Approximation of the density of a solution of a nonlinear SDE – application to parabolic SPDEs

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Abstract

This paper studies the approximation of the density $p_{t,i}(y)$ of the solution of the nonlinear limit-problem of a system of weakly interacting SDE’s via a convolution of the empirical measure of the system with a family of smooth mollifiers. The method, which mainly uses coupling techniques and Malliavin calculus, is also applied to the case of nonlinear white-noise driven parabolic SPDEs. © 1997 Elsevier Science B.V.

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1. Introduction

Consider the following nonlinear stochastic differential equation:

$$dX_t = b[X_t, m_t] \, dt + \sigma[X_t, m_t] \, dB_t,$$

where $X_t$ is a $d$-dimensional random variable, $B_t$ is a $m$-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$, $m_t$ is the law of the r.v. $X_t$, and $b, \sigma$ are functions defined on $\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$.

Such an equation can be viewed as the limit problem (i.e. when $n$ tends to infinity) for the following system of weakly-interacting diffusions:

$$(E_n) \quad dX^{i,n}_t = b[X^{i,n}_t, \mu^n_t] \, dt + \sigma[X^{i,n}_t, \mu^n_t] \, dB^i_t \quad 1 \leq i \leq n,$$

where $\mu^n_t (dz) = \frac{1}{n} \sum_{i=1}^n \delta_{X^{i,n}_t}(dz)$ is the empirical measure of the system, the $B^i$'s are independent Brownian motions on $\mathbb{R}^m$. More specifically, one can prove that, under certain conditions, the sequence of empirical measures $\mu^n$ converges in law towards $m$. Such a convergence is a particular case of the now well-known results concerning mean-field interacting particle systems, for which there exists an extensive literature (cf., for instance, Méléard, to appear; Sznitman, 1991 and the references therein). This convergence is equivalent to a phenomenon called propagation of chaos: any subsystem of $k$ particles $(X^{1,n}, \ldots, X^{k,n})$, where $k$ is a fixed integer, converges in law towards a random vector $(X^1, \ldots, X^k)$, where the $X^i$'s are independent copies of the solution of $(E)$. 

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McKean (1967) showed by analytical techniques that under certain conditions on \( b \) and \( \sigma \), the (unique) solution of (E) possesses a density \( p_t(y) \) which is \( C^\infty \) w.r.t. \( y \) and can be viewed as the solution of a certain infinite-dimensional partial differential equation (namely, the Fokker–Plank equation). The aim of this paper is to investigate the relationship between \( p_t \) and the convergence of the sequence of empirical measures \( \mu^n \).

A natural approach (see, for instance, Oelschläger, 1994; Bossy, 1995) is to regularize the (singular) measures \( \mu^n \) and to investigate the behaviour of \( \mu^n \star V^\varepsilon - p \) in a suitable normed space, where, for \( \varepsilon > 0 \), \( V^\varepsilon \) is a sequence of mollifiers defined by \( V^\varepsilon(z) = (1/\varepsilon^d) V(1/\varepsilon z) \), \( V \) being a symmetric density function. (We remark that Gaussian kernels satisfy these assumptions.)

Assuming that \( V \in W^{M,2} \), Oelschläger (1994) proved the convergence in probability of \( \| \mu^n \star V^\varepsilon - p \| \) towards zero, where \( \| \cdot \| \) is a Sobolev norm of \( W^{M,2} \) type and \( V := V^{1/n'} \), with \( V \in W^{M,2} \). On the other hand, Bossy (1995) established precise estimates of \( \mu^n \star V^\varepsilon - p \) in \( L^p \) spaces for a modified version of this problem, \( V^\varepsilon \) being the density of \( \mathcal{N}(0,\varepsilon^2) \).

Our purpose is to give estimates for \( \mu^n \star V^\varepsilon - p \) in certain spaces of \( W^{M,q} \) type, for smooth \( V^\varepsilon \)s, by using probabilistic techniques instead of the analytic ones employed by Oelschläger and Bossy. Furthermore, we contemplate finding a method which can be used in other contexts (which is not the case in Bossy, 1995; Oelschläger, 1994), such as the following: consider the nonlinear parabolic SPDE:

\[
(E') \quad \frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + f[u(x,t),m(x,t)] + g[u(x,t),m(x,t)] \frac{\partial^2 W}{\partial x \partial t},
\]

\( x \in [0;1], t > 0, \)

where \( W \) is a space-time white-noise (cf. Walsh, 1986), \( m(x,t) \) is the law of \( u(x,t) \), with Neumann (resp. Dirichlet) boundary conditions:

\[
\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0 \quad \text{(resp. } u(0,t) = u(1,t) = 0 \text{)},
\]

\( u(\cdot, 0) = u_0(\cdot) \in C^1([0,1], \mathbb{R}). \)

A solution of (E') is given by the following evolution equation:

\[
\begin{align*}
  u(x,t) &= G_t(x,u_0) + \int_0^t \int_0^1 f[u(y,s),m(y,s)] G_{t-s}(x,y) \, dy \, ds \\
  &\quad + \int_0^t \int_0^1 g[u(y,s),m(y,s)] G_{t-s}(x,y) W(dy, ds),
\end{align*}
\]

where \( G_t(x,y) \) is the fundamental solution of the heat equation with Neumann (resp. Dirichlet) boundary conditions and, if \( h \in C^0([0,1], \mathbb{R}), G_t(x,h) = \int_0^1 G_t(x,y)h(y) \, dy. \)

As it is the case for (E), (E') is naturally related to a propagation of chaos problem. More precisely, if one considers the following system of weakly interacting parabolic SPDEs: for \( x \in [0;1], t > 0, 1 \leq i \leq n \)

\[
(E'_n) \quad \frac{\partial u_{i,n}}{\partial t} = \frac{\partial^2 u_{i,n}}{\partial x^2} + f[u_{i,n}, \mu^n] + g[u_{i,n}, \mu^n] \frac{\partial^2 W^i}{\partial x \partial t},
\]
where $W^i$ are independent white-noises, $\mu_{\epsilon,t}^n(dz) = \frac{1}{n} \sum_{i=0}^{n} \delta_{\epsilon^{-2}(x,\epsilon)}(dz)$, then we proved in Morien (1995) that, under suitable assumptions on $f$ and $g$, the sequence $\mu^n$ converges in law towards $m$, where $m$ is the law of the solution of (E').

We remark that the last integral of (1.1) is an Itô stochastic integral. However, due to the presence of $G$, (1.1) is not a semimartingale decomposition, and therefore Itô's formula cannot be directly applied. Furthermore, the existence of a smooth density for $u(x,t)$ cannot be obtained by analytical methods as it is the case for the density of the solution of (E) insofar as it does not appear as the solution of a deterministic PDE. Hence, we must first and foremost prove the existence of a density for $u(x,t)$ and then, in order to obtain estimates for the SPDE case as well as for the SDE case in a single effort, we make use of Taylor’s expansions and employ a coupling with a system of independent copies of the solutions of the limit problems.

Our work is then divided as follows: in Section 2 we present the SDE case in a simple, yet comprehensive, manner. Precisely, for $q \in [1, +\infty]$ and for $h : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that for almost every $\omega \in \Omega$ and every $t \in [0, T]$, $h(\omega, t, \cdot) \in C^M(\mathbb{R}^d)$ with all its derivatives in $L^q(\mathbb{R}^d)$, we define the following norm:

$$
||h||_{(M,q)} := \sup_{t \leq T} \left( \mathbb{E} \left[ \int_{\mathbb{R}^d} \sum_{|\beta| \leq M} |\partial_{\beta}^\gamma h(\omega, t, y)|^q dy \right] \right)^{1/q}.
$$

The notation $\partial_{\beta}^\gamma h(\omega, t, y)$, where $\beta = (\beta_1, \ldots, \beta_d)$, means that one differentiates w.r.t. the coordinates of $y$ ($|\beta| = \beta_1 + \cdots + \beta_d$ being the length of the multiindex $\beta$).

Then we prove the following estimate (Theorem 2.1):

$$
||\mu^n * \epsilon^\delta - p||_{(M,q)} \leq C_{M,q} \left\{ \epsilon^2 + \frac{1}{\sqrt{n}} \cdot \frac{1}{\epsilon^{M+1-d/q}} \right\}.
$$

In Section 3 we first show the existence of a density for the solution of (E') (Section 3.1) and, in Section 3.2, its approximation via mollifiers is discussed, depending on the initial conditions taken. Precisely, if $q \in [1, +\infty]$, for $h : \Omega \times [0, 1] \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that for almost every $\omega \in \Omega$ and all $(x, t) \in [0, 1] \times [0, T]$, $h(\omega, x, t, \cdot) \in C^M(\mathbb{R}^d)$ with all its derivatives in $L^q(\mathbb{R}^d)$, we define the following norms:

$$
||h||_{(M,q)} := \sup_{(x,t)} \left( \mathbb{E} \left[ \int_{\mathbb{R}^d} \sum_{m=0}^{M} \left| \frac{\partial^m h}{\partial y^m}(\omega, x, t, y) \right|^q dy \right] \right)^{1/q}
$$

and

$$
||h||_{(M,q)} := \left[ \mathbb{E} \left( \int_{0}^{T} \int_{0}^{1} \int_{\mathbb{R}^d} \sum_{m=0}^{M} \left| \frac{\partial^m h}{\partial y^m}(\omega, x, t, y) \right|^q dy \, dx \, dt \right) \right]^{1/q}.
$$

Then we prove the following result (Theorem 3.1): first, under Neumann boundary conditions the following estimate holds for all $q \in [1, +\infty[$:

$$
||\mu^n * \epsilon^\delta - p||_{(M,q)} \leq C_{M,q} \left\{ \epsilon^2 + \frac{1}{\sqrt{n}} \cdot \frac{1}{\epsilon^{M+2-1/q}} \right\}.
$$

In the case of Dirichlet boundary conditions, the estimate above does not hold, because of the lesser regularity of the Green kernel $G$ involved, as it clearly appears
in the estimates given in the appendix. However, the approximation of the density can also be considered and the following result holds: for $q \in ]1,2[$

$$
\| \mu^n * V^c - p \|_{(M,q)} \leq C_{M,q} \left\{ \frac{1}{\sqrt{n}} + \frac{1}{\varepsilon^{M+2-1/q}} \right\}.
$$

The result in this case appears to be more limited than in the SDE case, since we only obtain an estimate in a Sobolev space of $W^{M,q}$-type with $q \in ]1,2[$. However these limitations are quite natural and were noticed earlier by Bally–Gyöngy–Pardoux in Bally et al. (1994).

Finally, if one wants uniform estimates in the case of Dirichlet conditions, one is compelled to restring the supremum on $[0,T] \times [a,1-a]$, where $0 < a < 1$.

2. The SDE case

2.1. Hypotheses and statement of the results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $B$ a Brownian motion on $\mathbb{R}^d$ defined on $\Omega, \mathcal{F}$, the filtration of $B$. We consider the following SDE:

$$(E) \quad dX_t = b[X_t, m_t] \, dt + \sigma[X_t, m_t] \, dB_t,$$

where $m_t$ is the law of $X_t$, and $b, \sigma$ satisfy the following assumptions:

(H.1) $b$ and $\sigma$ are linear w.r.t. the measure, i.e, if $\nu$ is a probability on $\mathbb{R}^d$:

$$
\begin{align*}
&b[\cdot, \cdot, \nu] = \int \! b(x, z) \nu(\,dz), \\
&\sigma[\cdot, \cdot, \nu] = \int \! \sigma(x, z) \nu(\,dz);
\end{align*}
$$

(H.2) The functions $(x, y) \mapsto b(x, y)$ and $(x, y) \mapsto \sigma(x, y)$ are $C^\infty$ in their first argument; they and their partial derivatives in the first argument are Lipschitz-continuous on $(\mathbb{R}^d)^2$; furthermore, for every multiindex $\beta$ with $|\beta| \geq 1$ and for every $(i,j) \in \{1, \ldots, d\} \times \{1, \ldots, m\}$

$$
\sup_{(x, y) \in (\mathbb{R}^d)^2} \{ \| \partial_\beta^y b(x, y) \| + \| \partial_\beta^y \sigma_{i,j}(x, y) \| \} < \infty;
$$

(H.3) The function $(x, y) \mapsto \sigma(x, y)$ satisfies a uniform strong ellipticity condition: if $a := \sigma \sigma^T$, then

$$
\exists c > 0, \quad \forall (x, y) \in (\mathbb{R}^d)^2, \quad \forall \xi \in \mathbb{R}^d \quad a(x, y) \xi \cdot \xi \geq c \| \xi \|^2.
$$

We remark that (H.3) implies that, for all probability measure on $\mathbb{R}^d$

$$
\forall x, \xi \in \mathbb{R}^d, \quad a[x, v] \xi \cdot \xi \geq c \| \xi \|^2.
$$

Under these hypotheses, $X_t$ is uniquely defined and possesses a density $p_t(y)$ which is $C^\infty$ on $\mathbb{R}^d$ (cf. McKean, 1967).
We then introduce the following mollifiers: Let $M$ be a fixed integer, $q$ some real number with $q \geq 1$ and $V \in C^\infty(\mathbb{R}^d)$ satisfying

(H.4) $V$ is the density of a probability on $\mathbb{R}^d$ whose marginals are symmetric and which possesses a moment of order $2q$;

(H.5) $\forall k \leq M + 1$, $(V^{(k)}) \in L^q(\mathbb{R}^d)$.

We then set, for $\varepsilon > 0$,

$$V^\varepsilon(z) = \frac{1}{\varepsilon^d} V \left( \frac{1}{\varepsilon} z \right),$$

and we consider the difference

$$(\mu_t^\varepsilon * V^\varepsilon)(y) - p_t(y) = \frac{1}{n} \sum_{k=1}^n [V^\varepsilon(X_t^{k,n} - y) - p_t(y)].$$

The result we prove in this section is the following:

**Theorem 2.1.** Let $\varepsilon \in [0,1[$. Under (H.1)–(H.5), for all $M \geq 0, q \in [1, \infty[$, there exists a constant $C_{M,q}$ such that

$$\|\mu_t^\varepsilon * V^\varepsilon - p\|_{(M,q)} \leq C_{M,q} \left\{ \varepsilon^2 + \frac{1}{\sqrt{n}} \frac{1}{\varepsilon^{M+d+1-d/q}} \right\}. \tag{2.1}$$

An immediate, yet remarkable, consequence of this theorem is the following convergence result: if one sets $V_n := V^{1/n^p}$ then we have:

**Corollary 2.1.** If $\rho < 1/2(M + d + 1)$, under the hypotheses of Theorem 2.1, for all $q \in [1, \infty[$, we have when $n$ tends to infinity:

$$\|\mu_t^\varepsilon * V_n - p\|_{(M,q)} \to 0.$$

**2.2. Proof of Theorem 2.1**

A natural idea so as to take advantage of the close relationship between equation (E) and system $(E_n)$, arising in all the literature concerning interacting systems, is to use a coupling technique. More precisely, let $(X_t^n)$ be the stochastic processes defined by the following equations:

$$dX_t^i = b[X_t^i, m_t] \, dt + \sigma[X_t^i, m_t] \, dB_t^i,$$

where the $B_t^i$s are the Brownian motions used to define the $X_t^{i,n}$s. In other words, the $X_t^i$s are independant copies of the solution of (E). The introduction of these processes is motivated by the following result (cf., for instance, Sznitman, 1991):

**Proposition 2.1.** If $p \geq 1$, there exists a constant $C_p$ such that, for all $k \leq n$

$$\mathbb{E} \left[ \sup_{t \leq T} \|X_t^{k,n} - X_t^k\|^{2p} \right] \leq C_p \frac{1}{n^p}.$$
We then write

\[
\begin{align*}
(\mu_t^n \ast V^\varepsilon)(y) - p_t(y) \\
= \frac{1}{n} \sum_{k=1}^n \left[ V^\varepsilon(X_t^{k,n} - y) - V^\varepsilon(X_t^k - y) \right] \\
+ \frac{1}{n} \sum_{k=1}^n \left[ V^\varepsilon(X_t^k - y) - p_t(y) \right] := T_1(t, y) + T_2(t, y).
\end{align*}
\]

We shall make use of the following a priori estimates of the density \( p_t \):

**Proposition 2.2.** If \( q \in [1, \infty[ \), then for any multiindex \( \beta \),

\[
\sup_{t \in T} \left( \int_{\mathbb{R}^d} |\partial_\beta p_t(y)|^q \, dy \right) < + \infty,
\]

estimates we shall prove in the last paragraph of this section, using Malliavin calculus.

- **Evaluation of \( \|T_1\|_{(M,q)} \).** Let \( \beta \) be a multiindex such that \( |\beta| \leq M \). In the sequel, we shall denote by \((e_1, \ldots, e_d)\) the canonical basis of \( \mathbb{R}^d \). Using Taylor's expansion, we have

\[
|\partial_\beta^y T_1(t, y)| = \left| \sum_{j=1}^d \frac{1}{n} \sum_{k=1}^n (X_t^{k,n} - X_t^k)(j) \cdot \int_0^1 \partial_{\beta+e_j}^y V^\varepsilon(X_t^k - y + v(X_t^{k,n} - X_t^k)) \, dv \right|,
\]

where \((X_t^{k,n} - X_t^k)(j)\) denotes the \( j \)th coordinate of \( X_t^{k,n} - X_t^k \).

Hence, by convexity arguments, integration, and Schwarz's inequality,

\[
\int_{\mathbb{R}^d} \mathbb{E}|\partial_\beta^y T_1(t, y)|^q \, dy \leq C_q \sum_{j=1}^d \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ |(X_t^{k,n} - X_t^k)(j)|^q \right. \\
\left. \cdot \int_{\mathbb{R}^d} \left( \int_0^1 \partial_{\beta+e_j}^y V^\varepsilon(X_t^k - y + v(X_t^{k,n} - X_t^k)) \, dv \right)^q \, dy \right] \\
\leq C_q \sum_{j=1}^d \frac{1}{n} \sum_{k=1}^n \left( \mathbb{E}|(X_t^{k,n} - X_t^k)(j)|^{2q} \right)^{1/2} \\
\times \left( \mathbb{E} \left[ \left( \int_{\mathbb{R}^d} \left( \int_0^1 \partial_{\beta+e_j}^y V^\varepsilon(X_t^k - y + v(X_t^{k,n} - X_t^k)) \, dv \right)^q \, dy \right)^2 \right] \right)^{1/2}.
\]
\[
\int_{\mathbb{R}^d} \left( \int_0^1 \partial_{\beta + \varepsilon}^{\varphi} \langle X_t^k - y + \varphi(X_t^{k,n} - X_t^k) \rangle \, dv \right)^q \, dy \\
\leq \int_{\mathbb{R}^d} \left( \int_0^1 \partial_{\beta + \varepsilon}^{\varphi} \langle X_t^k - y + \varphi(X_t^{k,n} - X_t^k) \rangle \right)^q \, dv \, dy = \int_{\mathbb{R}^d} \partial_{\beta + \varepsilon}^{\varphi} \langle V(z) \rangle^q \, d\tau,
\]

using the change of variables \( z = y - X_t^k - \varphi(X_t^{k,n} - X_t^k) \). Hence, applying Proposition 2.1, we get
\[
\int_{\mathbb{R}^d} \mathbb{E} |\partial_{\beta + \varepsilon}^{\varphi} T_1(t, y)|^q \, dy \leq \frac{C_q}{n^{q/2}} \int_{\mathbb{R}^d} |\partial_{\beta + \varepsilon}^{\varphi} V(z)|^q \, d\tau,
\]
where \( C_q \) does not depend on \( t \). Lastly,
\[
\int_{\mathbb{R}^d} |\partial_{\beta + \varepsilon}^{\varphi} V(z)|^q \, dz = \int_{\mathbb{R}^d} \left( \frac{1}{[\beta]^{d+1}} \partial_{\beta + \varepsilon}^{\varphi} \left( \frac{z}{\varepsilon} \right) \right)^q \, dz = \frac{1}{[\beta]^{d+1}} \int_{\mathbb{R}^d} |\partial_{\beta + \varepsilon}^{\varphi} V(z)|^q \, dx
\]
(setting \( x = z/\varepsilon \)). Hence, by (H.5),
\[
\int_{\mathbb{R}^d} \mathbb{E} |\partial_{\beta + \varepsilon}^{\varphi} T_1(t, y)|^q \, dy \leq \frac{C_{q,M}}{n^{q/2}} \frac{1}{[\beta]^{d+1}} \int_{\mathbb{R}^d} |\partial_{\beta + \varepsilon}^{\varphi} V(z)|^q \, dz
\]
which yields, since \( \varepsilon \in ]0, 1[ \),
\[
\|T_1\|_{(M,q)} \leq \frac{C_{q,M}}{\sqrt{n}} \frac{1}{[\beta]^{d+1}}.
\]

- **Evaluation of \( \|T_2\|_{(M,q)} \).** So as to deal with \( T_2 \), we use the following well-known result:

**Lemma 2.1.** (Rosenthal, 1940). *Let \( Y_1, \ldots, Y_n \) be independent, identically distributed, \( \mathbb{R}^d \)-valued r.v. with mean zero, such that \( \mathbb{E}(\|Y_i\|^q) < \infty \), where \( q \in ]1, +\infty[ \). Then there exists a (universal) constant \( C_q \) such that
\[
\mathbb{E} \left( \left\| \frac{1}{n} \sum_{i=1}^n Y_i \right\|^q \right) \leq \frac{C_q}{n^{q/2}} \mathbb{E} \|Y_1\|^q.
\]

We set \( T_2 = T_{21} + T_{22} \) with
\[
T_{21}(t, y) = \frac{1}{n} \sum_{k=1}^n [V(X_t^k - y) - \mathbb{E}[V(X_t^k - y)]],
\]
\[
T_{22}(t, y) = \frac{1}{n} \sum_{k=1}^n [\mathbb{E}[V(X_t^k - y)] - p_t(y)] = \mathbb{E}[V(X_t^1 - y)] - p_t(y).
\]

Notice that \( T_{22} \) is deterministic. Let us fix \( \beta \) such that \( \|\beta\| \leq M \). Using Lemma 2.1, we have
\[
\mathbb{E} |\partial_{\beta + \varepsilon}^{\varphi} T_{21}(t, y)|^q \leq \frac{C_q}{n^{q/2}} \mathbb{E} |\partial_{\beta + \varepsilon}^{\varphi} V(X_t^1 - y)|^q.
\]
and therefore,
\[
\int_{\mathbb{R}^d} \mathbb{E} |\partial^\gamma_{\beta} T_{21}(t, y)|^q \, dy \leq \frac{C_q}{n^{q/2}} \int_{B(x, y)} \int_{(\mathbb{R}^d)^q} |\partial_{\beta} V(x - y)|^q \, p_t(z) \, dz \, dy
\]
\[
= \frac{C_q}{n^{q/2}} \int_{\mathbb{R}} |\partial_{\beta} V(z)|^q \, dz.
\]
A similar calculation as for \( T_1 \) yields
\[
\int_{\mathbb{R}^d} \mathbb{E} |\partial^\gamma_{\beta} T_{21}(t, y)|^q \, dy \leq \frac{C_q}{n^{q/2}} \cdot \frac{1}{e^{(m+d)q-d}},
\]
where \( C_q \) does not depend on \( t \), and therefore, as \( \epsilon \in \mathbb{N} \),
\[
\| T_{21} \|_{(M,q)} \leq \frac{C_q}{\sqrt{n}} \cdot \frac{1}{e^{(M+d)-d/q}}.
\]
As for \( T_{22} \), we have
\[
\partial^\gamma_{\beta} T_{22}(t, y) = \int_{\mathbb{R}^d} V(z) \cdot (\partial_{\beta} p_t(y + \epsilon z) - \partial_{\beta} p_t(y)) \, dz.
\]
Using a Taylor's expansion with integral remainder, we get
\[
\partial_{\beta} p_t(y + \epsilon z) - \partial_{\beta} p_t(y) = \epsilon \sum_{j=1}^d \partial_{\beta + e_j} p_t(y) z_j + \epsilon^2 \sum_{1 \leq i, j \leq d} \int_0^1 (1 - v) \partial_{\beta + e_i + e_j} p_t(y + v\epsilon z) z_i z_j \, dv.
\]
Then, since the marginals of \( V \) are symmetric, we have \( \int_{\mathbb{R}^d} z_j V(z) \, dz = 0 \) for all \( j \), which, thanks to Hölder's inequality (w.r.t. the probability measure \( V(z) \, dz \)) yields,
\[
\int_{\mathbb{R}^d} |\partial^\gamma_{\beta} T_{22}(t, y)|^q \, dy
\]
\[
\leq C_q \epsilon^{2q} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| V(z) \cdot \int_0^1 \left| \sum_{1 \leq i, j \leq d} \partial_{\beta + e_i + e_j} p_t(y + v\epsilon z) z_i z_j \right|^q \, dv \right| dz \, dy
\]
\[
= C_q \epsilon^{2q} \sum_{1 \leq i, j \leq d} \left( \int_{\mathbb{R}^d} \left| \partial_{\beta + e_i + e_j} p_t(y) \right|^q \, dy \right) \cdot \left( \int_{\mathbb{R}^d} |z_i z_j|^q V(z) \, dz \right).
\]
We then apply Proposition 2.2, which finally gives
\[
\| T_{22} \|_{(M,q)} \leq C_q \epsilon^2.
\]
and the theorem is proved. \( \Box \)

2.3. Proof of Proposition 2.2

As said previously, the main tool we use is the Malliavin calculus related to the Brownian motion \( B \). We first recall the basics of this theory (we refer to Nualart...
Let \( m, s \geq 1 \) be integers, \( A \subseteq \mathbb{R}^d \) a product of bounded closed intervals. Let \( \dot{\lambda} \) denote the Lebesgue measure on \( A \) and \( \{W(A), \dot{\lambda}(A) < \infty\} \) be a Gaussian family of \( \mathbb{R}^m \)-valued r.v.

For \( F \in \mathcal{F} \), we define the first-order derivative \( DF \) of \( F \) as the following \( L^2(A, \mathbb{R}^m) \)-valued r.v.

\[
D_i F = \sum_{i=1}^{r} \delta_i f(W(h_1), W(h_2), \ldots, W(h_r)) h_i(t, x).
\]

Its coordinates are denoted by \( D_i^1 F \). Similarly, the derivative \( D^k F \) of order \( k \) is the random vector whose components are defined by

\[
(D^k_{1, \ldots, k} F)(j_1, \ldots, j_k) := D_{k_{j_1}}^{j_1} \cdots D_{k_{j_k}}^{j_k} F.
\]

Then, for \( p \geq 1 \) and \( k \in \mathbb{N} \), \( \mathbb{D}^{k,p}_m \) denotes the closure of \( \mathcal{F} \) w.r.t. the semi-norm

\[
\|F\|_{k,p} = \left[ \left( \mathbb{E}|F|^p \right) + \sum_{i=1}^{k} \left( \mathbb{E}\|D^i F\|_{L^2(A)}^p \right) \right]^{1/p},
\]

and \( \mathbb{D}^{\infty}_m = \bigcap_{p \geq 1} \bigcap_{k \in \mathbb{N}} \mathbb{D}^{k,p}_m \). To simplify the notations, we shall often write \( \| \cdot \|_p \) instead of \( \| \cdot \|_{0,p} \).

For \( F \in \mathcal{F} \), one also defines the Ornstein–Uhlenbeck operator \( L \) by

\[
L F = \sum_{i=1}^{r} \delta_i f(B(h_1), B(h_2), \ldots, B(h_r)) B(h_i)
\]

\[
- \sum_{i,j=1}^{r} \delta_{(i,j)} f(B(h_1), B(h_2), \ldots, B(h_r)) (h_i, h_j),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( L^2([0, T]; \mathbb{R}^m) \). The domain of \( L \) includes \( \mathbb{D}^{\infty}_m \).

For \( F = (F^{(1)}, \ldots, F^{(d)}) \in (\mathbb{D}^{\infty})^d \), we denote by \( \gamma_F \) the Malliavin covariance matrix associated to \( F \), i.e. the \( d \times d \) matrix defined by

\[
(\gamma_F)_{i,j} := \langle DF^{(i)}, DF^{(j)} \rangle = \sum_{l=1}^{m} \int_0^T D_l^{(i)} F^{(i)} D_l^{(j)} F^{(j)} dt.
\]

The determinant of \( \gamma_F \) will be denoted by \( \gamma_F \). A random vector \( F \) is said to satisfy the nondegeneracy assumption if the matrix \( \gamma_F \) is a.s. invertible and satisfies either of the following conditions:

\[
\frac{1}{\gamma_F} \in \bigcap_{p \geq 1} L^p(\Omega) \quad \text{or} \quad (\gamma_F)^{-1} \in \bigcap_{p \geq 1} L^p(\Omega).
\]
Now let \( F \in (D_m^\infty)^d \) satisfy the nondegeneracy assumption (2.2). Let \( g \) be a smooth function on \( \mathbb{R}^d \) with polynomial growth and let \( G \) in \( D_m^\infty \). We define the following r.v. \( H_\beta \), where \( \beta = (\beta_1, \ldots, \beta_d) \), recursively w.r.t. \(|\beta|\):

\[
H_\beta(F; G) = - \sum_{j=1}^d \{ G(D(j)F)_{i,j}^{-1} D(DF(j)) + (\gamma F)_{i,j}^{-1} (DG, DF(j)) + (\gamma F)^{-1} \cdot G \cdot LF(j) \},
\]

and

\[
H_{(\beta_1, \ldots, \beta_{i-1}, 1, \beta_{i+1}, \ldots, \beta_d)}(F; G) = H_\beta(F; H_{(\beta_1, \ldots, \beta_{i-1}, 1, \beta_{i+1}, \ldots, \beta_d)}(F; G)).
\]

We remark that the following identity holds:

\[
H_{\alpha^* + \beta}(F; G) = H_{\alpha^*}(F; H_\beta(F; G)). \tag{2.3}
\]

For such r.v. \( F, G \) and such smooth functions \( g \), the following integration-by-parts formula, which appears in Ikeda and Watababe (1989, p. 377) holds:

**Proposition 2.3.** For any multiindex \( \beta \)

\[
\mathbb{E}[(\partial_\beta g)(F)G] = \mathbb{E}[g(F)H_\beta(F; G)]. \tag{2.4}
\]

The following estimate is proved in Bally and Talay (1996):

**Proposition 2.4.** For any \( p \in ]1, \infty[ \) and any multiindex \( \beta \), there exists a constant \( C(p, \beta) > 0 \), and integers \( k(p, \beta), m(p, \beta), m'(p, \beta), N(p, \beta), N'(p, \beta) \) such that, for any \( F, G \) as in Proposition 2.3, one has

\[
\mathbb{E}[|H_\beta(F; G)|^p]^{1/p} \leq C(p, \beta) \|G\|_{N(p, \beta)} \|F\|_{N'(p, \beta)}, \tag{2.5}
\]

Now let \( F \) be a random vector satisfying the nondegeneracy assumption (2.2). It is then well-known (see, for instance, Nualart, 1995) that \( F \) possesses a density \( p_F \) which is indefinitely differentiable. Proposition 2.3 can then be used to obtain a representation formula for the density \( p_F \) and also for its partial derivatives, representation which will be the cornerstone of our proof of Proposition 2.2. Precisely, let us set \( \varepsilon = (1, 1, \ldots, 1) \in \mathbb{R}^d \) and, for \( F \) satisfying (2.2), \( H_\beta(F) := H_\beta(F; 1) \). Then the following result holds:

**Proposition 2.5.** Let \( F = (F^{(1)}, \ldots, F^{(d)}) \in (\mathbb{R}_m^\infty)^d \) be a random vector satisfying (2.2). Then its density \( p_F \) is given by

\[
p_F(x_1, \ldots, x_d) = \mathbb{E}[1_{\{F^{(1)} > x_1, \ldots, F^{(d)} > x_d\}} \cdot H_\varepsilon(F)]. \tag{2.6}
\]

Furthermore, its partial derivative \( \partial_\beta p_F \) is given by

\[
\partial_\beta p_F(x_1, \ldots, x_d) = (-1)^{|\beta|} \mathbb{E}[1_{\{F^{(1)} > x_1, \ldots, F^{(d)} > x_d\}} \cdot H_{\varepsilon + \beta}(F)]. \tag{2.7}
\]
The proof of (2.6) is a mere adaptation of that of Proposition 2.1.1, p. 78 in Nualart (1995) and is therefore omitted. As for (2.7) we give a short proof for the sake of completeness, in the case when $\beta = \epsilon_k$. We set $\phi_{1,y}(x) = \mathbf{1}_{\{y_i > x_1, \ldots, y_d > x_d\}}$ and

$$
\phi_{2,y}(x) = \int_{-\infty}^{y_1} \ldots \int_{-\infty}^{y_d} \phi_{1,z}(x) \, dz_1 \ldots dz_d,
$$

$$
\phi_{3,y}(x) = \int_{-\infty}^{y_1} \ldots \int_{-\infty}^{y_d} \phi_{2,z}(x) \, dz_1 \ldots dz_d.
$$

We then have, for $i = 2, 3$ $\partial \phi_{i,y}(x) = \phi_{i-1,y}(x)$. An easy computation gives

$$
\phi_{2,y}(x) = \prod_{j=1}^{d} (y_j - x_j) \cdot \phi_{1,y}(x), \quad \phi_{3,y}(x) = \frac{1}{2^d} \prod_{j=1}^{d} (y_j - x_j)^2 \cdot \phi_{1,y}(x),
$$

which proves that for any $y$ the function $\phi_{3,y}$ is $C^1$ on $\mathbb{R}^d$. Moreover,

$$
\partial_y \phi_{3,y}(x) = - \partial_y \phi_{2,y}(x) = (\phi_{3,y}(x))(y).
$$

Using (2.4) on (2.6) with $\alpha = \epsilon, \beta = 2\epsilon$, we have

$$
\hat{c} \cdot p_{\mathcal{F}}(x) = \mathbb{E}[\phi_{3,F}(x) H_{\mathcal{F}}(\mathcal{F})].
$$

Then Lebesgue’s theorem of derivation under the integral, (2.8) and (2.4) yield

$$
\hat{c} \cdot p_{\mathcal{F}}(x) = - \mathbb{E}[\partial_y \phi_{3,F}(x) \mathcal{F} H_{\mathcal{F}}(\mathcal{F})] = - \mathbb{E}[\phi_{3,F}(x) \cdot H_{\mathcal{F}+\epsilon_\alpha}(\mathcal{F})]
$$

$$
= - \mathbb{E}[\partial_y \phi_{3,F}(x) \mathcal{F} H_{\mathcal{F}+\epsilon_\alpha}(\mathcal{F})]
$$

which gives the required formula. (2.7) then follows by induction on $|\beta|$. \qed

Now, if $F \in \mathbb{D}_m^\infty$ satisfies (2.2), for any multiindex $\beta$, we have, using (2.7) and the techniques employed in Lemma 3.2 of Bally et al. (1994)

$$
|\hat{c}_\beta p_{\mathcal{F}}(x)| \leq \|H_{\mathcal{F}+\beta}(\mathcal{F})\|_2 \cdot (\mathbb{P}[|F^{(1)}| \geq |x_1|, \ldots, |F^{(d)}| \geq |x_d|])^{1/2}
$$

and, for all $q \geq 1$ and all $r, s$ such that $r q < 2 < s q$:

$$
\int_{\mathbb{R}^d} |\hat{c}_\beta p_{\mathcal{F}}(x)|^q \, dx \leq K_{r,s} \|H_{\mathcal{F}+\beta}(\mathcal{F})\|_2^q \left\{ \|F_1 F_2 \cdots F_d\|_s^{q/2} + \|F_1 F_2 \cdots F_d\|_s^{q/2} \right\}. \tag{2.10}
$$

We then prove Proposition 2.2. It is well-known that $\sup_{t} \|X_t\|_q < \infty$ a.s. for all $q > 1$. Therefore, so as to obtain that $\sup_{t \leq T} \left( \int_{\mathbb{R}^d} |\hat{c}_\beta p_{t}(y)|^q \, dy \right)^{1/q} < \infty$, it is sufficient to prove that $X_t$ satisfies the nondegeneracy assumption (2.2) and that $\sup_{t \leq T} \|H_{\beta+\epsilon_t}(X_t)\|_2 < \infty$. The latter part is obtained via estimate (2.5). As for the former part, it is easy to see that one has only to prove the following:

(1) $X_t \in \mathbb{D}_m^\infty$, $\forall t$, $p$, $\sup_{t \leq T} \|X_t\|_{k,p} < \infty$;

(2) $\forall t$, $p$, $\sup_{t \leq T} \left( \mathbb{E} \left[ \frac{1}{\partial_t \hat{c}_t} \right]^p \right) < \infty$,

where $\hat{c}_t$ denotes the determinant of the Malliavin covariance matrix of $X_t$. 

(1) is easily derived from the following adaptation of Theorems 2.2.1, p. 102 and 2.2.2, p. 105 in Nualart (1995): in fact, we have:

**Lemma 2.2.**

\[
\sup_{0 \leq r \leq t} \mathbb{E} \left[ \sup_{r \leq s \leq t} \left| D_{r} X_{s} \right|^{p} \right] < \infty,
\]

and the same sort of result for higher-order derivatives. Furthermore, \( D_{r} X_{t} \) satisfies for \( r < t \) the following integral equation:

\[
D_{r}^{(j)} X_{t} = \sigma_{r, j}[X_{r}, m_{r}] + \sum_{i=1}^{m} \sum_{k=1}^{d} \int_{r}^{t} \partial_{x_{i}}^{r} \sigma_{r, j}[X_{s}, m_{s}] D_{s}^{(j)} X_{s}^{(k)} \, dB_{s}^{(i)} + \sum_{k=1}^{d} \int_{r}^{t} \partial_{x_{k}}^{r} b[X_{s}, m_{s}] D_{s}^{(j)} X_{s}^{(k)} \, ds
\]

(2.11) if \( r < t \) (and is zero if not).

Indeed, in so far as \( b \) an \( \sigma \) satisfy (H.1) and (H.2) for every probability measure \( \nu \) on \( \mathbb{R} \), the functions \( x \mapsto b[x, \nu] \) and \( x \mapsto \sigma[x, \nu] \) have derivatives of all orders, and furthermore, \( \partial_{p}^{r} (b[x, \nu]) = (\partial_{p} b)[x, \nu] \) (and a similar formula for \( \sigma \)). Hence, the nonlinearity has no real effect on the differentiability of \( X_{t} \) in the Malliavin sense, and Eq. (2.11) can be obtained as in Theorem 2.2.1 in Nualart (1995).

As for (2), thanks to Lemma 2.3.1 of Nualart (1995), we know that it is sufficient to show that there exists a constant \( \lambda > 0 \) such that, for all \( p \geq 2 \):

\[
\sup_{t \leq T, u \in \mathbb{R}^{d}, \|u\| = 1} \mathbb{P}(\|Y_{t} u, u\| \leq \varepsilon) \leq C_{p} e^{\lambda p}.
\]

The proof of this last estimate, very similar (however, simpler) to that used in Proposition 3.2 below, is omitted.

### 3. The SPDE case

In this section, we consider the equation (E') given in the introduction. So as to mimic the SDE case, we assume that the functions \( f \) and \( g \) satisfy the following properties:

(H'.1) \( f \) are \( g \) are linear w.r.t. the measure, i.e.

\[
h[x, v] = \int_{\mathbb{R}} h(x, z) v(\,dz) \quad h \in \{f, g\}
\]

(H'.2) the functions \( (x, y) \mapsto f(x, y) \) and \( (x, y) \mapsto g(x, y) \) defining \( f[\cdot, \cdot] \) and \( g[\cdot, \cdot] \) are \( C^{\infty} \) in their first argument; they, and all their partial derivatives in their first argument,
are Lipschitz-continuous on $\mathbb{R}^2$; furthermore, $f$ and $g$ are bounded on $\mathbb{R}^2$ and for all $m \geq 1$

$$\sup_{(x,y) \in \mathbb{R}^2} \left\{ \left| \frac{\partial^m f}{\partial x^m}(x,y) \right| + \left| \frac{\partial^m g}{\partial x^m}(x,y) \right| \right\} < \infty,$$

(H'.3) the function $(x,y) \mapsto g(x,y)$ satisfies

$$\exists c > 0, \forall (x,y) \in \mathbb{R}^2, \ g^2(x,y) \geq c.$$

If one assumes the existence of a regular density $p_{t,x}$ for $u(x,t)$, then a natural question is, can one obtain the same kind of estimates for the difference

$$(\mu^n * V^\varepsilon)(y) - p_{t,x}(y) = \frac{1}{n} \sum_{k=1}^{n} \left[ V^\varepsilon(u^{k,n}(x,t) - y) - p_{t,x}(y) \right]$$

as in the SDE case? Of course, the prerequisite is the existence of $p_{t,x}$, which is not clear. The result we prove in this section is the following:

**Theorem 3.1.** (1) **Existence.** Under (H'.1)–(H'.3), for either Neumann or Dirichlet boundary conditions, for all $t \in [0,T]$, all $d \in \mathbb{N}$ and all $0 < x_1 \leq \cdots \leq x_d < 1$, the law of the random vector $(u(x_1,t), \ldots, u(x_d,t))$ has a smooth density w.r.t. the Lebesgue measure on $\mathbb{R}^d$.

(2) **Approximation: the Neumann case.** Let $\varepsilon \in [0,1]$. Under (H'.1)–(H.5) and for Neumann boundary conditions, for all $M \geq 0$, $q \in [1, +\infty]$, there exists a real number $C_{M,q}$ such that

$$\| \mu^n * V^\varepsilon - p \|_{(M,q)} \leq C_{M,q} \left\{ \varepsilon^2 + \frac{1}{\sqrt{n}} \frac{1}{\varepsilon^{M+2-1/q}} \right\}.$$

(3) **Approximation: the Dirichlet case.** Let $\varepsilon \in [0,1]$. Under (H'.1)–(H.5) and for Dirichlet boundary conditions, for all $M \geq 0$, $q \in [1,2]$, there exists a real number $C_{M,q}$ such that

$$\| \mu^n * V^\varepsilon - p \|_{(M,q)} \leq C_{M,q} \left\{ \varepsilon^2 + \frac{1}{\sqrt{n}} \frac{1}{\varepsilon^{M+2-1/q}} \right\}.$$

(ther norms $\| \cdot \|_{(M,q)}$ and $\| \cdot \|_{(M,q)}$ being those defined in introduction).

We first focus on part (1).

### 3.1. Existence of a smooth density

In this section, we prove part (1) of Theorem 3.1 using the Malliavin calculus associated with the white-noise $W$. From now on, $A_t$ will denote the product $[0,t] \times [0,1]$.

The method we use is based on Corollary 2.1.2, p. 91 in Nualart (1995): we first prove that the solution $u(x,t)$ of (E') is indefinitely differentiable w.r.t. the Malliavin calculus associated with the space–time white-noise $W$, and second, that the covariance matrix of the random vector $F = (u(x_1,t), \ldots, u(x_d,t))$ is in $\bigcap_{1 \leq p < \infty} L^p(\Omega)$. 
3.1.1. Differentiability of \( u(x, t) \)

First let us introduce some useful notations: let \( \varphi \) be a smooth function on \( \mathbb{R} \), \( F \in \mathbb{D}^\infty \) and \( M \geq 1 \). Then if \( \alpha = (\alpha_1, \ldots, \alpha_M) \in \mathbb{A}_T^M \), \( \alpha_k = (r_k, z_k) \), there exists for almost every \( \alpha \) an index \( i_0 \) such that \( \forall i \neq i_0, \ r_i > r_i \). We set \( \mathcal{Z} = (\mathcal{F}, \mathcal{E}) = \alpha_{i_0} \) and \( \mathcal{Z} = (\alpha_1, \ldots, \alpha_{i_0 - 1}, \alpha_{i_0 + 1}, \ldots, \alpha_M) \in \mathbb{A}_T^{M-1} \). For \( \alpha \in \mathbb{A}_T^M \), we also set

\[
\Delta_\alpha(\varphi)(F) = \sum_{m=2}^M \sum_{p_m} \varphi^{(m)}(F) \prod_{i=1}^m D_{p_i}^i F, \quad \Gamma_\alpha(\varphi)(F) = \sum_{m=1}^M \sum_{p_m} \varphi^{(m)}(F) \prod_{i=1}^m D_{p_i}^i F,
\]

where \( \lambda = |p_i| \) and \( \mathbb{P}_m \) denote the set of partitions of size \( m \), \( (p_1, \ldots, p_m) \), of \( \alpha \).

It is easy to prove recursively that, for all \( M \geq 1 \), if \( F \) is \( M \) times differentiable in the Malliavin sense, then, for \( \alpha \in \mathbb{A}_T^M \), \( D_2^M(\varphi(F)) = \Gamma_\alpha(\varphi)(F) \).

We prove the following result:

**Proposition 3.1.** For all \( (x, t) \in [0, 1] \times [0, T] \), \( u(x, t) \in \mathbb{D}^\infty \), with, for all \( q \in ]1, \infty[ \)

\[
\sup_{(x, t)} \mathbb{E}[\|D^M u(x, t)\|^q] \leq C_{M, q} < \infty.
\]

Moreover, its first derivative satisfies the following evolution equation:

\[
D_{r, z} u(x, t) = G_{t-r}(x, z)[u(z, r), m(z, r)]
\]

\[
+ \int_0^t \int_0^1 G_{t-s}(x, y) f'[u(y, s), m(y, s)] D_{r, z} u(y, s) W(dy, ds)
\]

\[
+ \int_s^t \int_0^1 G_{t-s}(x, y) f'[u(y, s), m(y, s)] D_{r, z} u(y, s) dy ds \quad (3.1)
\]

(and \( D_{r, z} u(x, t) = 0 \) if \( r > t \)).

**Proof.** So as to prove Proposition 3.1, we use the following Picard approximation:

\[ u_0(x, t) = \int_0^1 G_t(x, y) u_0(y) dy, \]

and

\[
u_n(x, t) = u_0(x, t) + \int_0^t \int_0^1 f[u_n(y, s), m(y, s)] G_{t-s}(x, y) dy ds
\]

\[
+ \int_0^t \int_0^1 g[u_n(y, s), m(y, s)] G_{t-s}(x, y) W(dy, ds) \quad (3.2)
\]

for which one easily has, for all \( p \in ]1, +\infty[ \), using the techniques of Walsh (1986):

\[
u_n(x, t) \to u(x, t) \text{ in } L^p \text{ uniformly in } (x, t),
\]

\[
\sup_n \sup_{(x, t)} \mathbb{E}[|u_n(x, t)|^p] \leq C_p < \infty. \quad (3.3)
\]
Now, thanks to (3.2), one shows without difficulty that \( u_n(x, t) \in \mathbb{D}^\infty \), by using the standard formulae of derivation in the Malliavin sense. Furthermore, one has

\[
D_x^{M}u_{n+1}(x, t) = G_{t-r}(x, z) \cdot I_{z}(g)(\tilde{z}, \tilde{r}) + \int_{\tilde{r}}^{t} \int_{0}^{1} G_{t-s}(x, y)A_{x}(y, s)W(dy, ds) \\
+ \int_{\tilde{r}}^{t} \int_{0}^{1} G_{t-s}(x, y)f_{1}[u_n(y, s), m(y, s)]D_x^{M}u_n(y, s)dy ds \\
+ \int_{\tilde{r}}^{t} \int_{0}^{1} G_{t-s}(x, y)g_{1}[u_n(y, s), m(y, s)]D_x^{M}u_n(y, s)dy ds
\]

Equation (3.4)

(and \( D_x^{M}u_{n+1}(x, t) = 0 \) if \( t < \tilde{r} \)). We introduce the following notations: for \( h = f, g \):

\[
\Gamma_{x}^{(n)}(h)(x, t) = \Gamma_{x}(h[., m(x, t)])(u_n(x, t)),
\]

\[
A_{x}^{(n)}(h)(x, t) = A_{x}(h[., m(x, t)])(u_n(x, t)).
\]

To prove Proposition 3.1, it only remains to establish the following lemma:

**Lemma 3.1.** For all \( M \geq 1 \), \( p \in ]3, \infty[ \), there exists a constant \( C_{p,M} \) such that

\[
\sup_n \sup_{(x, t)} \mathbb{E}\|D^M u_n(x, t)\|_{L^p(A_{x}^{(M)})}^{2p} \leq C_{p,M} < \infty.
\]

Indeed, if (3.5) holds, by virtue of Lemma 1.5.4 of Nualart (1995), since for all \( p > 1 \) and all \( q \geq 1 \), \( \sup_n \sup_{(x, t)} \|u_n(x, t)\|_{p,q} < \infty \), we have \( u(x, t) \in \mathbb{D}^\infty \) and Eq. (3.1) is obtained simply by differentiating Eq. (1.1).

**Proof of Lemma 3.1.** We proceed recursively on \( M \), using the techniques developed in Bally and Pardoux. The case of the first derivative is a simple adaptation of the proof of Theorem 2.4.3 p. 137 in Nualart (1995). For the general case, we assume that for every integer \( m < M \) and every \( p \in ]3, \infty[ \), we have \( \sup_n \sup_{(x, t)} \mathbb{E}\|D^m u_n(x, t)\|_{L^p(A_{x}^{(M)})}^{2p} < \infty \). We shall use Burkholder-Davis-Gundy inequalities for Hilbert-space valued martingales (cf. Méthivet (1982), E.2, p. 212) in the following form: if \((Q_{s,y})_{(s,y)\in A_{r}}\) is an adapted process in \(L^2(A_{r})\), then

\[
\mathbb{E} \left| \int_{A_{r}} \left( \int_{0}^{1} Q_{r,s}(r, z) W(dy, dz) \right)^{2} \right|^{p} \leq C_{p} \mathbb{E} \left| \int_{A_{r}} \left( \int_{0}^{1} Q_{r,s}^{2}(r, z) \right) dv du \right|^{p}.
\]

Thus,

\[
\mathbb{E}\|D^M u_{n+1}(x, t)\|_{L^p(A_{x}^{(M)})}^{2p} \leq C \left\{ \mathbb{E} \int_{A_{r}} G_{t-r}^{2}(x, \tilde{z}) \cdot \left( I_{\tilde{z}}(g)(\tilde{z}, \tilde{r}) \right)^{2} dx \right\}^{p}
\]
As the partial derivatives of $f$ and $g$ are bounded, we easily have

$$A_1(x, t) \leq C \sum_{m=1}^{M-1} \sum_{\mathcal{P}_m} \mathbb{E} \left[ \int_{A^{(m)}_t} \left( \int_{A^{(m)}_t} \left[ \int_{\mathbb{R}} D_{p_i}^2 u_n(x, y) \right] \right)^2 \right] \, dz, \quad (3.7)$$

where $\mathcal{P}_m$ denotes the set of partitions $p_1, \ldots, p_m$ of size $m$ of $\{1, 2, \ldots, M - 1\}$ and $\lambda_i = |p_i|$. Then, using Hölder’s and Schwarz’s inequality, as well as Lemma A.1(c) of the appendix (see Bally and Pardoux for details), we get

$$A_1(x, t) \leq C \sum_{m=1}^{M-1} \sum_{\mathcal{P}_m} \int_{A^{(m)}_t} \left( \mathbb{E} \left[ \int_{A^{(m)}_t} \left[ \int_{p_i} D_{p_i}^2 u_n(x, y) \right] \right] \right)^{2m-2} \, dz, \quad (3.7)$$

which yields the uniform boundedness of $A_1$. $A_2$ is similarly dealt with. As for $A_3$, the same methods gives

$$A_3(x, t) \leq C \int_0^t \int_0^1 G_{t-s}(x, y) \mathbb{E} \|D^M u_n(y, s)\|_{L^2(A^{(m)}_t)} \, dy \, ds.$$
Proof. Thanks to Lemma 2.2. in Nualart (1995) we only have to prove that there exists $\beta > \frac{1}{2}$ such that for all $p > 3$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$

$$\sup_{\|\xi\|=1} \mathbb{P}(\langle \sigma \xi, \xi \rangle \leq \varepsilon^p) \leq \varepsilon^p. \quad (3.9)$$

Let $\xi \in \mathbb{R}^d$ of norm 1. $\sigma$ is the matrix $\sigma_{ij} = \int_0^t \int_0^t D_{r,z} u(x_i, t)D_{r,z} u(y_j, t) \, dz \, dr$.

We choose $0 < \varepsilon < \frac{1}{4} \min_{i \neq j} |x_i - x_j|^2$. Then

$$\langle \sigma \xi, \xi \rangle = \int_0^t \int_0^t \left( \sum_{i=1}^d D_{r,z} u(x_i, t) \xi_i \right)^2 \, dz \, dr$$

$$\geq \sum_{j=1}^d \int_{t-\varepsilon}^t \int_{x_j - \sqrt{\varepsilon}}^{x_j + \sqrt{\varepsilon}} \left( \sum_{i=1}^d D_{r,z} u(x_i, t) \xi_i \right)^2 \, dz \, dr \geq \frac{1}{2} I_2(\xi)$$

with

$$I_1(\xi) = \sum_{j=1}^d \int_{t-\varepsilon}^t \int_{x_j - \sqrt{\varepsilon}}^{x_j + \sqrt{\varepsilon}} \left( \sum_{i \neq j}^d D_{r,z} u(x_i, t) \xi_i \right)^2 \, dz \, dr$$

$$I_2(\xi) = \sum_{j=1}^d \int_{t-\varepsilon}^t \int_{x_j - \sqrt{\varepsilon}}^{x_j + \sqrt{\varepsilon}} (D_{r,z} u(x_j, t))^2 \xi_j^2 \, dz \, dr.$$

We set

$$H_{r,z}(x,y) = \int_r^t \int_0^r G_{t-s}(x, y) g_1'[u(y, s), m(y, s)]D_{r,z} u(y, s) W(dy, ds)$$

$$+ \int_r^t \int_0^r G_{t-s}(x, y) f_1'[u(y, s), m(y, s)]D_{r,z} u(y, s) \, dy \, ds.$$

Then $I_2(\xi) \geq \frac{1}{2} I_4(\xi) - I_3(\xi)$, with

$$I_3(\xi) = \sum_{j=1}^d \int_{t-\varepsilon}^t \int_{x_j - \sqrt{\varepsilon}}^{x_j + \sqrt{\varepsilon}} (H_{r,z}(x_j, t))^2 \xi_j^2 \, dz \, dr,$$

$$I_4(\xi) = \sum_{j=1}^d \int_{t-\varepsilon}^t \int_{x_j - \sqrt{\varepsilon}}^{x_j + \sqrt{\varepsilon}} G_{t-s}^2(x_j, z) g^2[u(y, s), m(y, s)] \xi_j^2 \, dz \, dr.$$

Therefore, thanks to (H'.3) and Lemma A.2 (or Lemma A.4 if Dirichlet conditions are taken) of the appendix, we get $I_4(\xi) \geq C \sqrt{\varepsilon}$. Hence,

$$\langle \sigma \xi, \xi \rangle \geq C \sqrt{\varepsilon} - \sup_{\|\xi\|=1} \left( I_1(\xi) + \frac{1}{2} I_3(\xi) \right)$$

$$ \quad (3.10)$$
which yields

\[ \mathbb{P}(\sigma_{\xi, \xi} < \varepsilon) \leq \mathbb{P}\left( \sup_{\|\xi\| = 1} \left( I_1(\xi) + \frac{1}{2} I_3(\xi) \right) \geq C \sqrt{\varepsilon} \right). \]

Since we chose \( \beta > \frac{1}{2} \), there exists \( \varepsilon_0 > 0 \) such that, for all \( \varepsilon \leq \varepsilon_0 \), we have \( C \sqrt{\varepsilon} \geq \varepsilon^\beta \).

Therefore, using Tchebychev's inequality,

\[ \mathbb{P}(\sigma_{\xi, \xi} < \varepsilon) \leq \frac{Cq}{\varepsilon^q} \mathbb{E}\left( \sup_{\|\xi\| = 1} (|I_1(\xi)|^q + |I_3(\xi)|^q) \right). \]

We then check that \( \mathbb{E}(\sup_{\|\xi\| = 1} |I_k(\xi)|^q) \leq C_q \varepsilon^q \), for \( k = 1, 3 \) and \( q > \frac{3}{2} \). Indeed, (bounding \( \xi_2^2 \) by 1),

\[ I_1(\xi) \leq C \left( \sum_{j=1}^{d} \sum_{i \neq j} \left( \int_{t-\varepsilon}^{t} \int_{x_j - \sqrt{\varepsilon}}^{x_j + \sqrt{\varepsilon}} g^2[u(z,r), m(z,r)] G_{t-s}^2(x_i, z) \, dz \, dr \right) \right) = \sum_{j=1}^{d} \sum_{i \neq j} (a_{ij} + b_{ij}), \]

and thanks to Lemma A.5 of the appendix, setting \( \ell = \frac{1}{2} \min_{i \neq j} |x_i - x_j| \), we have \( a_{ij} \leq C \varepsilon^{-l/2} \). On the other hand, using (3.6), we easily get:

\[ \mathbb{E}[b_{ij}(\xi)]^q \leq C_q \left( \int_{t-\varepsilon}^{t} \int_{0}^{1} G_{t-s}^2(x_i, y) \, dy \, ds \right)^{q-1} \]

\[ \times \int_{t-\varepsilon}^{t} \int_{0}^{1} G_{t-s}^2(x_i, y) \mathbb{E} \left[ \int_{s}^{t} \int_{0}^{1} (D_{r,z} u(y,s))^2 \, dz \, dr \right] \, dy \, ds. \]

We then use the following lemma:

**Lemma 3.2.** For all \( q > 1 \), there exists \( C_q \) such that for all \( t > 0, s > 0, y \in [0,1] \)

\[ \mathbb{E} \left[ \int_{t-\varepsilon}^{t} \int_{0}^{1} (D_{r,z} u(y,s))^2 \, dz \, dr \right] \leq C_q \varepsilon^{q/2}. \]

The proof of the above is a mere adaptation of Lemma 4.3.2 in Morien (1995). Therefore,

\[ \mathbb{E}[b_{i,j}]^q \leq C_q \varepsilon^{q/2} \left( \int_{t-\varepsilon}^{t} \int_{0}^{1} G_{t-s}^2(x_i, y) \, dy \, ds \right)^{q} \leq C_q \varepsilon^q, \]
which gives the correct bound for $I_1$; as for $I_3$, we notice that $I_3 \leq C \sum_{j=1}^d b_{ij}$, which finally gives

$$\|\hat{\sigma}(x, z) \leq \sigma^\beta \| < \sum_{j=1}^d b_{ij},$$

Hence, choosing $\beta \in [1/2, 1]$ and $p > 3$, we obtain (3.9), and Proposition 3.2 is proved.

### 3.2. Approximation of the density

We finally prove the second part of Theorem 3.1. We shall use the notations of Section 2 (in the present section, $A_T$ denotes the product $[0, 1] \times [0, T]$), i.e.

$$T_1(x, t, y) := \frac{1}{n} \sum_{k=1}^n [V^x(u^{k,n}(x, t) - y) - V^x(u^k(x, t) - y)],$$

$$T_2(x, t, y) := \frac{1}{n} \sum_{k=1}^n [V^x(u^k(x, t) - y) - p_t(y)],$$

where, for $k \leq n$, $u^k(x, t)$ is the copy of the solution of (E') defined with white-noise $W^k$.

#### 3.2.1. The Neumann case

- **Evaluation of $\|T_1\|_{(M,q)}$.** The calculations used to evaluate

$$\sup_{(x,t)} \int \mathbb{E} \left| \frac{\partial^m T_1}{\partial x^m} (x, t, y) \right|^q \, dy \, dx \, dt$$

are rigorously identical to those employed in the corresponding part of Section 2, one only has to use Lemma 5.2.1 of Morien (1996) instead of Proposition 2.1.

- **Evaluation of $\|T_2\|_{(M,q)}$.** To evaluate $T_2$, the calculations are the same as in Section 2. As for $T_{22}$, we use the following estimates:

**Proposition 3.3.** If $q \in ]1, +\infty[$, then for all $m \geq 0$:

$$\sup_{(x,t) \in [0,1] \times [0,T]} \int_{\mathbb{R}} \left| p^{(m)}_{\nu,x}(y) \right|^q \, dy \, dx \, dt < \infty.$$

The method to prove Proposition 3.3 is the same as that used for Proposition 2.2 indeed, it suffices to show the following properties:

1. $(1')$ $u(x, t) \in \mathcal{D}^\infty$, $\forall k, p$, $\sup_{(x,t) \in [0,1] \times [0,T]} \|u(x,t)\|_{k,p} < \infty.$

2. $(2')$ $\forall p > 1$, $\sup_{(x,t) \in [0,1] \times [0,T]} \left( \mathbb{E} \left( \frac{1}{\|Du(x,t)\|_{\mathcal{L}^1(T)}} \right) \right)^p < \infty,$

and these properties are given by Propositions 3.1 and 3.2.
Remark. The previous method can also be applied in the case of Dirichlet boundary conditions, if one considers the supremum on \([a, 1 - a] \times [0, T]\) for a fixed \(\alpha \in ]0, 1[\).

3.2.2. The Dirichlet case

- **Evaluation of** \(\|T_1\|_{(M, p)}\). The calculations are conducted exactly as for \(\|T_1\|_{(M, q)}\).
- **Evaluation of** \(\|T_2\|_{(M, p)}\). To evaluate \(T_{21}\), the calculations are the same as in Section 2. The difference with the Neumann case resides in the treatment of \(T_{22}\), for which one cannot use Proposition 3.3. We use instead the following result.

**Proposition 3.4.** If \(q \in [1, 2[\), then for all \(m \geq 0\)

\[
\int_0^T \int_0^1 \int_\mathbb{R} |P_{t,x}^{(m)}(y)|^q \, dy \, dx \, dt < \infty.
\]

**Proof.** The proof of Proposition 3.4 follows the same lines as that of Proposition 2.2, the differences occurring when one has to bound the quantities \(H_{u(x,t)}^{(m)}\), for which one cannot have uniform estimates, but only integral ones on \(A_T\). So as to obtain these integral estimates, one uses the method of Lemmas 3.4 and 3.5 of Bally et al. (1994), which is based on the estimates of the Green kernel \(G\) given in Lemma A.3 of the appendix. Precisely one proves that for every \(0 < \alpha < 1\) and \(\varepsilon > 0\)

\[
\|H_{u(x,t)}^k\|_{q,p} \leq C_{q,p}(1 + (x \wedge (1 - x)))^{-(\varepsilon + (2 - \alpha)/2)}(1 + (1 + \alpha)/4)
\]

and then Proposition 3.4 is obtained by integrating (2.10).

**Appendix A**

In this section we recall some useful estimates concerning the Green kernels associated with Neumann or Dirichlet boundary conditions. The following result, in which \(G\) denotes either of the Green kernels, corresponds to Lemmas A. 2 and B. 1 of Bally–Millet–Sanz-Solé in Bally et al. (1995):

**Lemma A.1.** (a) Let \(h\) a \(2\beta\)-Hölder function, with \(\beta > 0\). Then, for all \(x, x', t, t'\)

\[
\left| \int_0^T G_t(x', y)h(y) \, dy - \int_0^T G_t(x, y)h(y) \, dy \right| \leq \|h\|_{L_2^\beta}(|t' - t|^\beta + |x' - x|^{2\beta}),
\]

where \(\|h\|_{L_2^\beta} = \sup_{x \neq y} \left( \frac{|h(y) - h(x)|}{|y - x|^{2\beta}} \right)\).

(b) For \(\beta \in ]\frac{1}{2}; 3[\), there exists \(C > 0\) such that for all \(x, y, t\) we have

\[
\int_0^T \int_0^1 |G_{t-r}(x, z) - G_{t-r}(y, z)|^\beta \, dz \, dr \leq C|x - y|^{3-\beta}.
\]

(c) For all \(\beta \in ]1; 3[\) there exists \(C > 0\) such that for all \((s, t)\) with \(s < t\) and for all \(x\) we have

\[
\int_s^t \int_0^1 |G_{t-r}(x, y)|^\beta \, dy \, dr \leq C|t - s|^{(3-\beta)/2}
\]
and

\[ \int_0^t \int_0^1 |G_{t-r}(x, y) - G_{s-r}(x, y)|^\beta \, dy \, dr \leq C|t - s|^{(3-\beta)/2}. \]

The following lemma corresponds to inequality (A.3) of Bally and Pardoux:

**Lemma A.2.** There exists a constant $C$ such that for all $x \in [0, 1]$, $\eta \in ]0, 1[$ and all $t > \eta$

\[ \int_{t-\eta}^t \int_{x-\sqrt{\eta}}^{x+\sqrt{\eta}} G^2_{t-s}(x, y) \, dy \, ds \geq C\sqrt{\eta}, \]

where $G$ denotes the Green kernel associated with Neumann boundary conditions.

The previous result does not hold in the case of Dirichlet boundary conditions, however, one has the following estimate (see Bally et al., 1994):

**Lemma A.3.** There exists a constant $C$ such that for all $x \in [0, 1]$, $\eta \in ]0, 1[$ and $t > \eta$

\[ \int_{t-\eta}^t \int_0^1 G^2_{t-s}(x, y) \, dy \, ds \geq C\sqrt{\eta}(1 - e^{-1/4\eta} - e^{-2(1-\sqrt{\eta})^2}) \quad \text{for } \frac{1}{2} \leq x < 1 \]

\[ \int_{t-\eta}^t \int_0^1 G^2_{t-s}(x, y) \, dy \, ds \geq C\sqrt{\eta}(1 - e^{-1/4\eta} - e^{-2(x/\eta)^2}) \quad \text{for } 0 < x \leq 1/2. \]

where $G$ denotes the Green kernel associated with Dirichlet boundary conditions.

The following corollary can then be deduced:

**Lemma A.4.** For every $x \in [0, 1]$, there exists a constant $C_x$ such that for all $\eta \in ]0, 1[$, $x \in [2, 1 - x]$ and $t > \eta$

\[ \int_{t-\eta}^t \int_0^1 G^2_{t-s}(x, y) \, dy \, ds \geq C_x \sqrt{\eta}, \]

where $G$ denotes the Green kernel associated with Dirichlet boundary conditions.

Finally, the following lemma, proved in Bally and Pardoux, gives a finer estimate for both kernels when one is far from zero.

**Lemma A.5.** There exists a constant $C$ such that for all $t, \varepsilon > 0$ such that $t - \varepsilon > 0$, we have

\[ \int_{t-\varepsilon}^t \int_{[x-l, x+l]^2} G^2_{t-s}(x, y) \, dy \, ds \leq Ce^{t/2\varepsilon}. \]

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