Note

Another Proof of the Folkman–Rado–Sanders Theorem

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The following result (finite union theorem) was obtained by Rado [1], Folkman, and Sanders [3] as an extension of Schur’s theorem [4] (we put \( n = \{0, 1, 2, \ldots, n-1\} \) for a positive integer \( n \)).

**Theorem.** For all positive integers \( n \) and \( r \), there exists a positive integer \( N = R(n, r) \) such that, if \( \mathcal{P}(N) \) is \( r \)-colored (that is, \( \chi: \mathcal{P}(N) \to r \)), then there exist pairwise disjoint nonempty subsets \( A_1, A_2, \ldots, A_n \) of \( N \) such that \( \bigcup_{i \in S} A_i \) is \( \chi \) monochrome. (That is, there exists \( j \in r \) such that \( \chi\left( \bigcup_{i \in S} A_i \right) = j \) whenever \( \emptyset \neq S \subseteq \{1, 2, \ldots, n\} \).)

The usual (and shorter) proof of this theorem is based on the Van der Waerden theorem. A proof in [3] is based on Ramsey’s theorem. The purpose of this note is to present another simple proof of this result. It appears much the same as the proof of the Schur’s theorem [4] using Ramsey’s theorem. The method of the proof presented here has some further generalizations which will appear elsewhere.

**Proof.** We proceed by induction on \( n \), the case \( n = 1 \) being immediate. Assume the statement is true for \( n \). We show it is true for \( 2n \).

Let \( N = R(n, r) \) and let \( k = r^{2n-1} \). Pick, by Ramsey’s theorem, a positive integer \( c \) such that \( c \rightarrow (3N)_2^N \). (That is, whenever \( \left\lfloor c \right\rfloor^N \) is \( k \)-colored there is some \( B \subseteq \left\lfloor c \right\rfloor^N \) such that \( \left\lfloor B \right\rfloor^N \) is monochrome.) Let \( R(2n, r) = cn \).

Let \( \chi \) be an \( r \)-coloring of \( \mathcal{P}(c \times N) \). Define \( \phi: \left\lfloor c \right\rfloor^N \to \chi(S \subseteq N, S \neq \emptyset) r \) by

\[
\phi\left( \{x_0, x_1, x_1, x_2, \ldots, x_{2N-1}\} \right)_S = \chi \left( \bigcup_{i \in S} (\left\lfloor x_{2i}, x_{2i+1} \right\rfloor) \times \{i\} \right).
\]

(Here \( \{x, y\} = \{z; z\} \) is an integer and \( x \leq z < y \) and \( B_{x, y} \) expresses the fact that the elements of \( B \) are written increasing order.) Then, \( \phi \) is a \( k \) coloring of \( \left\lfloor c \right\rfloor^N \) so pick a subset \( \{y_0, y_1, y_2, \ldots, y_{3N-1}\} \) of \( c \) such that \( \mathcal{P}(\{y_0, y_1, y_2, \ldots, y_{3N-1}\}) \) is \( \phi \) monochrome.
Define $\zeta: P(N) \to r$ by $\zeta(\emptyset) = 0$, and for $\emptyset \neq S \subseteq N$, $\zeta(S) = \chi(\bigcup_{i \in S} ([y_{3i}, y_{3i+1}] \times \{i\}))$. Since $N = R(n, r)$ there are disjoint nonempty subsets $B_1, B_2, \ldots, B_n$ of $N$ and $j \in r$ such that $\zeta(\bigcup_{i \in S} B_i) = j$ whenever $\emptyset \neq S \subseteq \{1, 2, \ldots, n\}$. For $m \in \{1, 2, \ldots, n\}$, let $A_m = \bigcup_{i \in B_m} ([y_{3i}, y_{3i+1}] \times \{i\})$ and $A_{n+m} = \bigcup_{i \in B_m} ([y_{3i+1}, y_{3i+2}] \times \{i\})$. Then, as is easily verified, $\chi(\bigcup_{i \in S} A_i) = j$ whenever $\emptyset \neq S \subseteq \{1, 2, \ldots, 2n\}$. (If an explicit verification is desired: Let $\emptyset \neq S \subseteq \{1, 2, \ldots, 2n\}$, let $L = S \cap \{1, 2, \ldots, n\}$, and let $M = (S \cap \{n + 1, n + 2, \ldots, 2n\}) \setminus n$. Thus $S = L \cup \{k + n: k \in M\}$. For $i \in \bigcup \{B_k: k \in M \cap L\}$, define $w_{2i} = y_{3i}$ and $w_{2i+1} = y_{3i+1}$. For all other $i \in N$, let $w_{2i} = y_{3i}$ and $w_{2i+1} = y_{3i+2}$. For all other $i \in N$, let $z_{2i} = y_{3i}$ and $z_{2i+1} = y_{3i+1}$. Let $T = \bigcup_{k \in L \cup M} B_k$. Then $\chi(\bigcup_{k \in S} A_k) = \chi(\bigcup_{k \in L \cup M} \bigcup_{i \in B_k} ([y_{3i}, y_{3i+1}] \times \{i\}) \cup \bigcup_{k \in L \cup M} \bigcup_{i \in B_k} ([y_{3i+1}, y_{3i+2}] \times \{i\})) = \chi(\bigcup_{i \in T} [w_{2i+1}] \times \{i\}) = \phi(\{w_0, w_1, \ldots, w_{2n-1}\}) = \phi(\{z_0, z_1, \ldots, z_{2n-1}\}) = \chi(\bigcup_{i \in T} [z_{2i}, z_{2i+1}] \times \{i\}) = \chi(\bigcup_{i \in T} [y_{3i}, y_{3i+1}] \times \{i\}) = \zeta(T) = j$.

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