\mathbf{q}$ is monotone on $\mathscr{T}$.

Now, it is obvious that a T_s-measure $\mathbf{m}$ on $\mathscr{T}$ is generated if and only if it can be extended to a T_s-measure on the generated T_s-tribe $(\mathscr{T}^\vee)^\wedge$ (since, if $\mathbf{m}$ is generated then (39) defines $\mathbf{m}$ on $(\mathscr{T}^\vee)^\wedge$, the converse following from Theorem 5.1). By contrast, monotonically irreducible T_s-measures, except for the trivial one, are not generated and, hence, they cannot be extended to $(\mathscr{T}^\vee)^\wedge$.

5.3 THEOREM. *If T_s is a fundamental t-norm with $s \in]0, \infty]$, if $\mathscr{T}$ is a T_s-tribe, and if $\mathbf{m}$ is a finite monotone T_s-measure on $\mathscr{T}$, then $\mathbf{m}$ can be uniquely decomposed in a monotonically irreducible and a generated T_s-measure; that is there exist a unique monotonically irreducible T_s-measure $\mathbf{m}^*$ on $\mathscr{T}$, a measure $\check{\mathbf{m}}$ on $\mathscr{T}^\vee$ (which is exactly the restriction of $\mathbf{m}$ to $\mathscr{T}^\vee$), and two $\check{\mathbf{m}}$-a.e. uniquely determined $\mathscr{T}^\vee$-measurable functions $g, h: X \to [0, 1]$, such that for all $A \in \mathscr{T}^\vee$ one has*

$$\mathbf{m}(A) - \mathbf{m}^*(A) = \int_{\{A>0\}} (g + h \cdot A)\, d\check{\mathbf{m}}. \tag{40}$$

Proof. If $s = +\infty$, then the result follows from Theorem 4.1 putting $g(x) = 0$, $h(x) = 1$ for $x \in X$ and $\mathbf{m}^* = 0$. Assume $s \in]0, \infty[$. In this case the theorem is proved in several steps.

Claim 1. If $\mathbf{p}$ is a finite monotone T_s-measure on $\mathscr{T}$, then there exists a unique finite monotone T_s-measure $|\mathbf{p}|$ on the generated T_s-tribe $(\mathscr{T}^\vee)^\wedge$ which is *monotonically maximal* in the sense that for any T_s-measure $\mathbf{p}'$ on $(\mathscr{T}^\vee)^\wedge$, for which $\mathbf{p} - \mathbf{p}'$ is monotone on $\mathscr{T}$, the difference $|\mathbf{p}| - \mathbf{p}'$ is also monotone on $(\mathscr{T}^\vee)^\wedge$.

In order to prove that, denote by $\mathscr{N}(\mathbf{p})$ the family of all T_s-measures $\mathbf{q}$ on $(\mathscr{T}^\vee)^\wedge$ such that $\mathbf{p} - \mathbf{q}$ is monotone on $\mathscr{T}$. This family is partially ordered by the *dominance relation* defined by

$$\mathbf{q} \geqslant \mathbf{q}' \Leftrightarrow \mathbf{q} - \mathbf{q}' \text{ is monotone on } (\mathscr{T}^\vee)^\wedge. \tag{41}$$

Let $\{\mathbf{q}_\alpha\}_{\alpha \in J}$ be a chain in $\mathscr{N}(\mathbf{p})$ with respect to the partial order (41) and define

$$\mathbf{q}(A) = \sup_{\alpha \in J} \mathbf{q}_\alpha(A) \qquad (A \in (\mathscr{T}^\vee)^\wedge).$$

Similarly as in the proof of Proposition 5.1, one can prove that $\mathbf{q}$ is a monotone finite T_s-measure on $(\mathscr{T}^\vee)^\wedge$. For $A, B \in \mathscr{T}$ with $A \leqslant B$ we have that

$$\mathbf{p}(A) - \mathbf{q}(A) = \inf_{\alpha \in J} [\mathbf{p}(A) - \mathbf{q}_\alpha(A)] \leqslant \inf_{\alpha \in J} [\mathbf{p}(B) - \mathbf{q}_\alpha(B)] = \mathbf{p}(B) - \mathbf{q}(B),$$

i.e., $\mathbf{p}-\mathbf{q}$ is monotone on $\mathscr{T}$. Clearly, $\mathbf{q} \geqslant \mathbf{q}_\alpha$ ($\alpha \in J$). Hence, each chain in $\mathscr{N}(\mathbf{p})$ has an upper bound in $\mathscr{N}(\mathbf{p})$ and, according to Zorn's lemma, $\mathscr{N}(\mathbf{p})$ has a maximal element denoted $|\mathbf{p}|$. It is obvious that $|\mathbf{p}|$ is the only T_s-measure with this property. Hence, Claim 1 is proved.

Let $|\mathbf{m}|$ be the T_s-measure existing by Claim 1 for $\mathbf{p}=\mathbf{m}$. Since $|\mathbf{m}|$ is defined on the generated tribe $(\mathscr{T}^\vee)^\wedge$, it can be represented according to Theorem 5.1 by

$$|\mathbf{m}|\,(A)=\int_{\{A>0\}} [w+(1-w)\cdot A]\, d\,|\mathbf{m}|^\vee,$$

where $|\mathbf{m}|^\vee$ is the restriction of $|\mathbf{m}|$ to $((\mathscr{T}^\vee)^\wedge)^\vee=\mathscr{T}^\vee$, and w is a $\mathscr{T}^\vee$-measurable function from X to $[0,1]$. As observed above (Remark 5.2 (ii)), the unique decomposition $(|\mathbf{m}|_\infty, |\mathbf{m}|_s)$ of $|\mathbf{m}|$ according to Proposition 5.1 is given by

$$|\mathbf{m}|_\infty\,(A)=\int_X (1-w)\cdot A\, d\,|\mathbf{m}|^\vee$$

and

$$|\mathbf{m}|_s\,(A)=\int_{\{A>0\}} w\, d\,|\mathbf{m}|^\vee. \tag{42}$$

Furthermore, let $(\mathbf{m}_\infty, \mathbf{m}_s)$ be the unique decomposition pair provided by Proposition 5.1 for $\mathbf{m}$, and let $f\colon X\to[0,1]$ be the function satisfying (34) and (35).

Claim 2. For all $A\in(\mathscr{T}^\vee)^\wedge$ we have

$$|\mathbf{m}|_\infty\,(A)=\int_X (1-f)\cdot A\, d\check{\mathbf{m}}. \tag{43}$$

To prove this consider $\mathbf{q}_\infty\colon(\mathscr{T}^\vee)^\wedge\to\mathbb{R}$ defined by

$$\mathbf{q}_\infty(A)=\int_X (1-f)\cdot A\, d\check{\mathbf{m}}.$$

Obviously, this is a finite nonnegative T_∞-measure on $(\mathscr{T}^\vee)^\wedge$ whose restriction to $\mathscr{T}$ coincides with $\mathbf{m}_\infty$. According to the definition of $|\mathbf{m}|$, the function $\mathbf{p}\colon\mathscr{T}\to\mathbb{R}$ given by $\mathbf{p}(A)=\mathbf{m}(A)-|\mathbf{m}|\,(A)$ is a monotone T_s-measure. Thus $\mathbf{m}-|\mathbf{m}|_\infty$ $(=\mathbf{p}+|\mathbf{m}|_s)$ is also a monotone T_s-measure on $\mathscr{T}$. Because of the maximality of $\mathbf{m}_\infty$ we have

$$\mathbf{m}_\infty(A)\geqslant|\mathbf{m}|_\infty\,(A) \qquad (A\in\mathscr{T}). \tag{44}$$

Taking into account Theorem 4.1 we can write for each $A \in (\mathscr{T}^{\vee})^{\wedge}$

$$\mathbf{q}_{\infty}(A) = \int_X A \, d\check{\mathbf{m}}_{\infty}$$

and

$$|\mathbf{m}|_{\infty}(A) = \int_X A \, d|\mathbf{m}|_{\infty}^{\vee},$$

where $\check{\mathbf{m}}_{\infty}$ and $|\mathbf{m}|_{\infty}^{\vee}$ are the restrictions of $\mathbf{m}_{\infty}$ and $|\mathbf{m}|_{\infty}$ to $\mathscr{T}^{\vee}$, respectively. Since by (44) we have also $\check{\mathbf{m}}_{\infty} \geqslant |\mathbf{m}|_{\infty}^{\vee}$, it follows that

$$\mathbf{q}_{\infty}(A) \geqslant |\mathbf{m}|_{\infty}(A) \qquad (A \in (\mathscr{T}^{\vee})^{\wedge}) \tag{45}$$

On the other hand, we know that $\mathbf{m} - \mathbf{m}_{\infty}$ and $\mathbf{m} - \mathbf{q}_{\infty}$ coincide on $\mathscr{T}$, and that the first one is monotone. Since $|\mathbf{m}|_{\infty}$ is a maximal T_{∞}-measure dominated (in the sense of (41)) by $|\mathbf{m}|$, it follows that $|\mathbf{m}|_{\infty} - \mathbf{q}_{\infty}$ is also monotone on $(\mathscr{T}^{\vee})^{\wedge}$, and this implies $|\mathbf{m}|_{\infty} \geqslant \mathbf{q}_{\infty}$ on $(\mathscr{T}^{\vee})^{\wedge}$, which combined with (45) proves Claim 2.

According to the definition of $|\mathbf{m}|$ (see Claim 1) we have that $\check{\mathbf{m}} \geqslant |\mathbf{m}|^{\vee}$ on $\mathscr{T}^{\vee}$. Therefore there exists a $[0, 1]$-valued Radon–Nikodym derivative u of $|\mathbf{m}|^{\vee}$ with respect to $\check{\mathbf{m}}$. Putting $g = w \cdot u$ and using (42) one gets

$$|\mathbf{m}|_s(A) = \int_{\{A>0\}} g \, d\check{\mathbf{m}} \qquad (A \in (\mathscr{T}^{\vee})^{\wedge}), \tag{46}$$

where g is a function with values in $[0, 1]$. From Claim 2 and (46) we deduce

$$\begin{aligned} \mathbf{m}(A) &= |\mathbf{m}|(A) + (\mathbf{m}_s(A) - |\mathbf{m}|_s(A)) \\ &= \mathbf{m}^*(A) + \int_{\{A>0\}} [g + (1-f) \cdot A] \, d\check{\mathbf{m}} \end{aligned} \tag{47}$$

with

$$\mathbf{m}^*(A) := \mathbf{m}_s(A) - |\mathbf{m}|_s(A) \qquad (A \in \mathscr{T}).$$

Claim 3. $\mathbf{m}^*$ is a finite monotonically irreducible T_s-measure on $\mathscr{T}$. It is clear that $\mathbf{m}^*$ is a finite monotone T_s-measure on $\mathscr{T}$. In order to show that it is also irreducible, it is sufficient to show that $|\mathbf{m}_s| = |\mathbf{m}|_s$ on $(\mathscr{T}^{\vee})^{\wedge}$, where $|\mathbf{m}_s|$ is the T_s-measure existing for $\mathbf{m}_s$ by Claim 1. First observe that $|\mathbf{m}_s|_{\infty} = \mathbf{0}$ on $(\mathscr{T}^{\vee})^{\wedge}$ because of Proposition 5.1 and of the fact, that $\mathbf{m} = \mathbf{m}_{\infty} + |\mathbf{m}_s|_{\infty} + |\mathbf{m}_s|_s$ (on $\mathscr{T}$) implies $|\mathbf{m}_s|_{\infty} = \mathbf{0}$ on $\mathscr{T}$ (by the maximality of $\mathbf{m}_{\infty}$), which in turn implies that the restriction of $|\mathbf{m}_s|_{\infty}$ to $\mathscr{T}^{\vee}$ is also identically zero. Now, observe that on $\mathscr{T}$ we have $\mathbf{m} - |\mathbf{m}_s| = \mathbf{m}_{\infty} + (\mathbf{m}_s - |\mathbf{m}_s|)$, where the right-hand side is a monotone T_s-measure on

$\mathscr{T}$. Thus, $|\mathbf{m}| - |\mathbf{m}_s|$ must be monotone on $(\mathscr{T}^\vee)^\wedge$ (cf. Claim 1). But by Claim 2 we have $\mathbf{m}_s - |\mathbf{m}|_s = \mathbf{m} - |\mathbf{m}|$ on $\mathscr{T}$ implying that $\mathbf{m}_s - |\mathbf{m}|_s$ is monotone on $\mathscr{T}$. Hence, because of the maximality of $|\mathbf{m}_s|$ we know that $|\mathbf{m}_s| - |\mathbf{m}|_s$ is monotone on $(\mathscr{T}^\vee)^\wedge$. Suppose that $|\mathbf{m}_s| - |\mathbf{m}|_s \neq 0$. Then the function $\bar{\mathbf{m}}$, which is defined by $\bar{\mathbf{m}}(A) = |\mathbf{m}|_\infty (A) + |\mathbf{m}_s| (A)$ $(A \in (\mathscr{T}^\vee)^\wedge)$, is a monotone T_s-measure on $(\mathscr{T}^\vee)^\wedge$ which dominates $|\mathbf{m}|$ (in the sense of (41)) and satisfies $\mathbf{m} - \bar{\mathbf{m}} = \mathbf{m}_s - |\mathbf{m}_s|$ on $\mathscr{T}$ (cf. Claim 2), where the right-hand side is monotone on $\mathscr{T}$. This contradicts the maximality of the T_s-measure $|\mathbf{m}|$. Claim 3 is completely proved.

Now, putting $h := 1 - f$ in (47), and taking into account Claim 3, we obtain a representation of the form (40) for $\mathbf{m}$. Suppose that $\mathbf{m} = \mathbf{m}' + \mathbf{p}'$ is another decomposition of $\mathbf{m}$ by a monotonically irreducible T_s-measure $\mathbf{m}'$ and a generated T_s-measure $\mathbf{p}'$. The generated measure $\mathbf{p}'$ can be extended in the canonical way to $(\mathscr{T}^\vee)^\wedge$. Since $\mathbf{m} - \mathbf{p}' = \mathbf{m}'$ is monotone on $\mathscr{T}$ it follows that $|\mathbf{m}| \gg \mathbf{p}'$ (cf. Claim 1). Hence on $\mathscr{T}$ we have $\mathbf{m}' = (\mathbf{m} - |\mathbf{m}|) + (|\mathbf{m}| - \mathbf{p}')$, which shows that there exists a generated T_s-measure, namely the difference $|\mathbf{m}| - \mathbf{p}'$, which differs from $\mathbf{m}'$ by a monotone T_s-measure, namely the difference $|\mathbf{m}| - \mathbf{p}'$, which differs from $\mathbf{m}'$ by a monotone T_s-measure on $\mathscr{T}$. Thus, $\mathbf{m}'$ cannot be monotonically irreducible. Consequently the representation (40) of $\mathbf{m}$ as a sum of a monotonically irreducible and of a generated T_s-measure is unique. This also shows that the generated component of the decomposition has to be the restriction of $|\mathbf{m}|$ to $\mathscr{T}$, whose unique decomposition provided by Proposition 5.1 is given by (43) and (46). Therefore the functions g and h involved in the representation (40) are $\check{\mathbf{m}}$-a.e. uniquely determined, completing the proof of the theorem. ▮

Combining Theorem 3.5 with Theorem 5.3 we deduce the following result:

5.4 COROLLARY. *If T_s is a fundamental t-norm with $s \in]0, \infty]$, if $\mathscr{T}$ is a T_s-tribe and if $\mathbf{m}$ is a finite monotone T_s-measure on $\mathscr{T}$, then there exists a unique finite nonnegative measure $\mathbf{p}$ on $\mathscr{T}^\vee$, a $\mathbf{p}$-a.e. uniquely determined $\mathscr{T}^\vee$-Markov kernel K from $X \times \mathscr{B}_1$ to $\mathbb{R}$ and a unique monotonically irreducible T_s-measure $\mathbf{m}^*$ on $\mathscr{T}$ such that for every $A \in \mathscr{T}$*

$$\mathbf{m}(A) = \mathbf{m}^*(A) + \int_X K(x, [0, A(x)[)\, d\mathbf{p}(x).$$

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