

On the enumeration of rooted trees with fixed size of maximal decreasing trees

Seunghyun Seo^a, Heesung Shin^{b,*}

^a Kangwon National University, Republic of Korea

^b Inha University, Republic of Korea

ARTICLE INFO

Article history:

Received 24 June 2011

Received in revised form 9 September 2011

Accepted 3 October 2011

Available online 26 October 2011

Keywords:

Rooted labeled trees

Refinement

Generating functions

Maximal decreasing subtrees

Bijections

ABSTRACT

Let \mathcal{T}_n be the set of rooted labeled trees on $\{0, \dots, n\}$. A maximal decreasing subtree of a rooted labeled tree is defined by the maximal subtree from the root with all edges being decreasing. In this paper, we study a new refinement $\mathcal{T}_{n,k}$ of \mathcal{T}_n , which is the set of rooted labeled trees whose maximal decreasing subtree has $k + 1$ vertices.

© 2011 Elsevier B.V. All rights reserved.

1. Introduction

For a nonnegative integer n , let \mathcal{T}_n be the set of rooted labeled trees on $[0, n] := \{0, \dots, n\}$. For a given rooted labeled tree T , a *maximal decreasing subtree* of T is defined by the maximal subtree from the root with all edges being decreasing, denoted by $MD(T)$. Fig. 1 illustrates the maximal decreasing subtree of a given tree T . Let $\mathcal{T}_{n,k}$ be the set of rooted labeled trees in \mathcal{T}_n whose maximal decreasing subtree has $k + 1$ vertices.

Within the scope of proven research, a maximal decreasing subtree first appeared in the paper [2] of Chauve, Dulucq, and Guibert, for constructing the bijection between $\mathcal{T}_{n,0}$ and the set of trees in \mathcal{T}_n with n being a leaf. Recently, Bergeron and Livernet [1] introduced it in order to analyze the free Lie algebra based on rooted labeled trees. None of them mentioned, however, the refined set $\mathcal{T}_{n,k}$ or considered the enumeration of $\mathcal{T}_{n,k}$.

In Section 2, we shall count the number of elements in $\mathcal{T}_{n,k}$. We shall also introduce a set of certain functions on $[n]$, which is equinumerous with $\mathcal{T}_{n,k}$. In Section 3, we shall decompose a rooted labeled tree into rooted subtrees, each maximal decreasing subtree of which is a single vertex. Then some formulas related to $|\mathcal{T}_{n,k}|$ are given from this decomposition. In Section 4, using the inverse of the matrix $\left[\binom{i+j}{j} \right]_{0 \leq i, j \leq n}$, $\mathcal{T}_{n,k}$ can be expressed as a linear combination of $\{(n+1)^n, (n+2)^n, \dots, (2n+1)^n\}$. In the last section, we discuss bijective proofs of our results.

2. The main results

First of all, let us count the number of elements in the set $\mathcal{T}_{n,k}$.

* Corresponding author.

E-mail addresses: shyunseo@kangwon.ac.kr (S. Seo), shin@inha.ac.kr (H. Shin).

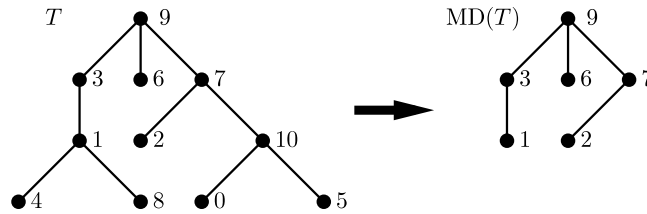


Fig. 1. The maximal decreasing subtree of T from the root 9.

Theorem 1. For nonnegative integers n and k , we have

$$|\mathcal{T}_{n,k}| = \sum_{m=k}^n \binom{n+1}{m+1} S(m+1, k+1) k! (n-k)^{n-m-1} (m-k),$$

where $S(n, k)$ is a Stirling number of the second kind.

Proof. Given a rooted labeled tree T , let V_1 be the union of the set of vertices in $MD(T)$ and the set of children of any vertex in $MD(T)$. Now, we will count the number of rooted labeled trees $T \in \mathcal{T}_{n,k}$ with $|V_1| = m + 1$.

First of all, the number of ways of selecting V_1 is equal to $\binom{n+1}{m+1}$. Make a partition of V_1 into $k + 1$ blocks, namely, B_1, \dots, B_{k+1} . The number of such partitions is equal to $S(m + 1, k + 1)$. Take the set V_0 consisting of the minimum m_i of each block B_i . Make a decreasing subtree on $V_0 = \{m_1, \dots, m_{k+1}\}$. Since it is well-known that the number of (unordered) increasing trees on $k + 1$ nodes is $k!$, there are exactly $k!$ ways of making a decreasing subtree on V_0 . Append vertices in $V_1 \setminus V_0$ to this decreasing subtree such that elements in $B_i \setminus \{m_i\}$ are children of m_i for $i = 1, \dots, k + 1$. It is well-known that the number of forests F on $[0, n] \setminus V_0$ such that $V_1 \setminus V_0$ is the set of all roots of F is equal to $(n - k)^{n-m-1} (m - k)$ (see [4, Prop. 5.3.2]). Since the range of m is $k \leq m \leq n$,

$$|\mathcal{T}_{n,k}| = \sum_{m=k}^n \binom{n+1}{m+1} S(m+1, k+1) k! (n-k)^{n-m-1} (m-k). \quad \square$$

For a nonnegative integer n , let \mathcal{F}_n be the set of functions from $[n]$ to $[n]$, where $[n] := \{1, \dots, n\}$ if n is a positive integer and $[0] := \emptyset$. Let $\mathcal{F}_{n,k}$ be the set of functions $f \in \mathcal{F}_n$ with $[k] \subset f([n])$, where $f([n])$ is the image of f .

Proposition 2. For nonnegative integers n and k , we have

$$|\mathcal{F}_{n,k}| = \sum_{m=k}^n \binom{n}{m} S(m, k) k! (n-k)^{n-m} \tag{1}$$

$$= \sum_{i \geq 0} (-1)^i \binom{k}{i} (n-i)^n. \tag{2}$$

Proof. Let $f^{-1}([k]) = A$ and $|A| = m$. The number of subsets A of $[n]$ of size m is equal to $\binom{n}{m}$. The function f can be decomposed into a surjection from A to $[k]$ in $S(m, k) k!$ ways and a function from $[n] \setminus A$ to $[n] \setminus [k]$ in $(n - k)^{n-m}$ ways. Since m runs through from k to n , the formula (1) holds.

Meanwhile, defining A_j by the set $\{f \in \mathcal{F}_n \mid f^{-1}(j) = \emptyset\}$, we have

$$\mathcal{F}_{n,k} = \mathcal{F}_n \setminus (A_1 \cup \dots \cup A_k).$$

By the principle of inclusion and exclusion, we have

$$|\mathcal{F}_{n,k}| = |\mathcal{F}_n| - |A_1 \cup \dots \cup A_k| = \sum_{I \subset [k]} (-1)^{|I|} (n - |I|)^n.$$

So, the formula (2) holds. \square

Theorem 3. For nonnegative integers n and k , we have $|\mathcal{T}_{n,k}| = |\mathcal{F}_{n,k}|$, i.e.,

$$\sum_{m=k}^n \binom{n+1}{m+1} S(m+1, k+1) k! (n-k)^{n-m-1} (m-k) = \sum_{i=0}^k (-1)^i \binom{k}{i} (n-i)^n. \tag{3}$$

Proof. Since $S(m + 1, k + 1)(k + 1)!$ is the number of surjective functions from $[m + 1]$ to $[k + 1]$, which is equal to $\sum_{j \geq 0} (-1)^j \binom{k+1}{j} (k + 1 - j)^{m+1}$ by the principle of inclusion and exclusion, it follows that

$$\begin{aligned} |\mathcal{T}_{n,k}| &= \sum_m \binom{n+1}{m+1} [S(m+1, k+1)k!] (n-k)^{n-m-1}(m-k) \\ &= \sum_m \binom{n+1}{m+1} \left[\frac{1}{k+1} \sum_{j \geq 0} (-1)^j \binom{k+1}{j} (k+1-j)^{m+1} \right] (n-k)^{n-m-1}(m-k). \end{aligned}$$

Separating the term $m - k$ to $(n - k) - (n - m)$ and changing the order of summations, we get

$$\begin{aligned} |\mathcal{T}_{n,k}| &= \sum_j \frac{(-1)^j}{k+1} \binom{k+1}{j} \sum_m \binom{n+1}{m+1} (k+1-j)^{m+1} (n-k)^{n-m} \\ &\quad - \sum_j \frac{(-1)^j}{k+1} \binom{k+1}{j} \sum_m (n+1) \binom{n}{m+1} (k+1-j)^{m+1} (n-k)^{n-m-1}. \end{aligned}$$

By the binomial theorem,

$$\begin{aligned} |\mathcal{T}_{n,k}| &= \sum_j \frac{(-1)^j}{k+1} \binom{k+1}{j} [(n+1-j)^{n+1} - (n+1)(n+1-j)^n] \\ &= \sum_j \frac{(-1)^j}{k+1} \binom{k+1}{j} (-j)(n+1-j)^n. \end{aligned}$$

Substituting $j = i + 1$ in the previous equation, the formula (3) holds. \square

3. Properties

A rooted labeled tree T is called a *local minimum tree* if $MD(T)$ consists of a single vertex. Note that $\mathcal{T}_{n,0}$ is the set of local minimum trees on $[0, n]$ and $|\mathcal{T}_{n,0}|$ is equal to n^n [2]. Also $\mathcal{T}_{n,n}$ is the set of decreasing trees on $[0, n]$ and $|\mathcal{T}_{n,n}|$ is equal to $n!$.

Given $T \in \mathcal{T}_{n,k}$, we can decompose T into $k + 1$ local minimum trees by removing k edges in $MD(T)$. This decomposition yields the following lemma.

Lemma 4. For nonnegative integers n and k , we have

$$|\mathcal{T}_{n,k}| = \frac{1}{k+1} \sum_{n_1 + \dots + n_{k+1} = n-k} \binom{n+1}{n_1+1, \dots, n_{k+1}+1} n_1^{n_1} \dots n_{k+1}^{n_{k+1}}.$$

Proof. It is enough to show the following formula:

$$|\mathcal{T}_{n,k}| = k! \left(\frac{1}{(k+1)!} \sum_{n_1 + \dots + n_{k+1} = n-k} \binom{n+1}{n_1+1, \dots, n_{k+1}+1} |\mathcal{T}_{n_1,0}| \dots |\mathcal{T}_{n_{k+1},0}| \right). \tag{4}$$

First of all, we will construct a tuple (T_1, \dots, T_{k+1}) of local minimum trees satisfying two conditions:

- (i) the tuple $(L(T_1), \dots, L(T_{k+1}))$ is an ordered partition of the set $[0, n]$ and
- (ii) $root(T_1) < root(T_2) < \dots < root(T_{k+1})$,

where $L(T)$ means a set of labels of vertices in T and $root(T)$ a label of the root of T .

Consider a tuple (S_1, \dots, S_{k+1}) of local minimum trees with the only condition (i). For a given sequence n_1, \dots, n_{k+1} of nonnegative integers with $n_1 + \dots + n_{k+1} = n - k$, the number of tuples (S_1, \dots, S_{k+1}) with $|L(S_i)| = n_i + 1$ is equal to

$$\binom{n+1}{n_1+1, \dots, n_{k+1}+1} |\mathcal{T}_{n_1,0}| \dots |\mathcal{T}_{n_{k+1},0}|.$$

So the number of all tuples (S_1, \dots, S_{k+1}) with the condition (i) is equal to

$$\sum_{n_1 + \dots + n_{k+1} = n-k} \binom{n+1}{n_1+1, \dots, n_{k+1}+1} |\mathcal{T}_{n_1,0}| \dots |\mathcal{T}_{n_{k+1},0}|.$$

From the condition (ii), the number of all tuples (T_1, \dots, T_{k+1}) is equal to

$$\frac{1}{(k+1)!} \sum_{n_1+\dots+n_{k+1}=n-k} \binom{n+1}{n_1+1, \dots, n_{k+1}+1} |\mathcal{T}_{n_1,0}| \cdots |\mathcal{T}_{n_{k+1},0}|.$$

Since the number of decreasing subtrees on $\{\text{root}(T_1), \dots, \text{root}(T_{k+1})\}$ is $k!$, we get the formula (4). \square

From Lemma 4, we can deduce the following result.

Theorem 5. We have three exponential generating functions:

$$1 + \sum_{n \geq 0} \sum_{k=0}^n |\mathcal{T}_{n,k}| \frac{t^{k+1}}{k!} \frac{x^{n+1}}{(n+1)!} = \exp \left(t \sum_{n \geq 0} n^n \frac{x^{n+1}}{(n+1)!} \right), \tag{5}$$

$$1 + \sum_{n \geq 0} \sum_{k=0}^n |\mathcal{T}_{n,k}| (k+1)t^{k+1} \frac{x^{n+1}}{(n+1)!} = \left(1 - t \sum_{n \geq 0} n^n \frac{x^{n+1}}{(n+1)!} \right)^{-1}, \tag{6}$$

$$\sum_{n \geq 0} \sum_{k=0}^n |\mathcal{T}_{n,k}| t^{k+1} \frac{x^{n+1}}{(n+1)!} = -\ln \left(1 - t \sum_{n \geq 0} n^n \frac{x^{n+1}}{(n+1)!} \right). \tag{7}$$

Proof. From Lemma 4, the left hand sides of the three formulas become

$$1 + \sum_{n \geq 0} \sum_{k=0}^n \sum_{n_i} \binom{n+1}{n_1+1, \dots, n_{k+1}+1} n_1^{n_1} \cdots n_{k+1}^{n_{k+1}} \frac{t^{k+1}}{(k+1)!} \frac{x^{n+1}}{(n+1)!}, \tag{8}$$

$$1 + \sum_{n \geq 0} \sum_{k=0}^n \sum_{n_i} \binom{n+1}{n_1+1, \dots, n_{k+1}+1} n_1^{n_1} \cdots n_{k+1}^{n_{k+1}} t^{k+1} \frac{x^{n+1}}{(n+1)!}, \tag{9}$$

$$\sum_{n \geq 0} \sum_{k=0}^n \sum_{n_i} \binom{n+1}{n_1+1, \dots, n_{k+1}+1} n_1^{n_1} \cdots n_{k+1}^{n_{k+1}} \frac{t^{k+1}}{k+1} \frac{x^{n+1}}{(n+1)!}, \tag{10}$$

where n_i means $n_1 + \dots + n_{k+1} = n - k$. Using the compositional formula for exponential structures [4, Theorem 5.5.4], the three formulas are of the form $F(tG(x))$ where

$$G(x) = \sum_{n \geq 0} n^n \frac{x^{n+1}}{(n+1)!}.$$

In the case of (8), the corresponding $F(x)$ is given by

$$F(x) = 1 + \sum_{i \geq 0} \frac{(i+1)!}{(i+1)!} \frac{x^{i+1}}{(i+1)!} = \exp(x).$$

In the case of (9), the corresponding $F(x)$ is given by

$$F(x) = 1 + \sum_{i \geq 0} \frac{(i+1)!}{1} \frac{x^{i+1}}{(i+1)!} = \frac{1}{1-x}.$$

In the case of (10), the corresponding $F(x)$ is given by

$$F(x) = \sum_{i \geq 0} \frac{(i+1)!}{(i+1)} \frac{x^{i+1}}{(i+1)!} = \ln \frac{1}{1-x}.$$

These complete the proof. \square

By the definition of $\mathcal{T}_{n,k}$, we have

$$\sum_{k=0}^n |\mathcal{T}_{n,k}| = (n+1)^n, \tag{11}$$

which can also be induced from $t = 1$ in (7). Similarly, putting $t = 1$ in (6) and applying the equation (5.67) in [4], we get

$$\begin{aligned} 1 + \sum_{n \geq 0} \sum_{k=0}^n |\mathcal{T}_{n,k}| (k+1) \frac{x^{n+1}}{(n+1)!} &= \left(1 - \sum_{n \geq 0} n^n \frac{x^{n+1}}{(n+1)!} \right)^{-1} \\ &= 1 + \sum_{n \geq 0} (n+2)^n \frac{x^{n+1}}{(n+1)!}. \end{aligned}$$

Thus we have

$$\sum_{k=0}^n (k+1) |\mathcal{T}_{n,k}| = (n+2)^n. \tag{12}$$

From (11) and (12), we are able to deduce the following.

Theorem 6. For a nonnegative integers n, k , and α , we have

$$\sum_{k=0}^n \binom{k+\alpha}{\alpha} |\mathcal{T}_{n,k}| = (n+1+\alpha)^n. \tag{13}$$

Proof. Since we have proved $|\mathcal{T}_{n,k}| = |\mathcal{F}_{n,k}|$ in Theorem 3, it is enough to show

$$\sum_{k=0}^n \binom{k+\alpha}{\alpha} |\mathcal{F}_{n,k}| = (n+1+\alpha)^n. \tag{14}$$

For $\alpha = 0$, let $\mathcal{G}_{n,k}$ be the set of functions g from $[n]$ to $[0, n]$ with $[0, k-1] \subset g([n])$ but $k \notin g([n])$. By the definition of $\mathcal{G}_{n,k}$,

$$\sum_{k=0}^n |\mathcal{G}_{n,k}| = (n+1)^n.$$

There is a simple bijection φ from $\mathcal{F}_{n,k}$ to $\mathcal{G}_{n,k}$ as follows: given an $f \in \mathcal{F}_{n,k}$, consider a function g from $[n]$ to $[0, n]$ defined by

$$g(i) = \begin{cases} f(i) - 1 & \text{if } f(i) \leq k \\ f(i) & \text{otherwise.} \end{cases}$$

Since the image of g includes $0, \dots, k-1$ but does not include k , the function g belongs to $\mathcal{G}_{n,k}$ and $\varphi(f) = g$ is well-defined. Since φ is reversible, it is a bijection. So it holds that

$$|\mathcal{F}_{n,k}| = |\mathcal{G}_{n,k}|$$

for all $0 \leq k \leq n$.

For $\alpha > 0$, let $\mathcal{H}_{n,k,\alpha}$ be the set of functions g from $[n]$ to $[-\alpha, n]$ with

$$|[-\alpha, k-1] \cap g([n])| = \alpha$$

and $k \notin g([n])$, where $[a, b] := \{a, \dots, b\}$. For every function h from $[n]$ to $[-\alpha, n]$, since the cardinality of the domain is less than the cardinality of the codomain by $\alpha + 1$, there exists a unique k satisfying the above conditions. Note that k is the $(\alpha + 1)$ st element in $[-\alpha, n] \setminus h([n])$. Thus

$$\sum_{k=0}^n |\mathcal{H}_{n,k,\alpha}| = (n+1+\alpha)^n.$$

Let \mathcal{A} be the set of all α -elements that are subsets of $[-\alpha, k-1]$. Clearly, $|\mathcal{A}| = \binom{k+\alpha}{\alpha}$. There is a bijection from $\mathcal{A} \times \mathcal{F}_{n,k}$ to $\mathcal{H}_{n,k,\alpha}$ as follows: for a given $(A, f) \in \mathcal{A} \times \mathcal{F}_{n,k}$, we take $(A, \varphi(f)) \in \mathcal{A} \times \mathcal{G}_{n,k}$. Consider the order-preserving bijection σ from $[0, n]$ to $[-\alpha, n] \setminus A$. Then we can define the function h from $[n]$ to $[-\alpha, n]$ by

$$h = \sigma \circ (\varphi(f))$$

and this function h is contained in $\mathcal{H}_{n,k,\alpha}$. Hence,

$$|\mathcal{H}_{n,k,\alpha}| = |\mathcal{A} \times \mathcal{F}_{n,k}| = |\mathcal{A}| \cdot |\mathcal{F}_{n,k}| = \binom{k+\alpha}{\alpha} |\mathcal{F}_{n,k}|$$

for all $0 \leq k \leq n$. \square

For example, let $n = 5, k = 2$, and $\alpha = 3$. Take $A = \{-2, -1, 1\} \in \binom{[-3,1]}{3}$ and $f = (f(1), \dots, f(5)) = (5, 2, 1, 3, 2) \in \mathcal{F}_{5,2}$. Then $g = \varphi(f) \in \mathcal{G}_{5,2}$ and $h = \sigma \circ (\varphi(f)) \in \mathcal{H}_{5,2,3}$ are given by

$$\begin{aligned} (g(1), \dots, g(5)) &= (5, 1, 0, 3, 1), \\ (h(1), \dots, h(5)) &= (5, 0, -3, 3, 0). \end{aligned}$$

Let us consider Eq. (13) or (14) for a negative integer α . In fact, the left hand sides of these equations are not well-defined even for $\alpha = -1$; however, the right hand sides are. Here we find, however, the coefficients of $|\mathcal{T}_{n,k}|$ that can replace the term $\binom{k-1}{-1}$ as follows.

Theorem 7. For a positive integer n , we have

$$\sum_{k=1}^n \frac{1}{k} |\mathcal{T}_{n,k}| = \sum_{k=1}^n \frac{1}{k} |\mathcal{F}_{n,k}| = n^n. \tag{15}$$

Proof. From Proposition 2 and Theorem 3, expanding $(n - i)^n$ by the binomial theorem, we have

$$\begin{aligned} |\mathcal{T}_{n,k}| &= |\mathcal{F}_{n,k}| = \sum_{i \geq 0} (-1)^i \binom{k}{i} (n - i)^n \\ &= \sum_{i \geq 0} (-1)^i \binom{k}{i} \sum_{j \geq 0} \binom{n}{j} n^{n-j} (-i)^j. \end{aligned}$$

Hence the left hand side of (15) is

$$\sum_{k=1}^n \frac{1}{k} |\mathcal{T}_{n,k}| = \sum_{j \geq 0} \sum_{k=1}^n \sum_{i \geq 0} \frac{1}{k} (-1)^i \binom{k}{i} \binom{n}{j} n^{n-j} (-i)^j.$$

We divide it into three cases; $j = 0, j = 1$, and $j > 1$.

In the case of $j = 0$,

$$\sum_{k=1}^n \sum_{i \geq 0} \frac{1}{k} (-1)^i \binom{k}{i} n^n = n^n \sum_{k=1}^n \frac{1}{k} (1 - 1)^k = 0. \tag{16}$$

In the case of $j = 1$,

$$\begin{aligned} \sum_{k=1}^n \sum_{i \geq 0} \frac{1}{k} (-1)^i \binom{k}{i} n^n (-i) &= n^n \sum_{k=1}^n \sum_{i \geq 0} (-1)^{i-1} \binom{k-1}{i-1} \\ &= n^n \sum_{k=1}^n (1 - 1)^{k-1} = n^n. \end{aligned} \tag{17}$$

In the case of $j > 1$,

$$\begin{aligned} \sum_{j > 1} \sum_{k=1}^n \sum_{i \geq 0} \frac{1}{k} (-1)^i \binom{k}{i} \binom{n}{j} n^{n-j} (-i)^j &= \sum_{j > 1} \sum_{i \geq 0} (-1)^i \binom{n}{j} n^{n-j} (-1)^j i^{j-1} \sum_{k=1}^n \binom{k-1}{i-1} \\ &= \sum_{j > 1} \sum_{i \geq 0} (-1)^{i+j} \binom{n}{j} n^{n-j} i^{j-1} \binom{n}{i} \\ &= \sum_{j > 1} (-1)^j \binom{n}{j} n^{n-j} \left[\sum_{i \geq 0} (-1)^i \binom{n}{i} i^{j-1} \right] = 0. \end{aligned} \tag{18}$$

Note that, by the principle of inclusion and exclusion, the expression $\sum_{i \geq 0} (-1)^i \binom{n}{i} i^{j-1}$ in (18) is the number of surjections from $[j - 1]$ to $[n]$. Since $j - 1 < n$, it is zero.

From (16)–(18), we finally obtain $\sum_{k=1}^n \frac{1}{k} |\mathcal{T}_{n,k}| = 0 + n^n + 0 = n^n$. \square

4. Inverse relation

For nonnegative integer n , let $A(n)$ be the square matrix of size $n + 1$ defined by

$$A(n) := \left[\binom{i+j}{i} \right]_{0 \leq i, j \leq n}.$$

Define two column vectors $t(n)$ and $p(n)$ by

$$t(n) := \begin{pmatrix} |\mathcal{T}_{n,0}| \\ |\mathcal{T}_{n,1}| \\ \vdots \\ |\mathcal{T}_{n,n}| \end{pmatrix} \quad \text{and} \quad p(n) := \begin{pmatrix} (n+1)^n \\ (n+2)^n \\ \vdots \\ (2n+1)^n \end{pmatrix}.$$

Then Eq. (13) can be interpreted as

$$A(n) t(n) = p(n).$$

The matrix $A(n)$ is nonsingular. Moreover, we can compute its inverse directly. Let $B(n)$ be the square matrix of size $n + 1$ defined by

$$B(n) := \left[(-1)^{i+j} \sum_{m=0}^n \binom{m}{i} \binom{m}{j} \right]_{0 \leq i, j \leq n}.$$

Theorem 8. For a nonnegative integer n , the two matrices $A(n)$ and $B(n)$ are inverses of each other.

Proof. Here is the calculation:

$$\begin{aligned} \sum_{l=0}^n A(n)_{i,l} B(n)_{l,j} &= \sum_{l=0}^n \binom{i+l}{i} (-1)^{l+j} \sum_{m=0}^n \binom{m}{l} \binom{m}{j} \\ &= \sum_{m=0}^n (-1)^{m-j} \binom{m}{j} \left[\sum_{l=0}^n \binom{i+l}{l} \binom{m}{m-l} (-1)^{m-l} \right]. \end{aligned}$$

Comparing the coefficients of q^m for $(1 - q)^{-(i+1)}(1 - q)^m = (1 - q)^{-(i+1-m)}$, we obtain

$$\sum_{l=0}^n \binom{i+l}{l} \binom{m}{m-l} (-1)^{m-l} = \binom{i}{m}.$$

Thus, we have

$$\begin{aligned} \sum_{l=0}^n A(n)_{i,l} B(n)_{l,j} &= \sum_{m=0}^n (-1)^{m-j} \binom{m}{j} \binom{i}{m} \\ &= \binom{i}{j} \sum_{m=0}^n (-1)^{m-j} \binom{i-j}{m-j} \binom{i}{j} (1 - 1)^{i-j} = \delta_{i,j} \end{aligned}$$

which completes the proof. \square

From the matrix identity $B(n) p(n) = t(n)$, we obtain another expression for $|\mathcal{T}_{n,k}|$.

Corollary 9. For nonnegative integers n and k , we have

$$|\mathcal{T}_{n,k}| = \sum_{0 \leq l \leq m \leq n} (-1)^{k+l} \binom{m}{k} \binom{m}{l} (n + 1 + l)^n. \tag{19}$$

5. Remarks

Since $|\mathcal{T}_{n,k}| = |\mathcal{F}_{n,k}|$, it is desired to construct a bijection between $\mathcal{T}_{n,k}$ and $\mathcal{F}_{n,k}$ for all $0 \leq k \leq n$. Also, it is natural to ask for a bijective proof of (15). Recently, Kim [3] constructed bijections for the above questions. It would be interesting to give a combinatorial explanation of (19).

Acknowledgments

The authors thank Frédéric Chapoton, who first asked this question while the second author was in Lyon, and the anonymous referees, for their valuable comments and suggestions for improving this paper. This study was supported by a 2008 Research Grant from Kangwon National University to the first author. Also, this work was supported by INHA UNIVERSITY Research Grant (INHA-42830) to the second author.

References

- [1] Nantel Bergeron, Muriel Livernet, A combinatorial basis for the free Lie algebra of the labelled rooted trees, *J. Lie Theory* 20 (1) (2010) 3–15.
- [2] Cedric Chauve, Serge Dulucq, Olivier Guibert, Enumeration of some labelled trees, in: *Formal Power Series and Algebraic Combinatorics*, Moscow, 2000, Springer, Berlin, 2000, pp. 146–157.
- [3] J.S. Kim, Bijections on rooted trees with fixed size of maximal decreasing subtrees. ArXiv e-prints, August 2011. [arXiv:1108.6038](https://arxiv.org/abs/1108.6038).
- [4] Richard P. Stanley, *Enumerative Combinatorics. Vol. 2*, in: *Cambridge Studies in Advanced Mathematics*, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.