Essential elements in connected $k$-polymatroids

Dennis Hall

Mathematics Department, Louisiana State University, Baton Rouge, LA, United States

A R T I C L E   I N F O

Article history:
Received 2 July 2012
Accepted 15 July 2012
Available online 15 October 2012

MSC:
05B35

Keywords:
Matroids
Polymatroids
Generalized parallel connection
Truncation

A B S T R A C T

It is a well-known result of Tutte that, for every element $x$ of a connected matroid $M$, at least one of the deletion and contraction of $x$ from $M$ is connected. This paper shows that, in a connected $k$-polymatroid, only two such elements are guaranteed. We show that this bound is sharp and characterize those 2-polymatroids that achieve this minimum. To this end, we define and make use of a generalized parallel connection for $k$-polymatroids that allows connecting across elements of different ranks. This study of essential elements gives results crucial to finding the unavoidable minors of connected 2-polymatroids, which will appear elsewhere.

© 2012 Elsevier Inc. All rights reserved.

1. Introduction

A classical result of Tutte is that, for every element $x$ of a connected matroid $M$, either $M\setminus x$ or $M/x$ is connected. This property of being able to either delete or contract any element while maintaining connectivity, however, does not hold for $k$-polymatroids. We call an element $x$ of a connected $k$-polymatroid essential if both its deletion and contraction from the $k$-polymatroid destroy connectivity. In this paper, we show that every $k$-polymatroid has at least two elements that are non-essential, show that this bound is sharp for each integer $k$ exceeding one, and characterize all 2-polymatroids with exactly two non-essential elements.

Additional motivation for this paper comes from the desire to find the unavoidable minors for connected 2-polymatroids, which is done in [2]. This study of essential elements turns out to be a crucial step in that endeavor. In fact, one may divide the class of unavoidable minors for connected 2-polymatroids into two categories: those that resemble circuits and cocircuits in matroids, and those that have exactly two non-essential elements.

E-mail address: dhall15@math.lsu.edu.
The main results, Theorems 4.3 and 4.9, are stated and proved in Section 4. The concepts of 2-sum and parallel connection for $k$-polymatroids, ideas that play an important role in the proofs of the main results, are studied in Section 3. Polymatroid-theoretic preliminaries are given in Section 2.

2. Polymatroids

Let $M$ be a matroid with ground set $E$ and rank function $r$. The pair $(E, r)$ is an example of a 1-polymatroid. In fact, the class of 1-polymatroids is exactly the class of matroids. For an arbitrary positive integer $k$, we now define a $k$-polymatroid noting that it is very much like a matroid but allows individual elements to have ranks up to $k$.

Much of the polymatroid-theoretic language in this paper follows [7]. Let $E$ be a finite set and $f$ be a function from the power set of $E$ into the integers. We say that $f$ is normalized if $f(\emptyset) = 0$; $f$ is submodular if $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$ for all $X, Y \subseteq E$; and $f$ is increasing if $f(X) \leq f(Y)$ whenever $X \subseteq Y \subseteq E$. We call the pair $(E, f)$ a polymatroid $Q$ if $f$ is normalized, submodular, and increasing. The set $E$ is called the ground set of $Q$ while $f$ is the rank function. For a positive integer $k$, a polymatroid $(E, f)$ is a $k$-polymatroid if $f(z) \leq k$ for all $z \in E$. For ease of notation, a rank-1 element of a $k$-polymatroid is called a point; a rank-0 element of a $k$-polymatroid is called a loop. If $a$ and $b$ are elements of a $k$-polymatroid such that $f((a, b)) = f(a) = f(b)$, then we say that $a$ and $b$ are parallel.

Examples

An important way to obtain a $k$-polymatroid from a matroid is as follows. Given a matroid $M$ with ground set $S$ and rank function $r$, we obtain a $k$-polymatroid $Q = (E, f)$ by taking $E$ to be some subset of the set of flats of $M$ of rank at most $k$ and letting $f(X) = r(\bigcup_{x \in X} x)$ for all subsets $X$ of $E$. Indeed, every $k$-polymatroid can be obtained in this way (see, for example, [3, 5]). This fundamental fact allows us, in particular, to think of 2-polymatroids as an arrangement of loops, points, and lines of a matroid.

Another natural class of 2-polymatroids arises from graphs. To see this, let $G$ be a graph and set $E = E(G)$. For a subset $X$ of $E$, define a function $f$ by $f(X) = |V(X)|$ where $V(X)$ is the set of vertices of $G$ that meet some edge of $X$. Then $(E, f)$ is a 2-polymatroid. We will call the 2-polymatroids that can be represented in this way Boolean and note that there is a one-to-one correspondence between the class of Boolean 2-polymatroids and the class of graphs without isolated vertices [8].

Finally, we consider $k$-polymatroids that are derived from other polymatroids. Let $Q_1 = (E, f_1)$ and $Q_2 = (E, f_2)$ be $k$-polymatroids on the same ground set. It is not difficult to check, then, that $(E, f)$ is a 2$k$-polymatroid where $f(Z) = f_1(Z) + f_2(Z)$ for all $Z \subseteq E$. We denote $(E, f)$ by $Q_1 + Q_2$ or, when $Q_1 = Q_2$, by $2Q_1$. We are not limited, however, to a sum of only 2 polymatroids. In particular, the sum of $k$ copies of the matroid $U_{n-1,n}$, denoted $kU_{n-1,n}$, is a $k$-polymatroid consisting of $n$ rank-$k$ elements placed freely in rank $kn - k$.

Duality and minors

One attractive feature of $k$-polymatroids is that there are notions of duality, deletion, and contraction that mimic many of the nice properties of the same notions in matroids. Let $Q = (E, f)$ be a $k$-polymatroid. For all subsets $X$ of $E$, let

$$f^*(X) = k|X| + f(E - X) - f(E).$$

Then $(E, f^*)$ is a $k$-polymatroid $Q^*$, which, following [7], we call the $k$-dual of $Q$.

For a subset $X$ of $E$, define $f_{Q\setminus X}$ and $f_{Q/X}$, for all subsets $A$ of $E - X$, by $f_{Q\setminus X}(A) = f(A)$ and $f_{Q/X}(A) = f(A \cup X) - f(A)$. Let $Q\setminus X = (E - X, f_{Q\setminus X})$ and $Q/X = (E - X, f_{Q/X})$. It is common to write $f\setminus X$ instead of $f_{Q\setminus X}$ and $f/X$ instead of $f_{Q/X}$. It is easy to verify that both of $Q\setminus X$ and $Q/X$ are $k$-polymatroids, and that $Q^\setminus X = (Q/X)^*$. We call $Q\setminus X$ and $Q/X$ the deletion and contraction of $X$ from $Q$. We note that the $k$-dual is the unique involutary operation on the class of $k$-polymatroids that interchanges deletion and contraction (see [10]).
Connectivity

Following Matúš [4], we say that a \( k \)-polymatroid \( Q = (E, f) \) is connected or 2-connected if \( f(X) + f(E - X) > f(E) \) for all nonempty proper subsets \( X \) of \( E \); otherwise, \( Q \) is disconnected. If \( f(X) + f(E - X) = f(E) \), then \( X \) is a separator; it is nontrivial if \( X \notin \{\emptyset, E\} \). When \( X \) is a nontrivial separator, \((X, E - X)\) is called a 1-separation of \( Q \). It is a quick exercise to see that \( Q \) is connected if and only if \( Q^* \) is connected (see [6]). We introduce the concept of 3-connectedness for \( k \)-polymatroids in the next section.

3. Parallel connection and 2-sum

Here, we expand upon the notion of parallel connection for polymatroids that is given in [4]. This operation for polymatroids is a generalization of that for matroids in that it consists of sticking together two polymatroids as freely as possible across a designated element of each. Below, we give a formal definition that mimics the language of parallel connection for matroids.

Suppose \( Q_1 = (E_1, f_1) \) and \( Q_2 = (E_2, f_2) \) are \( k \)-polymatroids on disjoint ground sets. Let \( Q_1 \oplus Q_2 = (E_1 \cup E_2, f) \) where \( f(Z) = f_1(Z \cap E_1) + f_2(Z \cap E_2) \) for all \( Z \subseteq E_1 \cup E_2 \). It is well known and easily checked that \( Q_1 \oplus Q_2 \) is a \( k \)-polymatroid. Following [1], we call it the direct sum of \( Q_1 \) and \( Q_2 \). Evidently, a \( k \)-polymatroid is 2-connected if and only if it cannot be written as a direct sum of two \( k \)-polymatroids with nonempty ground sets.

Next, suppose \( Q_1 = (E_1, f_1) \) and \( Q_2 = (E_2, f_2) \) are \( k \)-polymatroids with \( E_1 \cap E_2 = \{p\} \) and \( f_1(p) = f_2(p) \). Let \( P(Q_1, Q_2) = (E_1 \cup E_2, f) \) where, for all \( A \subseteq E \), if \( A_1 = A \cap E_1 \) and \( A_2 = A \cap E_2 \), then

\[
f(A) = \min\{f_1(A_1) + f_2(A_2), f_1(A_1 \cup A_2) + f_2(A_2 \cup p) - f_1(p)\}.
\]

A routine check shows that \( P(Q_1, Q_2) \) is a \( k \)-polymatroid. We call this \( k \)-polymatroid the parallel connection of \( Q_1 \) and \( Q_2 \) with respect to the basepoint \( p \). When \( Q_1 \) and \( Q_2 \) are matroids, this definition of parallel connection coincides with that for matroids. A limitation of our definition of \( P(Q_1, Q_2) \) is that it requires the basepoints to have the same rank. To rectify this, we extend the matroid operation of principal truncation (see, for example, [5, Section 7.3]).

Intuitively, the principal truncation of an element \( p \) is achieved by adding a point on \( p \) as freely as possible and then contracting the added point. To define this operation formally, let \( Q = (E, f) \) be a polymatroid with \( p \in E \) and let \( f_p \) be the function defined, for all subsets \( A \subseteq E \), by

\[
f_p(X) = \begin{cases} f(X) - 1, & \text{if } f(X \cup p) = f(X); \\ f(X), & \text{otherwise.} \end{cases}
\]

It is not difficult to check that \((E, f_p)\) is a polymatroid. We denote it by \( T_p(Q) \) and say that it is obtained from \( Q \) by truncating \( p \). This operation can be repeated. For a positive integer \( n \), we define \( T^n_p(Q) = T_p(T_p^{n-1}(Q)) \) where \( T_0^p(Q) = Q \). It is an easy exercise to verify that \( T^n_p(Q) \) has rank function \( f^n_p \) defined, for all \( X \subseteq E \), by

\[
f^n_p(X) = \begin{cases} \max\{f(X \cup p) - n, 0\}, & \text{if } f(X \cup p) - f(X) \leq n; \\ f(X), & \text{otherwise.} \end{cases}
\]

Suppose \( Q_1 = (E_1, f_1) \) and \( Q_2 = (E_2, f_2) \) are polymatroids with \( E_1 \cap E_2 = \{p\} \). Let \( n = f_2(p) - f_1(p) \). We expand the notion of parallel connection to this case by setting \( P(Q_1, Q_2) \) to be \( P(Q_1, T^n_p(Q_2)) \). When \( Q_1 \) and \( Q_2 \) are matroids such that \( p \) is a loop of \( Q_1 \) and a non-loop of \( Q_2 \), this definition coincides with that for matroids.

The following familiar properties of parallel connection hold for \( k \)-polymatroids.

**Proposition 3.1.** Let \( Q_1 = (E_1, f_1) \) and \( Q_2 = (E_2, f_2) \) be polymatroids such that \( E_1 \cap E_2 = \{p\} \). Then

(i) \( P(Q_1, Q_2) = Q_1/p \oplus Q_2/p \); and
(ii) for all \( e \in E_1 - p \),
\[
P(Q_1, Q_2)/e = P(Q_1/e, Q_2) \quad \text{and} \quad P(Q_1, Q_2)\setminus e = P(Q_1\setminus e, Q_2).
\]

**Proof.** The proof of this proposition is not significantly different from the proof of the corresponding result for matroids (see, for example, [5]) and is omitted. \( \square \)

The following result of Oxley and Whittle (see [6, Theorem 3.1]) is used throughout the paper.

**Lemma 3.2.** Let \( Q = (E, f) \) be a connected \( k \)-polymatroid where \( |E| \geq 2 \) and let \( A \) be a nonempty proper subset of \( E \). If
\[
f(A) + f(E - A) - f(E) < \min\{ f(X) + f(E - X) - f(E) : \emptyset \neq X \subseteq E \},
\]
then at least one of \( Q/A \) and \( Q \setminus A \) is connected. \( \square \)

From this lemma, we obtain the following result on non-essential elements. Recall that an element \( e \) of a connected \( k \)-polymatroid \( Q \) is non-essential if either \( Q \setminus e \) or \( Q/e \) is connected.

**Proposition 3.3.** If \( Q = (E, f) \) is a connected \( k \)-polymatroid and \( e \in E \) such that \( f(e) = 1 \), then \( e \) is non-essential.

**Proof.** This is an immediate consequence of Lemma 3.2. \( \square \)

**Theorem 3.4.** Suppose \( Q_1 = (E_1, f_1) \) and \( Q_2 = (E_2, f_2) \) are \( k \)-polymatroids such that \( E_1 \cap E_2 = \{p\} \) where \( f_1(p) = f_2(p) \). Then both \( Q_1 \) and \( Q_2 \) are connected if and only if \( P(Q_1, Q_2) \) is connected. Further, if \( P(Q_1, Q_2)\setminus p \) is connected, then \( P(Q_1, Q_2) \) is connected.

**Proof.** If \((X, Y \cup p)\) is a 1-separation of \( Q_1 \), then it is not difficult to check that \((X, E_2 \cup Y)\) is a 1-separation of \( P(Q_1, Q_2) \) and that \((X, (E_2 - p) \cup Y)\) is a 1-separation of \( P(Q_1, Q_2)\setminus p \). On the other hand, suppose \((X, Y \cup p)\) is a 1-separation of \( P(Q_1, Q_2) \), and \( f_3 \) is the rank function for \( P(Q_1, Q_2) \). Let \( X_i = X \cap E_i \) and \( Y_i = Y \cap E_i \) for each \( i \in \{1, 2\} \), and observe that
\[
f_3(X) = \min\{ f_1(X_1) + f_2(X_2), f_1(X_1 \cup p) + f_2(X_2 \cup p) - f_1(p) \};
\]
\[
f_3(Y \cup p) = f_1(Y_1 \cup p) + f_2(Y_2 \cup p) - f_1(p); \quad \text{and}
\]
\[
f_3(E_1 \cup E_2) = f_1(E_1) + f_2(E_2) - f_1(p).
\]
If \( f_1(X_1) + f_2(X_2) \leq f_1(X_1 \cup p) + f_2(X_2 \cup p) - f_1(p) \), then since \( f_3(X) + f_3(Y \cup p) = f_3(E_1 \cup E_2) \), we have
\[
f_1(X_1) + f_2(X_2) + f_1(Y_1 \cup p) + f_2(Y_2 \cup p) = f_1(E_1) + f_2(E_2).
\]
As \( f_1(X_1) + f_1(Y_1 \cup p) \geq f_1(E_1) \) for each \( i \in \{1, 2\} \), it follows that \((X_i, Y_i \cup p)\) is a 1-separation for each \( i \in \{1, 2\} \). On the other hand, if \( f_1(X_1) + f_2(X_2) > f_1(X_1 \cup p) + f_2(X_2 \cup p) - f_1(p) \), then, as \( f_3(X) + f_3(Y \cup p) = f_3(E_1 \cup E_2) \), we have
\[
f_1(X_1 \cup p) + f_2(X_2 \cup p) + f_1(Y_1 \cup p) + f_2(Y_2 \cup p) - f_1(p) = f_1(E_1) + f_2(E_2).
\]
From submodularity again, it follows that \( f_2(p) = f_1(p) = 0 \), and thus \( Q_1 \) and \( Q_2 \) are disconnected. \( \square \)
In addition to parallel connection, we make use of the 2-sum operation. Let \( Q_1 \) and \( Q_2 \) be \( k \)-polymatroids on ground sets \( E_1 \) and \( E_2 \), respectively, with \( E_1 \cap E_2 = \{ p \} \). If \( f_1(p) = f_2(p) = 1 \) and \( p \) is not a separator for either \( Q_1 \) or \( Q_2 \), then the 2-sum of \( Q_1 \) and \( Q_2 \) is defined to be \( P(Q_1, Q_2) \setminus p \) and denoted \( Q_1 \oplus_2 Q_2 \). The following shows some fundamental connectivity properties of this 2-sum operation.

**Corollary 3.5.** Suppose \( Q_1 = (E_1, f_1) \) and \( Q_2 = (E_2, f_2) \) are \( k \)-polymatroids such that \( E_1 \cap E_2 = \{ p \} \) where \( f_1(p) = f_2(p) = 1 \). Then the following are equivalent.

(i) \( Q_1 \) and \( Q_2 \) are both connected;
(ii) \( Q_1 \oplus_2 Q_2 \) is connected;
(iii) \( P(Q_1, Q_2) \) is connected.

**Proof.** Using Theorem 3.4, we have only to show that (iii) implies (ii). From Proposition 3.1, we observe that \( P(Q_1, Q_2) \setminus p \) is disconnected. Since \( f_1(p) = f_2(p) = 1 \), we use Proposition 3.3 to see that \( p \) is non-essential and therefore that \( P(Q_1, Q_2) \setminus p = Q_1 \oplus_2 Q_2 \) is connected. \( \Box \)

We say that a \( k \)-polymatroid \( Q \) is 3-connected if and only if it cannot be written as a 2-sum of a pair of \( k \)-polymatroids each with fewer elements than \( Q \). The following proposition allows us to give an alternative definition.

**Proposition 3.6.** Suppose \( Q = (E, f) \) is a \( k \)-polymatroid for which there exists a partition \( (X_1, X_2) \) of \( E \) such that \( f(X_1) + f(X_2) = f(E) + 1 \) and \( \min(|X_1|, |X_2|) \geq 2 \). Then there are polymatroids \( Q_1 \) and \( Q_2 \) on ground sets \( X_1 \cup p \) and \( X_2 \cup p \), respectively, where \( p \) is a new point not in \( E \), such that \( Q = Q_1 \oplus_2 Q_2 \).

**Proof.** For \((i, j) \in \{(1, 2), (2, 1)\} \), let \( Q_i = (X_i \cup p, f_i) \) where \( f_i \) is defined, for all \( A \subseteq X_i \cup p \), by

\[
    f_i(A) = \begin{cases} 
        f((A - p) \cup X_j) - f(X_j) + 1 & \text{if } p \in A; \\
        f(A) & \text{if } p \notin A.
    \end{cases}
\]

It is routine to check that \( f_i \) is a \( k \)-polymatroid. Let \( f_3 \) be the rank function of \( P(Q_1, Q_2) \). Since \( Q_1 \oplus_2 Q_2 = P(Q_1, Q_2) \setminus p \), it suffices to show that \( f_3(A) = f(A) \) for all subsets \( A \) of \( E \). Choose such a subset \( A \), let \( A_i = A \cap X_i \) for \( i \in \{1, 2\} \), and note that

\[
    f_3(A) = \min\{f_1(A_1) + f_2(A_2), f_1(A_1 \cup p) + f_2(A_2 \cup p) - f_1(p)\}
    = \min\{f(A_1) + f(A_2), f(A_1 \cup X_2) + f(A_2 \cup X_1) - f(E)\}.
\]

Observe that if \( U \) and \( V \) are disjoint subsets of \( E \) with \( S \subseteq U \) and \( T \subseteq V \), then

\[
    f(U) + f(V) + f(S \cup T) \geq f(U) + f(S \cup V) + f(T)
    \geq f(U \cup V) + f(S) + f(T).
\]

Rearranging this inequality provides that

\[
    f(U) + f(V) - f(U \cup V) \geq f(S) + f(T) - f(S \cup T).
\]

Since \( f(X_1) + f(X_2) = f(E) + 1 \), we have from (3.1) that

\[
    f(A_1 \cup X_2) \in \{f(A_1) + f(X_2), f(A_1) + f(X_2) - 1\};
\]
with $f(A_2 \cup X_1)$ behaving similarly. If $f(A_1 \cup X_2) = f(A_1) + f(X_2)$, then another application of (3.1) shows that

$$f(A_1) + f(A_2) = f(A_1 \cup A_2).$$

From submodularity, we have that $f(A_1 \cup X_2) + f(A_2 \cup X_1) - f(E) \geq f(A_1 \cup A_2)$, and it follows that $f_3(A) = f(A_1) + f(A_2) = f(A)$, as desired. By symmetry, then, we have only to consider when $f(A_1 \cup X_2) = f(A_1) + f(X_2) - 1$ and $f(A_2 \cup X_1) = f(A_2) + f(X_1) - 1$. In this case, we observe that

$$f(A_1) + f(A_2) = f(A_1 \cup X_2) + f(X_1 \cup A_2) - f(X_1) - f(X_2) + 2$$

$$= f(A_1 \cup X_2) + f(X_1 \cup A_2) - f(E) + 1$$

$$\geq f(E) + f(A) - f(E) + 1$$

$$= f(A) + 1.$$

From this, it follows that

$$f_3(A) = f(A_1 \cup X_2) + f(X_1 \cup A_2) - f(E) = f(A_1) + f(A_2) - 1,$$

and with an application of (3.1), that

$$f(A_1) + f(A_2) = f(A) + 1.$$

Combining these equations yields that $f_3(A) = f(A)$ and the conclusion holds. \qed

**Corollary 3.7.** A $k$-polymatroid $Q = (E, f)$ is 3-connected if and only if for any partition $(X, Y)$ of $E$ with $f(X) + f(Y) = f(E) + 1$, either $|X| = 1$ or $|Y| = 1$.

From this, it is clear that a $k$-polymatroid $Q$ is 3-connected if and only if $Q^*$ is 3-connected. Our final result shows that 2-summing commutes for $k$-polymatroids. We omit the proof since it involves a routine, but tedious, exhaustive case-check.

**Proposition 3.8.** For $i \in \{1, 2, 3\}$, let $Q_i = (E_i, f_i)$ be a $k$-polymatroid for which $E_1 \cap E_2 = \{p_1\}$ and $E_2 \cap E_3 = \{p_2\}$ with $f_1(p_1) = f_2(p_1) = f_2(p_2) = f_3(p_2) = 1$. Then $Q_1 \oplus_2 (Q_2 \oplus_2 Q_3) = (Q_1 \oplus_2 Q_2) \oplus_2 Q_3$.

4. **Non-essential elements**

Recall that an element $e$ of a connected $k$-polymatroid $Q$ is non-essential if either $Q \setminus e$ or $Q/e$ is connected. Tutte showed in [9] that every element of a connected matroid is non-essential. We expand this result to $k$-polymatroids by determining the number of non-essential elements that are guaranteed to exist in any $k$-polymatroid. To do so, we make extensive use of the truncation operation defined in the previous section.

**Lemma 4.1.** Let $Q = (E, f)$ be a connected $k$-polymatroid with $e \in E$. Then $T_e(Q)$ is connected if and only if $Q$ is connected with $f(e) > 1$.

**Proof.** Let $(A, B)$ be a partition of $E$ with $e \in A$ and $B$ nonempty. Suppose $T_e(Q)$ is connected. Then certainly $f(e) > 1$ or else $e$ would be a loop in $T_e(Q)$. To show that $Q$ is connected, observe that
\[ f(A) + f(B) = f_e(A) + f(B) + 1 \]
\[ \geq f_e(A) + f_e(B) + 1 \]
\[ > f_e(E) + 1 \]
\[ = f(E). \]

We now assume that \( Q \) is connected with \( f(e) > 1 \). Then
\[ f_e(A) + f_e(B) \geq f(A) + f(B) - 2 \geq f(E) - 1 = f_e(E), \]
and it thus suffices to consider the case when both \( f_e(B) = f(B) - 1 \) and \( f(A) + f(B) = f(E) + 1 \).

From the first of these equations, we have \( f(B \cup e) = f(B) \) and so, from the second equation, get \( f(A) + f(B \cup e) = f(E) + 1 \). It follows from submodularity that
\[ f(E) + f(e) \leq f(A) + f(B \cup e) = f(E) + 1, \]
and therefore \( f(e) \leq 1 \). \( \square \)

**Lemma 4.2.** Let \( Q = (E, f) \) be a \( k \)-polymatroid with \( e \in E \) and disjoint sets \( C, D \subseteq E - e \) such that \( f(C \cup e) > f(C) \). Then \( T_e(Q) \setminus D/C = T_e(Q) \setminus D/C \).

**Proof.** Let \( X \subseteq E - (C \cup D) \). It is straightforward to show that \( f_e \setminus D/C(X) = (f \setminus D/C)e(X) \). \( \square \)

**Theorem 4.3.** Every connected \( k \)-polymatroid having at least two elements has at least two non-essential elements.

**Proof.** Let \( Q = (E, f) \) be a connected \( k \)-polymatroid with \( |E| \geq 2 \). We proceed by induction on the rank of \( Q \). If \( f(E) = 0 \), then \( Q \) is not connected and we are done. Thus we assume the theorem holds for polymatroids of rank less than that of \( Q \). If possible, choose \( e \in E \) such that \( f(E - e) < f(E) \). If each \( e \in E \) satisfies \( f(E - e) = f(E) \), then choose \( e \in E \) such that \( f(e) = \max\{f(x) : x \in E\} \). If \( f(e) = 1 \), then \( Q \) consists entirely of rank-1 elements and so consists entirely of non-essential elements by Proposition 3.3. Otherwise, we use Lemma 4.1 to see that \( T_e(Q) \) is a connected \( k \)-polymatroid with at least two elements and rank one less than the rank of \( Q \). By induction, then, we may pick two elements \( a, b \in E \) that are non-essential in \( T_e(Q) \). By combining Lemmas 4.1 and 4.2, we note that if an element of \( E - e \) is non-essential in \( T_e(Q) \), then it is non-essential in \( Q \). Therefore we need only show that either \( e \) is non-essential in \( Q \), or there are two elements \( x, y \in E - e \) that are non-essential in \( T_e(Q) \). Clearly, if \( a, b \in E - e \), then we are done. Thus assume that \( e \) is non-essential in \( T_e(Q) \). If \( f(E - e) < f(E) \), then it is not difficult to show that \( e \) is essential in \( Q \). Hence assume that \( f(E - x) = f(E) \) for all \( x \in E \). If \( T_e(Q)/e \) is connected, then, as \( T_e(Q)/e = Q/e \), the theorem holds. Hence we may assume that \( T_e(Q) \setminus e \) is connected. If \( |E| = 2 \), then the result is obvious and so \( T_e(Q) \setminus e \) is a connected \( k \)-polymatroid with at least two elements. Let \( x \) and \( y \) be non-essential in \( T_e(Q) \).

If \( T_e(Q) \setminus \{e, x\} \) is connected, then \( T_e(Q) \setminus \{x\} \) is connected unless
\[ f_e(e) + f_e(E - \{e, x\}) = f_e(E - x) = f_e(E). \]  (4.1)

In this case, suppose \( (A \cup x, B) \) partitions \( E - e \) nontrivially such that
\[ f/e(A \cup x) + f/e(B) = f/e(E - e). \]

Then
\[ f(A \cup \{e, x\}) + f(B \cup e) = f(E) + f(e). \]  (4.2)
Observe, however, that (4.1) implies that \( f(e) + f(E - \{e, x\}) = f(E) \) and thus, since \( B \subseteq E - \{e, x\} \), that \( f(e) + f(B) = f(B \cup e) \). Applying this to (4.2) shows that \((A \cup \{e, x\}, B)\) is a 1-separation of \( Q \), a contradiction. It remains to consider the case when \( T_e(Q) \setminus e/x \) is connected. By a similar argument to the above, we have that \((e, E - x)\) is the only possible 1-separation of \( T_e(Q)/x \). If \( Q/x \) is connected, we are done. Thus assume that \((A \cup e, B)\) is a 1-separation of \( Q/x \). Since \((f/x)_e(A \cup e) = f/x(A \cup e) - 1\) and \((f/x)_e(E - x) = f/x(E - x) - 1\), it follows that

\[
(f/x)_e(A \cup e) + f/x(B) = (f/x)_e(E - x). \tag{4.3}
\]

Now, either \( f/x(B) = (f/x)_e(\{e\}) \) or \( f/x(B) = (f/x)_e(B) + 1 \). Observe that if \( f((x, e)) = f(x) \), then \( Q \setminus e \) is connected and we are done. Thus \( f((x, e)) > f(x) \) and we have, from Lemma 4.2, that \( T_e(Q)/x = T_e(Q)/x \). Thus if \( f/x(B) = (f/x)_e(B) + 1 \), then (4.3) becomes

\[
f_e/x(A \cup e) + f_e/x(B) = f_e/x(E - x) - 1,
\]

contradicting the submodularity of \( T_e(Q)/x \). On the other hand, if \( f/x(B) = f_e/x(B) \), then

\[
f_e/x(A \cup e) + f_e/x(B) = f_e/x(E - x).
\]

As \((e, E - x)\) is the only possible 1-separation of \( T_e(Q)/x \), it follows that \( A = \emptyset \). Then, since \((A \cup e, B)\) is a 1-separation of \( Q/x \),

\[
f/x(e) + f/x(E - \{e, x\}) = f/x(E - x).
\]

Since \( f/x(E - \{e, x\}) = f/x(E - x) \), it follows that \( f((x, e)) = f(e) \) and thus \( Q \setminus x \) is connected. \( \square \)

We now know that every connected \( k \)-polymatroid has at least two non-essential elements. The next example shows that this bound is sharp.

**Example 4.4.** Choose integers \( k \geq 1 \) and \( n \geq 1 \). Let \( E \) be a set with \(|E| = k\) and choose distinct elements \( a, b \notin E \). Take \( M = (E \cup \{a, b\}, r) \) to be a matroid isomorphic to \( U_{1,k+1} \oplus U_{0,1} \) where \( b \) is the loop and \( Q = (E \cup \{a, b\}, f) \) to be an \( n \)-polymatroid isomorphic to \( nU_{k,k+1} \oplus U_{0,1} \) where \( a \) is the loop. Then the \((n + 1)\)-polymatroid \( M + Q \) has \( a \) and \( b \) as its only non-essential elements. If \( n = 1 \), then we denote \( M + Q \) by \( S_k \) for each \( k \). The 2-polymatroid \( S_k \) is shown geometrically in Fig. 1 for \( k \in \{1, 2, 3\} \).

**Lemma 4.5.** If \( Q = (E, f) \) is a connected 2-polymatroid and, for some \( e \in E \), both \( f \setminus e \) and \( f / e \) are not connected, then \( f(E) = f(E - e), f(e) = 2, \) and \( f(X) + f(E - X) - f(E) = 1 \) for some set \( X \subseteq E \).

**Proof.** This is an immediate consequence of Lemma 3.2. \( \square \)

If \( Q = (E, f) \) is a 2-polymatroid and \( x \in E \) such that \( f(E - x) = f(E) - 1 \), then \( f^*(e) = 1 \) and we say that \( e \) is a **copoint**. In the following theorem, we show that the polymatroids given in Example 4.4
when \( n = 1 \) are the only 3-connected 2-polytopes with exactly 2 non-essential elements and no copoints. After obtaining this result, it is not difficult to remove the no-copoints requirement, which is done in Corollary 4.7.

**Theorem 4.6.** If \( Q \) is a 3-connected 2-polytope with at least three elements, no copoints, and exactly two non-essential elements, then \( Q \) is isomorphic to \( S_k \) for some \( k \).

**Proof.** Let \( Q = (E, f) \) be a 3-connected 2-polytope with an essential element \( a \). Choose a nontrivial partition \((X, Y)\) of \( E - a \) with \( |X| \) maximal such that \( f(X \cup a) + f(Y \cup a) = f(E) + 2 \). A partition of this type is a 1-separation of \( Q/a \) and thus exists. Similarly, choose a partition \((A, B)\) of \( E - a \) with \( |A| \) maximal such that \( f(A) + f(B) = f(E) \). Then

\[
2f(E) + 2 = f(A) + f(B) + f(X \cup a) + f(Y \cup a) 
\geq f(A \cup X \cup a) + f(B \cup Y) + f(B \cup Y \cup a) + f(A \cap X) \tag{4.4}
\]

and

\[
2f(E) + 2 = f(A) + f(B) + f(X \cup a) + f(Y \cup a) 
\geq f(A \cup Y \cup a) + f(B \cup X) + f(B \cup X \cup a) + f(A \cap Y). \tag{4.5}
\]

Since \( Q \) is 3-connected, we get from (4.4) that at least one of \( |A \cap X| \) and \( |B \cap Y| \) is less than 2. In fact, for some \( k \geq 1 \),

\[
(|A \cap X|, |B \cap Y|) \in \{(0, k), (0, 0), (1, 1), (k, 0)\}.
\]

If \( |A \cap X| = 0 \) and \( |B \cap Y| \) is nonzero, then (4.5) tells us that, since neither \( A \cap Y \) nor \( B \cap X \) may be empty, both must be singletons. Thus \( A = (X \cap a) \cup (Y \cap a) = A \cap Y \) and \( A \) is a singleton. However, \( B \) then contains at least two elements, contradicting the maximality of \( A \).

Next, we assume both \( A \cap X \) and \( B \cap Y \) are empty. Again, from (4.5), we get that \( |A \cap Y| = |B \cap X| = 1 \). Let \( x \in B \cap X \) and \( y \in A \cap Y \). Since \( f(A \cup Y \cup a) + f(B \cap X) = f(E) + 1 \), we have that \( f((a, y)) + f(x) = f(E) + 1 \). As \( Q \) has no copoints, it follows that \( f(x) = 1 \) and similarly that \( f(y) = 1 \). It follows, since \( f(A) + f(B) = f(E) \), that \( f(E) = 2 \), so \( f((a, y)) = f((a, x)) = 2 \). If \( f((x, y)) = 1 \), then \( f((x, y)) + f(a) = f(E) + 1 \), which is impossible since \( f(a) = 2 \). Therefore, \( Q \cong S_1 \).

Now, we assume \( |A \cap X| = |B \cap Y| = 1 \) and let \( A \cap X = \{x\}; B \cap Y = \{y\} \). From (4.5) and the maximality of \( A \) and \( X \), we have \( |A \cap Y| = |B \cap X| \leq 1 \). If \( |A \cap Y| = |B \cap X| = 0 \), then, similarly to the previous case, we have that \( Q \cong S_1 \). We thus assume \( |A \cap Y| = |B \cap X| = 1 \) and let \( \{w\} = B \cap X \) and \( \{z\} = A \cap Y \). From this, we may use (4.4) and (4.5) to get \( f(w) = f(z) = f(x) = f(y) = 1 \). As rank-1 elements are always non-essential, this contradicts that \( Q \) has exactly two non-essential elements.

Finally, we consider the case when \( |B \cap Y| = 0 \) and \( A \cap X \) is nonempty. Arguing as above, we find that each of \( B \cap X \) and \( A \cap Y \) consists of a single rank-one element, which we call \( x \) and \( y \), respectively. Using (4.4), (4.5), and the 3-connectedness of \( Q \), we are able to find that \( f((a, y)) = 2 \), \( f((a, x, y)) = 3 \), \( f(E - (a, x, y)) = f(E) - 1 \), \( f(E - (a, x, y)) = f(E), f(E - (a, x, y)) = f(E) - 1 \), and \( f((a, x)) = 3 \). Indeed, as \( x \) and \( y \) are points, they are the sole non-essential elements of \( Q \). Thus we may choose \( b \in E - \{a, x, y\} \) and note that \( b \) must be essential. If we repeat the previous steps of this proof using \( b \) instead of \( a \), we come to the conclusion that \( b \) satisfies \( f((b, y)) = 2 \), \( f((b, x, y)) = 3 \), \( f(E - (b, x)) = f(E) - 1 \), \( f(E - (b, y)) = f(E), f(E - (b, x, y)) = f(E) - 1 \), and \( f((b, x)) = 3 \). As \( b \) was chosen arbitrarily, we have that these equations are satisfied for all \( p \in E - \{x, y\} \).

Since, for each \( p \in E - \{x, y\} \), we have that \( f((p, y)) = 2 \), it follows that \( f(E - x) \leq |E| - 1 \). It follows that \( f(E) \leq |E| - 1 \). If possible, choose a minimal set \( P \subseteq E - \{x, y\} \) for which \( f(P) \leq |P| \) and let \( b \in P \). By the minimality of \( P \), we have \( f(P - b) \geq |P - b| + 1 = |P| \geq f(P) \) and thus \( f(P - b) = f(P) \). Recall, however, that \( f((x, f(E - (b, x)) = f(E)) \). Since \( P - b \subseteq E - \{b, x\} \), it follows that \( f(E - x) = f(E) \).
Corollary 4.7. Every 3-connected 2-polymatroid on at least three elements with exactly two non-essential elements can be obtained from some $S_n$ by performing a sequence of element expansions.

**Proof.** Suppose $Q = (E, f)$ is such a 2-polymatroid having $\{x_1, x_2, \ldots, x_n\}$ as its set of copoints. Let $R = T_{x_1}(T_{x_2}(\cdots T_{x_n}(Q))\cdots)$. It is not difficult to check that $R$ is 3-connected and we can use Lemmas 4.1 and 4.2 to see that $R$ has exactly two non-essential elements. From Theorem 4.6, we have that $R$ is isomorphic to $S_n$ for some $n$. The conclusion follows. □

We conclude by characterizing all those 2-polymatroids with exactly two non-essential elements. The following proposition will be helpful to this end.

**Proposition 4.8.** Let $Q_1 = (E_1, f_1)$ and $Q_2 = (E_2, f_2)$ be connected $k$-polymatroids such that $E_1 \cap E_2 = \{p\}$ and $f_1(p) = f_2(p) = 1$. An element $x$ in $(E_1 \cup E_2) - p$ is non-essential in either $Q_1$ or $Q_2$ if and only if $x$ is non-essential in $Q_1 \oplus Q_2$.

**Proof.** From Proposition 3.1,

$$(Q_1 \oplus Q_2) \backslash x = P(Q_1, Q_2) \backslash \{x, p\} = P(Q_1 \backslash x, Q_2) \backslash p = (Q_1 \backslash x) \oplus Q_2.$$ 

Similarly, $(Q_1 \oplus Q_2) / x = (Q_1 / x) \oplus Q_2$. By combining these equations with Corollary 3.5, we obtain the proposition. □

The connected 2-polymatroids with exactly two non-essential elements consist of the members of $\{S_1, S_2, \ldots\}$ along with paths of 2-sums of such 2-polymatroids where the basepoints of the 2-sums are non-essential in both summands.

**Theorem 4.9.** Let $Q$ be a connected 2-polymatroid with at least three elements. Then $Q$ has exactly two non-essential elements if and only if, for some $n \geq 1$, there is a sequence $Q_1, Q_2, \ldots, Q_n$ of 2-polymatroids such that

(i) each $Q_i$ is isomorphic to some member of $\{U_{1,2} + U_{1,1}, S_1, S_2, \ldots\}$;
(ii) if either $n = 1$ or $2 \leq i \leq n - 1$, then $Q_i$ is isomorphic to some member of $\{S_1, S_2, \ldots\}$;
(iii) the ground sets of $Q_1, Q_2, \ldots, Q_n$ are disjoint except that, for each $i$ in $\{1, 2, \ldots, n - 1\}$, the sets $E(Q_i)$ and $E(Q_{i+1})$ meet in a single rank-1 element; and
(iv) $Q \cong Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n$.

**Proof.** If we have a sequence satisfying the four conditions, Proposition 4.8 implies that $Q$ has exactly two non-essential elements. For the converse, we proceed by induction on the rank of $E$. If $f(E) = 1$, then, since $|E| > 2$ and $Q$ is connected, it follows that $Q$ consists of $|E|$ points, each of which must be non-essential, a contradiction. Thus assume $f(E) > 1$ and that the conclusion holds for 2-polymatroids of rank less than $f(E)$. If $Q$ is 3-connected, then, from Corollary 4.7, there are three possibilities:
n = 1 with \( Q_1 \) isomorphic to some member of \( \{ S_1, S_2, \ldots \} \); n = 2 with \( Q_1 \) isomorphic to \( U_{1,2} + U_{1,1} \) and \( Q_2 \) isomorphic to some member of \( \{ S_1, S_2, \ldots \} \); or n = 3 with both \( Q_1 \) and \( Q_3 \) isomorphic to \( U_{1,2} + U_{1,1} \) and \( Q_2 \) isomorphic to some member of \( \{ S_1, S_2, \ldots \} \). We thus assume that \( Q \) is not 3-connected. Choose a nontrivial partition \((X, Y)\) of \( E \) such that \( f(X) + f(Y) = f(E) + 1 \) and \( 2 \leq |X| \leq |Y| \). If \( f(X) = 1 \), then each member of \( X \) is a point and is thus non-essential. As \( Q \) has exactly two non-essential elements, \( X \) consists of two points which are necessarily parallel. However, \( Q \setminus x \), where \( x \in X \), is connected with two non-essential elements. Clearly these two non-essential elements are also non-essential in \( Q \), a contradiction. Therefore \( f(X) > 1 \) and thus \( f(Y) < f(E) \). Similarly, \( f(X) < f(E) \). We now use Proposition 3.6 to choose 2-polymatroids \( Q_1 \) and \( Q_2 \) on ground sets \( X \cup p \) and \( Y \cup p \), respectively, where \( p \) is a point not in \( E \) and \( Q = Q_1 \oplus Q_2 \). Moreover, the ranks of \( Q_1 \) and \( Q_2 \) are each less than that of \( Q \). If \( x \) is a non-essential element of \( Q_1 \) that meets \( E \), then, by using Proposition 4.8, \( x \) is a non-essential element of \( Q_2 \). Thus each of \( Q_1 \) and \( Q_2 \) has \( p \) as a non-essential element and has exactly one other non-essential element. By induction, \( Q_1 \) and \( Q_2 \) satisfy the four conditions in the theorem. It follows immediately that the 2-sum of \( Q_1 \) and \( Q_2 \), that is \( Q \), satisfies the four conditions. \( \square \)

Acknowledgment

The author thanks James Oxley for suggesting the study of 2-polymatroids and for his valuable advice in the preparation of this paper.

References