The topological type of the $\alpha$-sections of convex sets

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Abstract

An $\alpha = (\alpha_1, \ldots, \alpha_k)$ section of a family $\{K_1, \ldots, K_k\}$ of convex bodies in $\mathbb{R}^d$ is a transversal halfspace $H^+$ for which $\text{Vol}_d(K_i \cap H^+) = \alpha_i \cdot \text{Vol}_d(K_i)$ for every $1 \leq i \leq k$. Our main result is that for any well-separated family of strictly convex sets, the space of $\alpha$-sections is diffeomorphic to $S^{d-k}$.

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1. Introduction

A halfspace in $\mathbb{R}^d$ can be parametrized by its outer unit normal vector and by the signed distance of its bounding hyperplane from the origin, that is, for $(v, t) \in S^{d-1} \times \mathbb{R}$ let

$$H^+(v, t) = \{x \mid \langle v, x \rangle \leq t \} \quad \text{and} \quad H(v, t) = \{x \mid \langle v, x \rangle = t \}.$$  

With this correspondence we identify halfspaces with points of the cylinder $S^{d-1} \times \mathbb{R}$, and we regard the cylinder as a differentiable submanifold of $\mathbb{R}^{d+1}$.

Let $\mathcal{F} = \{K_1, \ldots, K_k\}$ be a family of well-separated convex bodies, that is, for each choice of the points $p_i \in K_i$ ($1 \leq i \leq k$) the set $\{p_i\}_1^k$ is affinely independent. A halfspace $H^+$ is transversal to $\mathcal{F}$ if the bounding hyperplane $H$ is transversal to $\mathcal{F}$. Given a point $\alpha = (\alpha_1, \ldots, \alpha_k) \in I^k$ of the unit cube $I^k$, we say that a transversal halfspace $H^+$ is an $\alpha$-section of the family $\mathcal{F}$.

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if $\text{Vol}_d(K_i \cap H^+ \cap H^-) = \alpha_i \cdot \text{Vol}_d(K_i)$ for every $1 \leq i \leq k$. Very recently I. Bárány, A. Hubard, J. Jeronimo [1] proved the following

**Theorem BHJ.** Let $\mathcal{F} = \{K_1, \ldots, K_d\}$ be a family of well-separated convex bodies in $\mathbb{R}^d$, and let $\alpha = (\alpha_1, \ldots, \alpha_d) \in I^d$. Then there are exactly two $\alpha$-sections of $\mathcal{F}$.

S.E. Cappell, J.E. Goodman, J. Pach, R. Pollack, M. Sharir, R. Wenger in [3] investigated the topological structure of the set of common tangents of a family of convex sets which plays important role in certain geometric algorithms. In terms of $\alpha$-sections they proved the following

**Theorem CGPPSW.** Let $\mathcal{F} = \{K_1, \ldots, K_k\}$ be a family of well-separated strictly convex bodies in $\mathbb{R}^d$, and let $\alpha = (\alpha_1, \ldots, \alpha_k) \in I^k$ be a $0$–$1$ vector (that is, a vertex of $I^k$). Then the set of $\alpha$-sections of $\mathcal{F}$ is homeomorphic to $S^{d-k}$.

In this paper we shall prove the analogous result for arbitrary $\alpha$-sections of a family of strictly convex sets. Actually we get more, using differential topological methods, we prove diffeomorphism instead of homeomorphism and so we will have a stronger form of the Theorem CGPPSW as well. Our main result is the following

**Theorem 1.** Let $\mathcal{F} = \{K_1, \ldots, K_k\}$ be a well-separated family of strictly convex bodies in $\mathbb{R}^d$, and let $\alpha = (\alpha_1, \ldots, \alpha_k) \in I^k$ be fixed. Then the set of $\alpha$-sections of $\mathcal{F}$ is $C^1$-diffeomorphic to $S^{d-k}$.

### 2. Preliminaries

For a family $\mathcal{F}$ of convex bodies denote by $T(\mathcal{F})$ the set of common transversal halfspaces of $\mathcal{F}$ and by $T(\mathcal{F}, \alpha) \subset T(\mathcal{F})$ the set of all $\alpha$-sections of the family $\mathcal{F}$. For a single strictly convex body $K \subset \mathbb{R}^d$ the set of transversal halfspaces of $K$ can be described by the well-known support function of $K$ (which is defined on the whole $\mathbb{R}^d$)

$$h_K(v) = \max_{x \in K} \langle v, x \rangle.$$

It is clear that $(v, t)$ is a boundary point of $T(K)$ iff $H(v, t) \cap \text{int} K = \emptyset$ and $H(v, t) \cap K \neq \emptyset$, that is $H(v, t)$ is a supporting hyperplane. For each $v$ there are exactly two supporting hyperplanes and the corresponding halfspaces are $H^+(v, h_K(v))$ and $H^+(v, -h_K(-v))$. So the transversal halfspaces form the strip

$$T(K) = \{(v, t) \mid -h_K(-v) \leq t \leq h_K(v)\}$$

on the cylinder $S^{d-1} \times \mathbb{R}$. Thus $T(K)$ is a $d$-dimensional submanifold with boundary and the boundary is the disjoint union of the sets

$$M_1 = \{(v, h_K(v)) \mid v \in S^{d-1}\},$$

$$M_0 = \{(v, -h_K(-v)) \mid v \in S^{d-1}\}.$$

In what follows, we shall use some ideas of differential topology which can be found in the books [2,4]. We shall work with $C^1$-submanifolds of $\mathbb{R}^{d+1}$ and all submanifolds will be
considered with the standard Riemannian structure, that is, we use the euclidean scalar product in the tangent space.

Since $K$ is strictly convex, the support function is $C^1$-differentiable (see [6]) and its gradient is

$$\nabla h_K(v) = p_K(v),$$

where $p_K(v)$ is the unique common point of $K$ and the supporting hyperplane $H(v, h_K(v))$. This gives that $M_0, M_1$ are $(d - 1)$-dimensional $C^1$-submanifolds and $T(K)$ is a $C^1$-submanifold with boundary. Moreover, the projection onto the first coordinate is a diffeomorphism between $M_0, M_1$ and $S^{d-1}$.

We determine a normal vector field of the submanifolds $M_0, M_1$ in the tangent space of the cylinder. The vector field

$$(-\nabla h_K(v), 1) = (-p_K(v), 1)$$

is a normal field to the graph $\{(v, h_K(v)) \mid v \in \mathbb{R}^d \setminus 0\}$ of the support function and pointing upward. If we project this field orthogonally into the tangent space of the cylinder then we obtain the normal field of $M_1$ pointing outward of $T(K)$:

$$(-p_K(v) + \langle p_K(v), v \rangle v, 1) = (-p_K(v) + h_K(v)v, 1).$$

The same way we obtain a normal field for $M_0$:

$$(-p_K(-v) + \langle p_K(-v), v \rangle v, 1) = (-p_K(-v) - h_K(-v)v, 1),$$

which points into $T(K)$.

Consider now the function $f_K : T(K) \rightarrow I = [0, 1]$

$$f_K(v, t) = \frac{\text{Vol}_d(K \cap H^+(v, t))}{\text{Vol}_d K}.$$

The continuous differentiability of $f_K$ on the interior of $T(K)$ was investigated in [5] but different parametrization was used there. For the sake of the completeness we present here a bit different proof. For this we need the following lemma whose proof is standard:

**Lemma 2.** If $K$ is strictly convex then the map $(v, t) \rightarrow K \cap H(v, t)$ is continuous (from the topology of $T(K)$ to the Hausdorff topology).

**Lemma 3.** The function $f_K$ is $C^1$-differentiable on the manifold $T(K)$ and its gradient in the tangent space of the cylinder is

$$\nabla f_K = \frac{\text{Vol}_{d-1}(K \cap H(v, t))}{\text{Vol}_d K}(-c(K \cap H(v, t)) + tv, 1),$$

where $c(S)$ is the center of gravity of the set $S$. 
Proof. The tangent hyperplane $T(v_0, t_0)(S^{d-1} \times \mathbb{R})$ of the cylinder at the point $(v_0, t_0)$ has normal vector $(v_0, 0)$ and so

$$T(v_0, t_0)(S^{d-1} \times \mathbb{R}) = \{(w, t) \in \mathbb{R}^{d+1} \mid \langle w, v_0 \rangle = 0\}.$$  

We shall prove the differentiability of $f_K$ by showing that it is differentiable along smooth curves on the cylinder.

The line $\{(v_0, t) \mid t \in \mathbb{R}\}$ is the common part of the cylinder and the tangent space $T(v_0, t_0)(S^{d-1} \times \mathbb{R})$ and it has direction vector $(0, 1)$. It is well known that

$$\frac{df_K(v_0, t)}{dt} = \frac{\text{Vol}_{d-1}(K \cap H(v_0, t))}{\text{Vol}_d K}.$$

Now let $(w, 0) \in T(v_0, t_0)(S^{d-1} \times \mathbb{R})$ be a unit vector of the tangent space and consider the following curve on the cylinder

$$(v(\alpha), t_0) = (v_0 \cos \alpha + w \sin \alpha, t_0).$$

Clearly, $(v(0), t_0) = (v_0, t_0)$ and $(v(0), t_0)' = (w, 0)$. To calculate the volume, let $L_0$ be the $(d - 2)$-dimensional subspace of $\mathbb{R}^d$ orthogonal to the vectors $v_0$ and $w$ and for $y \in L_0$ let $M_y$ be the 2-dimensional subspace through the point $y$ and orthogonal to $L_0$. Then, by the Fubini theorem, we have that

$$\text{Vol}_d(K \cap H^+(v(\alpha), t_0)) = \int_{L_0} \text{Vol}_2(K \cap H^+(v(\alpha), t_0) \cap M_y) \, dy,$$

where the area in $M_y$ is calculated in the coordinate system $\{v_0, w\}$ with coordinates $(\lambda, \mu)$. Now we introduce new parametrizations in $M_y$ on the complement of the interior of the circle with center $y$ and radius $|t_0|$ (see Fig. 1). For $(\alpha, \varphi_1)$ ($\varphi_1 > 0$) resp. $(\alpha, \varphi_2)$ ($\varphi_2 > 0$) let $l(\alpha)$ be the tangent line of the circle at the point $T$ where the angle of $\overrightarrow{OT}$ and $t_0v_0$ is $\alpha$ and let $P$ resp. $P'$ be
the point on $l(\alpha)$ on the positive resp. negative halfline and $|PT| = \varrho_1$ resp. $|PT| = \varrho_2$ (a halfline is positive if its direction has acute angle with $w$; for small $\alpha$ it is uniquely defined). Calculating the Jacobian, we have that for a set $S$ the area is

\[ \text{Vol}_2(S) = \int \int \varrho_1 d\varrho_1 d\alpha = \int \int \varrho_2 d\varrho_2 d\alpha. \]

For small $\alpha$, the symmetric difference of the sets $H^+(v(\alpha), t_0) \cap M_y$ and $H^+(v_0, t_0) \cap M_y$ is the disjoint union of the sectors, $H_1$, $H_2$ (see Fig. 2). Let $H_3$ be the region bounded by the two tangent lines and the arc of the circle between the two tangent points. Then $H_1 \cup H_3$ resp. $H_2 \cup H_3$ is exactly the region swept by the positive resp. negative halfline of $l(\alpha)$ and we have that

\[
\text{Vol}_2\left( K \cap H^+(v(\alpha), t_0) \cap M_y \right) - \text{Vol}_2\left( K \cap H^+(v_0, t_0) \cap M_y \right) \\
= - \text{Vol}_2(H_1 \cap K) + \text{Vol}_2(H_2 \cap K) \\
= - \left( \text{Vol}_2(H_1 \cap K) + \text{Vol}_2(H_3 \cap K) \right) + \text{Vol}_2(H_2 \cap K) + \text{Vol}_2(H_3 \cap K) \\
= - \int_{0}^{\alpha} \int_{l(\alpha) \cap K} \varrho d\varrho d\alpha
\]

where, taking into account the signs, $\varrho$ is the signed distance from the tangent point on the line $l(\alpha)$. By integration,

\[
\int_{l(\alpha) \cap K} \varrho d\varrho = \frac{1}{2} \left( \varrho(q_1(\alpha))^2 - \varrho(q_2(\alpha))^2 \right)
\]
where \( q_1, q_2 \) are the points of intersection of \( l(\alpha) \) with \( \text{bd} \ K \). Lemma 2 implies that \( \varrho(q_i(\alpha)) \) are continuous functions of both \( \alpha \) and \( y \), so, by the bounded convergence theorem, we can change the integration and the limit and we have that

\[
\lim_{\alpha \to 0} \frac{1}{\alpha} \left( -\int_0^\alpha \int_{l(\alpha) \cap K} \varrho \, d\varrho \, d\alpha \, dy \right) = -\int_{L_0} \int_{l(0) \cap K \cap \mathcal{M}_y} \varrho \, d\varrho \, dy.
\]

But \( \varrho = \langle w, x \rangle \), \( \langle w, v_0 \rangle = 0 \) and \( d\varrho \, dy \) is the volume form in the hyperplane \( H(v_0, t_0) \). Substituting these into the above formulas we finally have that

\[
d\text{Vol}_d(K \cap H(v_0, t_0)) \bigg|_{\alpha = 0} = -\int_{H(v_0, t_0) \cap K} \langle w, x \rangle \, dx = \left\langle w, -\int_{H(v_0, t_0) \cap K} x \, dx \right\rangle
\]

\[
= \text{Vol}_{d-1}(K \cap H(v_0, t_0)) \left\langle w, -c(K \cap H(v_0, t_0)) + t_0 v_0 \right\rangle
\]

The center of gravity of a set is in the set, so \( c(K \cap H(v_0, t_0)) \in H(v_0, t_0) \), which gives that

\[
\left\langle -c(K \cap H(v_0, t_0)) + t_0 v_0, v_0 \right\rangle = 0,
\]

that is, the vector \(( -c(H(v_0, t_0) \cap K) + t_0 v_0, 0 \) is in the tangent space and for any \(( w, 0) \) satisfies the above equality. Combining this with the formula for the partial derivative with respect to \( t \), we finally have

\[
\text{grad } f_K(v_0, t_0) = \frac{\text{Vol}_{d-1}(K \cap H(v_0, t_0))}{\text{Vol}_{d}(K \cap H(v_0, t_0))} (-c(K \cap H(v_0, t_0)) + t_0 v_0, 1).
\]

Lemma 2 implies that \( \text{grad } f_K(v_0, t_0) \) is continuous (the continuity of \( c(K \cap H(v_0, t_0)) \) comes from the strictly convexity of \( K \)).

**Remark 4.** Lemma 3 shows that \( \text{grad } f_K = 0 \) only in the boundary points of \( \mathcal{T}(K) \) and we know that \( 0 < f_K < 1 \) in the interior points of \( \mathcal{T}(K) \), which gives that the level surfaces \( f_K^{-1}(\alpha) \) are \( C^1 \)-submanifolds for each \( 0 < \alpha < 1 \). We have seen earlier that \( f_K^{-1}(0) = M_0 \) and \( f_K^{-1}(1) = M_1 \) are also \( C^1 \)-submanifolds. Thus each level surface \( f_K^{-1}(\alpha) \) \( (0 \leq \alpha \leq 1) \) is a \((d - 1)\)-dimensional \( C^1 \)-submanifold. Moreover, the gradient of \( f_K \) can be written in the following form

\[
\text{grad } f_K(v, t) = \frac{\text{Vol}_{d-1}(K \cap H(v, t))}{\text{Vol}_{d}(K \cap H(v, t))} (-x_i(v, t) + t v, 1),
\]

where

\[
x_i(v, t) = \begin{cases} p_{K_i}(v), & \text{if } (v, t) \in \partial \mathcal{T}(K), \\ c(K_i \cap H(v, t)), & \text{otherwise}. \end{cases}
\]
The vector field \((-x_1(v,t) + tv, 1)\) is a continuous, nonzero tangent vector field on \(T(K)\) and it is orthogonal to the level surface \(f_K^{-1}(f_K(v,t))\).

The most common way to prove the diffeomorphism of level surfaces would be the following

**Smooth cobordism theorem.** (See [4].) Let \(M\) be a compact \(C^1\)-submanifold with boundary and assume \(\partial M = M_a \cup M_b\) where \(M_a\) and \(M_b\) are disjoint closed sets. Suppose there exists a \(C^1\)-map \(f : M \rightarrow [a, b]\) such that \(\text{grad} f(p) \neq 0\) on \(M\) and \(f^{-1}(a) = M_a, f^{-1}(b) = M_b\). Then \(M_a\) and \(M_b\) are diffeomorphic.

However, in our case the gradient of \(f\) may vanish on the boundary of \(M\). We prove the following variant of the smooth cobordism theorem.

**Lemma 5.** Let \(M\) be a compact \(C^1\)-submanifold with boundary and assume \(\partial M = M_0 \cup M_1\) where \(M_0\) and \(M_1\) are disjoint closed sets. Suppose there exists a \(C^1\)-map \(f : M \rightarrow [0, 1]\) such that \(f^{-1}(0) = M_0, f^{-1}(1) = M_1\) and

\[
\text{grad} f(p) = g(p)X(p)
\]

where \(g : M \rightarrow \mathbb{R}\) is a continuous function, \(g(p) \geq 0\) and \(g^{-1}(0) = \partial M\), and \(X(p)\) is a non-vanishing continuous tangent vector field on \(M\) which is orthogonal to \(M_0, M_1\) and for \(p \in M_0\) points inward and for \(p \in M_1\) points outward. Then each level surface \(f^{-1}(s)\) is diffeomorphic to \(f^{-1}(1)\).

**Proof.** For each \(0 < s_0 < s_1 < 1\), the smooth cobordism theorem can be applied for the manifold \(f^{-1}([s_0, s_1])\) and for \(f\) and we obtain that the level surfaces \(f^{-1}(s)\) are diffeomorphic to each other for all \(0 < s < 1\). So it is enough to prove that there are \(0 < s_0, s_1 < 1\) such that \(f^{-1}(s_i)\) is diffeomorphic to \(f^{-1}(i)\) for \(i = 0, 1\).

We shall use the construction of the collaring theorem (see [4, Theorem 2.1, p. 152.]). We choose a \(C^\infty\) vector field \(Y(p)\) on a neighborhood of \(M_1\) which is nowhere tangent to \(M_1\) and points out of \(M\). There is an open neighborhood \(W\) of \(M_1\) which is \(C^\infty\) diffeomorphic to \(M_1 \times (-\varepsilon, 0]\) by the map

\[
(p, t) \mapsto \xi(p, t),
\]

where \(\xi\) is the integral curve of the vector field \(Y\):

\[
\frac{d\xi(p, t)}{dt} = Y(\xi(p, t)), \quad \xi(p, 0) = p.
\]

The transversality of \(Y\) gives that the scalar product \((X(p), Y(p)) > 0\) on \(M_1\) so we may suppose that the scalar product is positive on \(M_1 \times (-\varepsilon, 0]\). Along each integral curve \(\xi(p, t)\) \((p \in M_1, t \in (-\varepsilon, 0])\) of the vector field \(Y(p)\), the function \(f\) is strictly increasing because

\[
f'(\xi(p, t)) = \langle \text{grad} f(\xi(p, t)), \xi'(p, t) \rangle = g(\xi(p, t)) \langle X(\xi(p, t)), Y(\xi(p, t)) \rangle \geq 0
\]

and it is 0 only at \(t = 0\). By compactness, this gives that there is a \(\delta > 0\) such that \(f(\xi(p, -\varepsilon/2)) \leq 1 - \delta\) for each \(p \in M_1\). The manifold \(M_\delta = \{\xi(p, -\varepsilon/2)\}\) is diffeomorphic
with $M_1$. The condition $f^{-1}(1) = M_1$ and the compactness of $M$ implies that there is an $s_1 \in (1 - \delta, 1)$ such that the level surface $f^{-1}(s_1)$ is the subset of $\xi(M_1 \times (-\varepsilon/2, 0])$. The monotonicity of $f$ along integral curves gives that each integral curve $\xi(p, t)$ meets the level surface $f^{-1}(s_1)$ in exactly one point. To finish the proof we apply the following result for $f^{-1}(s_1)$, $M_\varepsilon$, $Y$ and for $\xi(M_1 \times (-\varepsilon, 0]) \setminus M_1$ (see [4, Exercise 11, p. 156]):

**Let $Y$ be a $C^1$ vector field on the $C^1$ manifold $N$. Let $V_0, V_1$ be $C^1$-submanifolds of $N$ which are transverse to $Y$. Assume that $\partial V_0 = \partial V_1 = \partial M = \emptyset$. Suppose that every integral curve through a point of either of the submanifolds intersects the other at a unique point. Then $V_0$ and $V_1$ are $C^1$-diffeomorphic.**

The case $f^{-1}(0)$ can be handled in a similar way.  

3. The proof of Theorem 1

We divide the proof into two parts. First we prove the statement under the condition that there is an $\alpha$-section for each $\alpha$ (Lemma 6). In the second part we prove the existence of $\alpha$-sections (Lemma 7). This will be done by an induction, where in the induction step we use the result of the first part.

**Lemma 6.** Suppose that for any well-separated family $\mathcal{F} = \{K_1, \ldots, K_k\} (1 \leq k \leq d)$ of strictly convex sets in $\mathbb{R}^d$ and for any $\alpha \in I^k$ there exists an $\alpha$-section. Then the set of $\alpha$-sections $T(\mathcal{F}, \alpha)$ is a $C^1$-submanifold of the cylinder and it is $C^1$-diffeomorphic to $S^{d-k}$.

**Proof.** The set of $\alpha$-sections of $\mathcal{F}$ can be written as the intersection of the level surfaces of the functions $f_{K_i}$

$$T(\mathcal{F}, \alpha) = f_{K_1}^{-1}(\alpha_1) \cap \cdots \cap f_{K_k}^{-1}(\alpha_k).$$

From Lemma 3 and Remark 4 we know that each level surface of each $f_{K_i}$ is a $(d - 1)$-dimensional $C^1$-submanifold and at an arbitrary point $(v, t) \in T(\mathcal{F}, \alpha)$ a normal vector of the level surface $f_{K_i}^{-1}(\alpha_i)$ is

$$X_{K_i}(v, t) := (-x_i(v, t) + tv, 1)$$

where

$$x_i(v, t) = \begin{cases} p_{K_i}(v), & \text{if } (v, t) \in \partial T(K_i), \\ c(K_i \cap H(v, t)), & \text{otherwise}. \end{cases}$$

These vectors are linearly independent, because if

$$\sum \lambda_i (-x_i(v, t) + tv, 1) = (0, 0)$$

then
\[ \sum \lambda_i = 0, \]
\[ \sum \lambda_i (-x_i(v, t) + tv) = -\sum \lambda_i x_i(v, t) = 0. \]

But \( x_i(v, t) \in K_i \) so the well-separatedness of the family implies that \( \lambda_i = 0 \) for all \( i \).

The linear independence of the normal vectors means that the level surfaces intersects each other transversally, which implies (see [2, Chapter II, Theorem 7.7]) that the intersection of the surfaces \( f_{K_i}^{-1}(\alpha_i) \) is a \((d-k)\)-dimensional \( C^1 \)-submanifold of the cylinder.

Now we shall prove that for different \( \alpha \) the manifolds \( T(F, \alpha) \) are diffeomorphic to each other. This we do by investigating \( \alpha \)'s which differ only in one coordinate.

Consider the segment \( \alpha(s) = [(\alpha_1, \ldots, \alpha_{i-1}, s, \alpha_{i+1}, \ldots, \alpha_k) \mid 0 \leq s \leq 1] \) in \( I^k \). The set of all \( \alpha(s) \)-sections can be written in the following form
\[ M = f_{K_1}^{-1}(\alpha_1) \cap \cdots \cap T(K_i) \cap \cdots \cap f_{K_k}^{-1}(\alpha_k). \]

As above, we have that the intersection \( \bigcap_{j \neq i} f_{K_j}^{-1}(\alpha_j) \) is a \((d-k+1)\)-dimensional \( C^1 \)-submanifold and, by the transversality of the level surfaces, its intersection with \( T(K_i) \) is a compact \((d-k+1)\)-dimensional \( C^1 \)-submanifold with boundary which is the disjoint union of \( f_{K_i}^{-1}(0) \cap M \) and \( f_{K_i}^{-1}(1) \cap M \).

Consider now the restriction \( f_{K_i}|_M \) of \( f_{K_i} \) onto \( M \). It is known that
\[ f_{K_i}|_M : M \to [0, 1] \]
is \( C^1 \)-differentiable and \( \text{grad}(f_{K_i}|_M) \) is the image of \( \text{grad} f_{K_i} \) under the orthogonal projection \( \pi \) onto the tangent space of \( M \). By Lemma 3 and Remark 4, the function \( f_{K_i} : T(K_i) \mapsto [0, 1] \) satisfies the conditions of Lemma 5 and the gradient has the form
\[ \text{grad} f_{K_i}(v, t) = g_{K_i}(v, t)X_{K_i}(v, t). \]

From this we have that
\[ \text{grad} f_{K_i}|_M(v, t) = \pi(\text{grad} f_{K_i}(v, t)) = g_{K_i}(v, t)\pi(X_{K_i}(v, t)). \]

The tangent space of \( M \) at \((v, t)\) is the orthogonal complement of the normal vectors \( \{X_{K_j}(v, t)\}_{j \neq i} \). The linear independence of the vectors \( \{X_{K_j}(v, t)\}_{j \neq i} \) implies that \( \pi(X_{K_j}(v, t)) \) is never zero and at the boundary points of \( M \) it is orthogonal to the boundary. So the function \( f_{K_i}|_M : M \to [0, 1] \) satisfies the conditions of Lemma 5, which gives that for each \( s \) the \((d-k)\)-dimensional \( C^1 \)-submanifold \( (f_{K_i}|_M)^{-1}(s) = \pi(\mathcal{F}, \alpha(s)) \) is diffeomorphic to \( (f_{K_i}|_M)^{-1}(1) = T(F, (\alpha_1, \ldots, 1, \ldots, \alpha_k)) \). Repeating the above procedure to the other coordinates of \( \alpha \), we finally have that \( T(F, \alpha) \) is diffeomorphic to \( T(F, (1, \ldots, 1)) \).

Theorem CGPPSW shows that \( T(F, (1, \ldots, 1)) \) is homeomorphic to \( S^{d-k} \). Using our method, we prove more, namely, that \( T(F, (1, \ldots, 1)) \) is diffeomorphic to \( S^{d-k} \). This time we shall deform the sets one by one into an appropriate ball showing that the manifold of \((1, \ldots, 1)\)-sections changes diffeomorphically.

For short, let \( S(F) = T(F, (1, \ldots, 1)) \). For the single set \( K_1 \) we saw in the preliminaries, that \( S(K_1) \) can be expressed by the support function \( h_{K_1} \)
\[ S(K_1) = \{(v, h_{K_1}(v)) \mid v \in S^{d-1} \}. \]
We deform the first set into a ball. For $0 \leq s \leq 1$ define the set $K_1(s) = sK_1 + (1 - s)B_1$, where $B_1 \subset \text{int} K_1$ is a ball. The sets $K_1(s)$ are strictly convex and, by $K_1(s) \subseteq K_1$ we have, that the family $\mathcal{F}(s) = \{K_1(s), K_2, \ldots, K_k\}$ is well-separated. The union of $S(K_1(s))$ is the strip on the cylinder

$$S = \bigcup_s S(K_1(s)) = \bigcup_s \{(v, h_{K_1(s)}(v))\} = \bigcup_s \{(v, sh_{K_1}(v) + (1-s)h_{B_1}(v))\} = \{(v, t) \mid h_{B_1}(v) \leq t \leq h_{K_1}(v)\},$$

which is clearly a $d$-dimensional $C^1$-submanifold with boundary. For $(v, t) \in S$ there is a unique $s$ for which $t = sh_{K_1}(v) + (1-s)h_{B_1}(v)$:

$$s(v, t) = \frac{t - h_{B_1}(v)}{h_{K_1}(v) - h_{B_1}(v)}.$$ 

Actually, the function $s(v, t)$ is defined on $\mathbb{R}^{d+1} \setminus \{(0, t) \mid t \in \mathbb{R}\}$ and $C^1$-differentiable. So the gradient of $s$ in the tangent space of the cylinder can be calculated in the following way: we determine the gradient of the extended function and then project it orthogonally into the tangent space of the cylinder. Finally we obtain that grad $s = (a, b)$ where

$$a = -((h_{K_1}(v) - t) \text{grad} h_{B_1}(v) + (t - h_{B_1}(v)) \text{grad} h_{K_1}(v)) + tv,$$

$$b = \frac{1}{h_{K_1}(v) - h_{B_1}(v)},$$

which is never zero.

By the condition of the lemma we have that

$$S(\mathcal{F}(s)) = S(K_1(s)) \cap S(K_2) \cap \cdots \cap S(K_k) \neq \emptyset$$

and so, as above, it is a $(d - k)$-dimensional $C^1$-submanifold. For different $s$ these manifolds are disjoint and their union can be written in the following way

$$N = \bigcup_s S(\mathcal{F}(s)) = S \cap S(K_2) \cap \cdots \cap S(K_k),$$

which, by the transversality of the surfaces $S(K_1(s)), S(K_2), \ldots, S(K_k)$, is a $C^1$-submanifold with boundary. The restriction of the function $s$ onto $N$ is $C^1$-differentiable and the gradient of it is never zero. This time the smooth cobordism theorem is applicable for the function $s|_N : N \rightarrow [0, 1]$, which gives that level surfaces $(s|_N)^{-1}(1) = S(\mathcal{F}(1)) = S(\mathcal{F})$ and $(s|_N)^{-1}(0) = S(\mathcal{F}(0))$ are diffeomorphic. Repeating the above deformation for $K_2, K_3, \ldots$, finally we have a ball system $\mathcal{B} = \{B_1, \ldots, B_k\}$ with $S(\mathcal{F})$ is diffeomorphic with $S(\mathcal{B})$, which is clearly diffeomorphic to $S^{d-k}$. □

To finish the proof of Theorem 1 we shall prove the existence of the $\alpha$-sections. We emphasize that this statement is valid for any family of well-separated convex bodies (i.e. compact convex sets with nonempty interior).
Lemma 7. For any well-separated family $\mathcal{F}$ of $k$ convex bodies in $\mathbb{R}^d$ and for any $\alpha \in I^k$ there exists an $\alpha$-section.

Proof. Consider the family $\mathcal{F} = \{K_1, \ldots, K_k\}$ and an element $\alpha = (\alpha_1, \ldots, \alpha_k) \in I^k$.

We may suppose that each set is strictly convex. For, each convex body $K_i$ can be approximated in the Hausdorff metric by a sequence $\{K_i^n\}$ of strictly convex bodies [6]; there is a $\delta > 0$ such that if $K_i^n$ is closer to $K_i$ than $\delta$ then the family $\{K_1^n, \ldots, K_k^n\}$ is also well-separated; if $H^+(v_n, t_n)$ is an $\alpha$-section of the family $\{K_1^n, \ldots, K_k^n\}$ then the continuity of the volume in the Hausdorff metric gives that any limit point of the sequence $H^+(v_n, t_n)$ will be an $\alpha$-section of the family $\mathcal{F}$.

The proof is by induction on the number of the sets. The statement is clear for $k = 1$. Suppose that it is true for any family $\mathcal{F}$ with $|\mathcal{F}| < k$. Let $\mathcal{F}' = \mathcal{F} \setminus \{K_k\}$, $\alpha' = (\alpha_1, \ldots, \alpha_{k-1})$. The family $\mathcal{F}'$ is clearly well-separated. By the induction hypothesis and by Lemma 6 we have that $T(\mathcal{F}', \alpha')$ is a $C^1$-submanifold, which is $C^1$-diffeomorphic to $S^{d-k+1}_0$, so it is arcwise connected (because $k \leq d$).

For any point $p = (p_1, \ldots, p_{k-1}) \in K_1 \times \cdots \times K_{k-1}$, the affine subspace $L(p) = \text{aff}\{p_1, \ldots, p_{k-1}\}$ is disjoint from $K_k$ (by the well-separatedness of $\mathcal{F}$). Let $x \in K_k$, $y \in L(p)$ be the closest points of $K_k$ and $L(p)$. These points are unique, because $K_k$ is strictly convex. The unit vector $v(p) = \frac{x - y}{||x - y||}$ is orthogonal to the subspace $L(p)$ and depends continuously on $p$. So the halfspace $H^+(v(p), t(p))$ containing $L(p)$ depends continuously on the point $p$ and contains $K_k$ in its interior. Now we apply a modification of the idea of [1]. For each $K_i$ ($1 \leq i \leq k - 1$) find the parallel position of the halfspace $H^+(v(p), t(p))$ such that it will be an $\alpha_i$-section of $K_i$ and take the center of gravity $c_i(p)$ of the section of $K_i$ with the bounding hyperplane of the translated halfspace. This way we obtain a continuous mapping $p \mapsto (c_1(p), \ldots, c_{k-1}(p))$ from $K_1 \times \cdots \times K_{k-1}$ into itself. By the Brouwer fixed point theorem there is a fixed point $p_1$ which means that $H^+(v(p_1), t(p_1))$ is an $\alpha'$-section of $\mathcal{F}'$ containing $K_k$. The same way, using the halfspaces $H^+(-v(p), -t(p))$, we get an $\alpha'$-section $H^+(v(p_0), t(p_0))$ of $\mathcal{F}'$ which is disjoint to $K_k$. The space $T(\mathcal{F}', \alpha')$ is arcwise connected, so there is a continuous arc $w : I \to T(\mathcal{F}', \alpha')$ between $(v(p_0), t(p_0))$ and $(v(p_1), t(p_1))$. Clearly, $f_{K_k}(v(p_0), t(p_0)) = 0$ and $f_{K_k}(v(p_1), t(p_1)) = 1$. By continuity, we have that there is a $z \in I$ where $f_{K_k}(v(z)) = \alpha_k$, that is the family $\mathcal{F}$ has an $\alpha$-section. □

References