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Quasi-convex density and determining subgroups of compact abelian groups

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A R T I C L E I N F O

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A B S T R A C T

For an abelian topological group G, let $\widehat{G}$ denote the dual group of all continuous characters endowed with the compact open topology. Given a closed subset X of an infinite compact abelian group G such that $w(X) < w(G)$, and an open neighborhood U of 0 in $\mathbb{T}$, we show that $|\{\chi \in \widehat{G}\colon\ \chi(X) \subseteq U\}| = |\widehat{G}|$. (Here, $w(G)$ denotes the weight of G.) A subgroup D of G determines G if the map $r\colon \widehat{G} \to \widehat{D}$ defined by $r(\chi) = \chi \restriction_D$ for $\chi \in \widehat{G}$, is an isomorphism between $\widehat{G}$ and $\widehat{D}$. We prove that

$$w(G) = \min\{|D|\colon\ D \text{ is a subgroup of } G \text{ that determines } G\}$$

for every infinite compact abelian group G. In particular, an infinite compact abelian group determined by a countable subgroup is metrizable. This gives a negative answer to a question of Comfort, Raczkowski and Trigos-Arrieta (repeated by Hernández, Macario and Trigos-Arrieta). As an application, we furnish a short elementary proof of the result from [S. Hernández, S. Macario, F.J. Trigos-Arrieta, Uncountable products of determined groups need not be determined, J. Math. Anal. Appl. 348 (2008) 834–842] that a compact abelian group G is metrizable provided that every dense subgroup of G determines G.

All topological groups are assumed to be Hausdorff, and all topological spaces are assumed to be Tychonoff. As usual, $\mathbb{R}$ denotes the group of real numbers (with the usual topology), $\mathbb{Z}$ denotes the group of integer numbers, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the circle group (with the usual topology), $\mathbb{N}$ denotes the set of natural numbers, ω denotes the first infinite cardinal, and $w(X)$ denotes the weight of a space X. If A is a subset of a space X, then $\overline{A}$ denotes the closure of A in X.

1. Introduction

For spaces X and Y, we denote by $C(X, Y)$ the space of all continuous functions from X to Y endowed with the *compact open topology*, that is, the topology generated by the family

$$\{[K, U]\colon\ K \text{ is a compact subset of } X \text{ and } U \text{ is an open subset of } Y\}$$

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as a subbase, where

$$[K, U] = \{g \in C(X, Y): \ g(K) \subseteq U\}. \tag{1}$$

For an abelian topological group G, we denote by $\widehat{G}$ the Pontryagin–van Kampen dual of G, namely the group of all continuous characters $\chi : G \to \mathbb{T}$ endowed with the compact open topology. Clearly, $\widehat{G}$ is a closed subgroup of $C(G, \mathbb{T})$. In particular, a base of neighborhoods of 0 in $\widehat{G}$ is given by the sets

$$W(K, U) = \{\chi \in \widehat{G}: \ \chi(K) \subseteq U\} = [K, U] \cap \widehat{G},$$

where K is a compact subset of G and U is an open neighborhood of 0 in $\mathbb{T}$.

We identify $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the real interval $(-1/2, 1/2]$ in the obvious way, and write

$$\mathbb{T}_+ = \{x \in \mathbb{T}: \ -1/4 \leqslant x \leqslant 1/4\}.$$

Definition 1.1. Let G be an abelian topological group.

(i) For $E \subseteq G$ and $A \subseteq \widehat{G}$, define the *polars*

$$E^{\triangleright} = \{\chi \in \widehat{G} \mid \chi(E) \subseteq \mathbb{T}_+\} \quad \text{and} \quad A^{\triangleleft} = \{x \in G \mid \chi(x) \in \mathbb{T}_+ \text{ for all } \chi \in A\}.$$

(ii) A set $E \subseteq G$ is said to be *quasi-convex* if $E = E^{\triangleright\triangleleft}$.
(iii) The *quasi-convex hull* $Q_G(E)$ of $E \subseteq G$ is the smallest quasi-convex subset of G containing E.
(iv) Following [7,8], we will say that $E \subseteq G$ is *qc-dense* (an abbreviation for *quasi-convexly dense*) provided that $Q_G(E) = G$, or equivalently, if $E^{\triangleright} = \{0\}$.

Obviously, $E \subseteq E^{\triangleright\triangleleft}$. Therefore, a set $E \subseteq G$ is quasi-convex if and only if for every $x \in G \setminus E$ there exists $\chi \in E^{\triangleright}$ such that $\chi(x) \notin \mathbb{T}_+$.

The notion of quasi-convexity was introduced by Vilenkin [17] as a natural counterpart for topological groups of the fundamental notion of convexity from the theory of topological vector spaces (we refer the reader to [1,2] for additional information).

The proof of the following fact is straightforward.

Fact 1.2. Let $f : G \to H$ be a continuous homomorphism of topological abelian groups such that $f(G)$ be dense in H. If a subset X of G is qc-dense in G, then $f(X)$ is qc-dense in H.

Definition 1.3. Following [5,6], we say that a subgroup D of an abelian topological group G *determines* G if the restriction homomorphism $r : \widehat{G} \to \widehat{D}$ (defined by $r(\chi) = \chi \restriction_D$ for $\chi \in \widehat{G}$) is an isomorphism between the topological groups $\widehat{G}$ and $\widehat{D}$.

This notion is relevant to extending the Pontryagin–van Kampen duality to non-locally compact groups [1,3]. Indeed, if G is locally compact and abelian, then every subgroup D that determines G must be dense in G; in particular, no proper locally compact subgroup of G can determine G. (We note that the original definition in [5,6] assumed upfront that D is dense in G.) When D is dense in G, the restriction homomorphism $r : \widehat{G} \to \widehat{D}$ is always a continuous isomorphism.

The ultimate connection between the notions of determined subgroup and qc-density is established in the next fact. This fact is a particular case of a more general fact stated without proof (and in equivalent terms) in [6, Remark 1.2(a)] and [12, Corollary 2.2].

Fact 1.4. For a subgroup D of a compact abelian group G the following conditions are equivalent:

(i) D determines G;
(ii) There exists a compact subset of D which is qc-dense in G.

Definition 1.5. According to [5,6], an abelian topological group G is said to be *determined* if every dense subgroup of G determines G.

Chasco [3, Theorem 2] and Außenhofer [1, Theorem 4.3] proved that all metrizable abelian groups are determined. Comfort, Raczkowski and Trigos-Arrieta established the following amazing inverse of this theorem for compact groups: Under the Continuum Hypothesis CH, every determined compact abelian group is metrizable ([5, Corollary 4.9] and [6, Corollary 4.17]). Quite recently, Hernández, Macario and Trigos-Arrieta removed the assumption of CH from their result [12, Corollary 5.11]. We note that this theorem becomes an immediate consequence of our main result, see Corollary 2.6.

Fact 1.6. (See [6, Corollary 3.15].) If $f : G \to H$ is a continuous surjective homomorphism of compact abelian groups and G is determined, then H is determined as well.

The following question remained the last principal unsolved problem in the theory of compact determined groups:

Question 1.7. (See [5, Question 7.1], [6, Question 7.1], [12, Question 5.12].)

(a) Is there a compact group G with a countable dense subgroup D such that $w(G) > \omega$ and D determines G?
(b) What if $G = \mathbb{T}^\kappa$?

We completely resolve this question in Corollary 2.5. In fact, we even solve the most general version of this question with ω replaced by an arbitrary cardinal, see Corollary 2.4.

2. Main results

Definition 2.1. If X is a subset of a compact abelian group G, then $r_X^G : \widehat{G} \to C(X, \mathbb{T})$ denotes the "restriction map" defined by $r_X^G(\chi) = \chi \restriction_X$ for $\chi \in \widehat{G}$.

Observe that $C(X, \mathbb{T})$ is a topological group and r_X^G is a continuous group homomorphism.

Theorem 2.2. *Let X be a closed subset of an infinite compact abelian group G such that $w(X) < w(G)$. Then for every open neighborhood U of 0 in $\mathbb{T}$ one has $|W(X, U)| = |\widehat{G}|$.*

The proof of Theorem 2.2 is postponed until Section 3.

Corollary 2.3. *If a closed subspace X of an infinite compact abelian group G is qc-dense in G, then $w(X) = w(G)$.*

Proof. Let U be an open neighborhood of 0 in $\mathbb{T}$ such that $U \subseteq \mathbb{T}_+$. Since X is qc-dense in G, we have $W(X, U) \subseteq X^\triangleright = \{0\}$. Now Theorem 2.2 yields $w(X) \geqslant w(G)$. The reverse inequality $w(X) \leqslant w(G)$ is trivial. □

Our next corollary constitutes a major breakthrough in the theory of compact determined groups.

Corollary 2.4. *If a subgroup D of an infinite compact abelian group G determines G, then $|D| \geqslant w(G)$.*

Proof. According to Fact 1.4, D contains a compact subset X that is qc-dense in G, so $|D| \geqslant |X| \geqslant w(X)$ (see, for example, [10, Theorem 3.1.21]). Finally, $w(X) = w(G)$ by Corollary 2.3. □

Even the particular case of Corollary 2.4 provides a complete answer to Question 1.7:

Corollary 2.5. *A compact abelian group determined by a countable subgroup is metrizable.*

Corollary 2.6. *(See [12, Corollary 5.11].) Every determined compact abelian group is metrizable.*

Proof. Assume that G is a non-metrizable determined compact abelian group. Then $w(G) \geqslant \omega_1$, and so we can find a continuous surjective group homomorphism $h : G \to K = H^{\omega_1}$, where H is either $\mathbb{T}$ or $\mathbb{Z}(p)$ for some prime number p (see, for example, [6, Theorem 5.15 and Discussion 4.14]). As a continuous homomorphic image of the determined group G, the group K is determined by Fact 1.6. Since K is separable (see, for example, [10, Theorem 2.3.15]), there exists a countable dense subgroup D of K. Since K is determined, we conclude that D must determine K. Therefore, K must be metrizable by Corollary 2.5, a contradiction. □

Useful properties of determined groups can be found in [4].

A *super-sequence* is a non-empty compact Hausdorff space X with at most one non-isolated point x^* [9]. When X is infinite, we will call x^* the *limit* of X and say that X *converges to* x^*. Observe that a convergent sequence is a countably infinite super-sequence.

Being an immediate consequence of [1, Theorem 4.3 or Corollary 4.4], the following result is essentially due to Außenhofer:

Fact 2.7. (See [1].) Every dense subgroup D of an infinite compact metric abelian group G contains a sequence converging to 0 that is qc-dense in G.

In particular, every infinite compact metric abelian group has a qc-dense sequence converging to 0. Our next theorem extends this result to all compact abelian groups by replacing converging sequences with super-sequences.

Theorem 2.8. *Every infinite compact abelian group contains a qc-dense super-sequence converging to* 0.

The proof of Theorem 2.8 is postponed until Section 5.

Corollary 2.9. *Every infinite compact abelian group G has a (dense) subgroup D which determines G such that $|D| \leqslant w(G)$.*

Proof. Apply Theorem 2.8 to find a super-sequence X that is qc-dense in G. Let D be the subgroup of G generated by X. Clearly, $|X| = w(X) \leqslant w(G)$. Since G is infinite, $w(G)$ must be infinite, and therefore $|D| \leqslant \omega + |X| \leqslant w(G)$. Finally, D determines G by Fact 1.4. □

Our next corollary provides another major advance in the theory of compact determined groups:

Corollary 2.10. *If G is an infinite compact abelian group, then*

$$w(G) = \min\{|D|:\ D \text{ is a subgroup of } G \text{ that determines } G\}.$$

Proof. Combine Corollaries 2.4 and 2.9. □

We have been kindly informed by Chasco that our next corollary was independently proved by Bruguera and Tkachenko:

Corollary 2.11. *Every infinite compact abelian group G contains a proper (dense) subgroup D which determines G.*

Proof. Let D be a subgroup of G as in the conclusion of Corollary 2.9. Since G is an infinite compact group, we have $|D| \leqslant w(G) < 2^{w(G)} = |G|$. Therefore, D must be a proper subgroup of G. □

Remark 2.12. A common strengthening of Fact 2.7 and Theorem 2.8 is impossible. Indeed, *every non-metrizable compact abelian group G contains a dense subgroup D such that no super-sequence $S \subseteq D$ is qc-dense in G*. To see this, apply Corollary 2.6 to get a dense subgroup D of G that does not determine G, and then notice that any super-sequence $S \subseteq D$ (being compact) cannot be qc-dense in G by Fact 1.4.

3. Proof of Theorem 2.2

Fact 3.1. (See [10, Proposition 3.4.16].) If X is a compact space and Y is a space, then $w(C(X,Y)) \leqslant w(X) + w(Y) + \omega$.

Proof of Theorem 2.2. Consider first the case when $w(X) < \omega$. Then X must be finite. Note that the set $W(X,U)$ is an open neighborhood of 0 in the initial topology $\mathcal{T}$ of $\widehat{G}$ with respect to the family $\{\eta_x:\ x \in X\}$ of evaluation characters $\eta_x : \widehat{G} \to \mathbb{T}$ defined by $\eta_x(\pi) = \pi(x)$ for every $\pi \in \widehat{G}$. Since topologies generated by characters are totally bounded, finitely many translates of $W(X,U)$ cover the whole group $\widehat{G}$. Since $\widehat{G}$ is infinite, this yields $|W(X,U)| = |\widehat{G}|$.

From now on we will assume that $w(X) \geqslant \omega$. The inequality $|W(X,U)| \leqslant |\widehat{G}|$ being trivial, it suffices to check that $|\widehat{G}| \leqslant |W(X,U)|$.

Let r_X^G be the map from Definition 2.1, and let $H = r_X^G(\widehat{G})$. Note that $\ker r_X^G \subseteq W(X,U)$, so $|\ker r_X^G| \leqslant |W(X,U)|$. If $|\ker r_X^G| = |\widehat{G}|$, we are done. Assume now that $|\ker r_X^G| < |\widehat{G}|$. Since $\widehat{G}$ is infinite, we obtain

$$|\widehat{G}| = \left|\widehat{G}/\ker r_X^G\right| = \left|r_X^G(\widehat{G})\right| = |H|. \tag{2}$$

Let N be the subgroup of H generated by the open subset $[X,U] \cap H$ of H. Then N is a clopen subgroup of H, so the index of N in H cannot exceed $w(H)$, which gives

$$|H| = |N| + |H/N| \leqslant |N| + w(H) \leqslant \left|[X,U] \cap H\right| + \omega + w(H). \tag{3}$$

Since $w(H) \leqslant w(C(X,\mathbb{T})) \leqslant w(X) + \omega$ by Fact 3.1, and $w(X) + \omega = w(X)$ by our assumption, we obtain from (3) that

$$|H| \leqslant \left|[X,U] \cap H\right| + w(X). \tag{4}$$

As $w(X) < w(G) = |\widehat{G}| = |H|$ by (2), and $|H| = |\widehat{G}| \geqslant \omega$, from (4) it follows that

$$\left|[X,U] \cap H\right| = |H|. \tag{5}$$

Finally, note that $[X,U] \cap H = r_X^G(W(X,U))$, which yields that

$$\left|[X,U] \cap H\right| = \left|r_X^G\big(W(X,U)\big)\right| \leqslant \left|W(X,U)\right|. \tag{6}$$

Combining (2), (5) and (6), we obtain the inequality $|\widehat{G}| \leqslant |W(X,U)|$. □

4. Characterization of qc-dense subsets and determining subgroups of compact abelian groups in terms of $C(X,\mathbb{T})$

Lemma 4.1. *Suppose that U is an open neighborhood of 0 in $\mathbb{T}$ and X is a compact subset of a compact abelian group G such that $W(X,U)=\{0\}$. Then:*

(i) *There exists $n\in\mathbb{N}$ such that the sum*
$$K_n=\big(X\cup\{0\}\big)+\big(X\cup\{0\}\big)+\cdots+\big(X\cup\{0\}\big)$$
of n many copies of the set $X\cup\{0\}$ is qc-dense in G;
(ii) *The subgroup of G generated by X must be dense in G.*

Proof. (i) There exists $n\in\mathbb{N}$ such that
$$V_n=\{x\in\mathbb{T}\colon\ kx\in\mathbb{T}_+ \text{ for all } k=1,2,\dots,n\}\subseteq U. \tag{7}$$

Let $\chi\in K_n^{\triangleright}$. Fix $x\in X$. Let $k=1,2,\dots,n$ be arbitrary. Since $0\in X\cup\{0\}$, one has $kx\in K_n$, and so $k\chi(x)=\chi(kx)\in\mathbb{T}_+$. This yields $\chi(x)\in V_n\subseteq U$ by (7). Since $x\in X$ was chosen arbitrarily, it follows that $\chi\in W(X,U)$. Since $W(X,U)=\{0\}$, this gives $\chi=0$. Therefore, $K_n^{\triangleright}=\{0\}$, and so K_n is qc-dense in G.

(ii) Suppose that the smallest subgroup N of G containing X is not dense in G. Then we can choose $\chi\in\widehat{G}$ such that $\chi(\overline{N})=\{0\}$ and $\chi(y)\neq 0$ for some $y\in G\setminus\overline{N}$. So $\chi\in W(X,U)$ and yet $\chi\neq 0$, in contradiction with our assumption. □

We refer the reader to Definition 2.1 for the notation used in item (ii) of our next theorem.

Theorem 4.2. *For a closed subset X of a compact abelian group G the following conditions are equivalent*:

(i) *$W(X,U)=\{0\}$ for some open neighborhood U of 0 in $\mathbb{T}$;*
(ii) *r_X^G is an isomorphism between the topological groups $\widehat{G}$ and $H=r_X^G(\widehat{G})$.*

Proof. (i) → (ii) Let U be as in (i). Since $\ker r_X^G\subseteq W(X,U)=\{0\}$, we conclude that r_X^G is an injection. Since X is compact,
$$\big\{r_X^G(0)\big\}=r_X^G\big(\{0\}\big)=r_X^G\big(W(X,U)\big)=H\cap\big\{g\in C(X,\mathbb{T})\colon\ g(X)\subseteq U\big\}$$
is an open subset of H. Since H is a subgroup of $C(X,\mathbb{T})$, we conclude that H is discrete. Therefore, r_X^G is an open map onto its image.

(ii) → (i) The assumption from item (ii) implies that H is a discrete subgroup of $C(X,\mathbb{T})$. Since X is compact, we can find an open neighborhood U of 0 in $\mathbb{T}$ such that $H\cap[X,U]=\{r_X^G(0)\}$. This yields $W(X,U)=\{0\}$. □

Corollary 4.3. *If a closed subset X of a compact abelian group G is qc-dense in G, then r_X^G is an isomorphism between the topological groups $\widehat{G}$ and $r_X^G(\widehat{G})$.*

Proof. Choose an open neighborhood U of 0 with $U\subseteq\mathbb{T}_+$. Since X is qc-dense in G, we have $W(X,U)\subseteq X^{\triangleright}=\{0\}$, and so we can apply Theorem 4.2. □

Corollary 4.4. *Let X be a closed subset of a compact abelian group G such that r_X^G is an isomorphism between the topological groups $\widehat{G}$ and $r_X^G(\widehat{G})$. Then there exists $n\in\mathbb{N}$ such that the sum*
$$K_n=\big(X\cup\{0\}\big)+\big(X\cup\{0\}\big)+\cdots+\big(X\cup\{0\}\big)$$
of n many copies of the set $X\cup\{0\}$ is (compact and) qc-dense in G.

Proof. Apply Theorem 4.2 to find an open neighborhood U of 0 in $\mathbb{T}$ such that $W(X,U)=\{0\}$. Then apply Lemma 4.1(i) to obtain the required $n\in\mathbb{N}$. □

Corollary 4.5. *For a subgroup D of a compact abelian group G the following conditions are equivalent*:

(i) *D determines G*;
(ii) *There exists a compact set $X\subseteq D$ such that r_X^G is an isomorphism between the topological groups $\widehat{G}$ and $r_X^G(\widehat{G})$.*

Proof. (i) → (ii) Since D determines G, there exists a compact set $X\subseteq D$ which is qc-dense in G (Fact 1.4). Then r_X^G is an isomorphism between the topological groups $\widehat{G}$ and $r_X^G(\widehat{G})$ (Corollary 4.3).

(ii) → (i) Let X be as in item (ii). Apply Corollary 4.4 to get $n\in\mathbb{N}$ and K_n as in the conclusion of this corollary. Clearly, K_n is a compact subset of D. Since K_n is qc-dense in G, D determines G by Fact 1.4. □

5. Proof of Theorem 2.8

The following definition is an adaptation to the abelian case of [9, Definition 4.5]:

Definition 5.1. Let $\{G_i\colon i \in I\}$ be a family of abelian topological groups. For every $i \in I$, let X_i be a subset of G_i. Identifying each G_i with a subgroup of the direct product $G = \prod_{i\in I} G_i$ in the obvious way, define $X = \bigcup_{i\in I} X_i \cup \{0\}$, where 0 is the zero element of G. We will call X the *fan* of the family $\{X_i\colon i \in I\}$ and will denote it by $\mathrm{fan}_{i\in I}(X_i, G_i)$.

The proof of the following lemma is straightforward.

Lemma 5.2. *Let $\{G_i\colon i \in I\}$ be a family of abelian topological groups. For every $i \in I$, let X_i be a sequence converging to 0 in G_i. Then $\mathrm{fan}_{i\in I}(X_i, G_i)$ is a super-sequence in $G = \prod_{i\in I} G_i$ converging to 0.*

Lemma 5.3. *Let $\{G_i\colon i \in I\}$ be a family of abelian topological groups. For each $i \in I$ let X_i be a qc-dense subset of G_i. Then $X = \mathrm{fan}_{i\in I}(X_i, G_i)$ is qc-dense in $G = \prod_{i\in I} G_i$.*

Proof. Let $\chi : G \to \mathbb{T}$ be a non-trivial continuous character. There exist a non-empty finite subset J of I and a family $\{\chi_j \in \widehat{G}_j\colon j \in J\}$ such that $\chi(g) = \sum_{j\in J} \chi_j(g(j))$ for $g \in G$. Since $J \neq \emptyset$, we can fix $j_0 \in J$. Since X_{j_0} is qc-dense in G_{j_0}, there exists $x \in X_{j_0} \subseteq X$ such that $\chi_{j_0}(x) \notin \mathbb{T}_+$. Finally, note that

$$\chi(x) = \sum_{j\in J} \chi_j\big(x(j)\big) = \chi_{j_0}\big(x(j_0)\big) + \sum_{j\in J\setminus\{j_0\}} \chi_j\big(x(j)\big) = \chi_{j_0}(x) + \sum_{j\in J\setminus\{j_0\}} \chi_j(0) = \chi_{j_0}(x) \notin \mathbb{T}_+.$$

Therefore, $\chi \notin X^{\triangleright}$. This gives $X^{\triangleright} = \{0\}$, and so X is qc-dense in G. □

$\mathbb{P}$ denotes the set of prime numbers, and for $p \in \mathbb{P}$, the symbol $\mathbb{Z}_p$ denotes the group of p-adic integers. We denote by $\mathbb{Q}$ the group of the rational numbers equipped with the discrete topology. The next lemma is probably known, but we include its proof for the reader's convenience.

Lemma 5.4. *Every infinite compact abelian group of weight κ is isomorphic to a quotient group of the group $\widehat{\mathbb{Q}}^{\kappa} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p^{\kappa}$.*

Proof. Let H be an infinite compact abelian group such that $w(H) = \kappa$. Clearly, κ is infinite and $X = \widehat{H}$ is a discrete abelian group of size κ [13, Theorem (24.15)]. Let $Y = X \oplus \bigoplus_{\kappa}(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})$. By [11, Theorem 24.2] there exists a divisible abelian group D containing Y such that no proper subgroup of D containing Y is divisible. According to the text immediately following [11, Theorem 24.2], $r_0(D) = r_0(Y)$ and $r_p(D) = r_p(Y)$ for every prime p, where $r_0(N)$ and $r_p(N)$ denote the free-rank and the p-rank of an abelian group N, respectively (see, for example, [11, §16]). Since $r_0(Y) = \kappa$ and $r_p(Y) = \kappa$ for every prime p, we conclude that

$$D \cong \bigoplus_{\kappa}(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}) \cong \Big(\bigoplus_{\kappa} \mathbb{Q}\Big) \oplus \bigoplus_{p\in\mathbb{P}} \Big(\bigoplus_{\kappa} \mathbb{Z}(p^{\infty})\Big) \tag{8}$$

by the structure theorem for divisible abelian groups (see [11, Theorem 23.1]). Consider the compact group $G = \widehat{D}$. By (8), $G \cong \widehat{\mathbb{Q}}^{\kappa} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p^{\kappa}$. According to [13, Theorem (24.5)], $H \cong \widehat{X} \cong G/X^{\perp}$, where $X^{\perp} = \{\chi \in \widehat{D}\colon \chi(X) = \{0\}\}$. Therefore, H is isomorphic to a quotient group of $\widehat{\mathbb{Q}}^{\kappa} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p^{\kappa} \cong G$. □

Proof of Theorem 2.8. Let H be an infinite compact abelian group. Define $\kappa = w(H)$ and $G = \widehat{\mathbb{Q}}^{\kappa} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p^{\kappa}$. By Lemma 5.4 there exists a surjective continuous homomorphism $f : G \to H$. Clearly, $G = \prod_{i\in I} G_i$, where $|I| = \kappa$ and each G_i is either $\widehat{\mathbb{Q}}$ or $\mathbb{Z}_p$ for a suitable $p \in \mathbb{P}$. By Fact 2.7, for every $i \in I$ there exists a sequence X_i converging to 0 which is qc-dense in G_i. Applying Lemmas 5.2 and 5.3, we conclude that $X = \mathrm{fan}_{i\in I}(X_i, G_i)$ is a super-sequence in G converging to 0 such that X is qc-dense in G. Since $f : G \to H$ is a surjection, $S = f(X)$ is qc-dense in H by Fact 1.2. Being the image of a super-sequence X in G converging to 0, S is super-sequence in H converging to 0 [9, Fact 4.3]. Finally, $|S| \leqslant |X| \leqslant \omega \cdot |I| = \kappa$. □

6. Final remarks

A subspace X of a topological group G *topologically generates* G if G is the smallest closed subgroup of G that contains X.

Remark 6.1.

(i) Item (ii) of Lemma 4.1 can be restated as follows: *A qc-dense subset of a compact abelian group G topologically generates G*. Therefore, for a subset X of a compact abelian group G, one has the following implications:

$$X \text{ is dense in } G \longrightarrow X \text{ is qc-dense in } G \longrightarrow X \text{ topologically generates } G. \tag{9}$$

(ii) The first arrow in (9) cannot be reversed. Let S be any qc-dense sequence S in $\mathbb{T}$ (for example, the sequence $S = \{\frac{1}{2n}\colon n \in \mathbb{N}\}$ is *qc*-dense in $\mathbb{T}$; other examples can be found in [8]). Clearly, S is not dense in $\mathbb{T}$.

(iii) The last arrow in (9) cannot be reversed either. Indeed, it follows from the results in [9] that $\mathbb{T}^{\mathfrak{c}}$ contains a converging sequence (i.e., countably infinite super-sequence) topologically generating $\mathbb{T}^{\mathfrak{c}}$. This sequence, however, cannot be qc-dense in $\mathbb{T}^{\mathfrak{c}}$ by Corollary 2.3.

Let G be a topological group with the identity e. If a discrete subset X of G topologically generates G and $X \cup \{e\}$ is closed in G, then X is called a *suitable set for G* [14]. Hofmann and Morris proved the following fundamental theorem.

Fact 6.2. (See [14,15].) Every locally compact group G has a suitable set.

See also [16] for a "purely topological" proof of this result based on Michael's selection theorem.

Remark 6.3.

(i) Clearly, if S is a super-sequence in a topological group G that converges to e and topologically generates G, then $S \setminus \{e\}$ is a suitable set for G.

(ii) Let G be a compact abelian group. If G is finite, then $G \setminus \{0\}$ is obviously a suitable set for G. When G is infinite, Theorem 2.8 guarantees the existence of a qc-dense super-sequence S in G converging to 0. By the emphasized text in Remark 6.1(i), S topologically generates G. Applying item (i), we conclude that $S \setminus \{0\}$ is a suitable set for G. This argument shows that *Theorem* 2.8 *implies the particular case of Fact* 6.2 *for compact abelian groups G*.

(iii) A (super-)sequence S topologically generating a compact abelian group G need not be qc-dense in G; see Remark 6.1(iii). Since $X = S \setminus \{0\}$ is a suitable set for G by item (i), it follows that a suitable set X for a compact abelian group G need not be qc-dense in G. Therefore, *Fact* 6.2 *does not imply Theorem* 2.8.

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