Differential systems, the mapping over period for which is represented by a product of three exponential matrixes

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Abstract

For some periodic differential systems the sufficient conditions of a representation of the mapping over period as a product of a three exponential matrix of the special aspect are obtained. Obtained results are used for research of problems of the existence and stability of periodic solutions of nonlinear systems and the asymptotic orbital stability of cycles of autonomous differential systems.

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1. Introduction

As is well known, most of differential systems cannot be integrated in quadratures (in finite terms). Even so, some differential systems can be investigated on the qualitative level using the reflecting function introduced in [1].

Consider the system

$$\dot{x} = X(t, x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n,$$

(1)

with a continuously differentiable right-hand side and with a general solution $\varphi(t; t_0, x_0)$. For each such system, the reflecting function (RF) is defined as $F(t, x) := \varphi(-t; t, x)$. If system (1)
is $2\omega$-periodic with respect to $t$, and $F$ is its RF, then $F(-\omega, x) = \varphi(\omega; -\omega, x)$ is the mapping of this system over the period $[-\omega, \omega]$.

A function $F(t, x)$ is a reflecting function of system (1) if and only if it is a solution of the system of partial differential equations (called a basic relation, BR)

$$F_t' + F_x' X(t, x) + X(-t, F) = 0$$

with the initial condition $F(0, x) \equiv x$.

Each continuously differentiable function $F$ that satisfies the condition

$$F(-t, F(t, x)) \equiv F(0, x) \equiv x$$

is a RF of the whole class of systems of the form (see [2, p. 71])

$$\dot{x} = -\frac{1}{2} F_x'(-t, F) F_t' + F_x'(-t, F) S(t, x) - S(-t, F),$$

where $S$ is an arbitrary vector function such that the solutions of system (2) are uniquely determined by their initial conditions. Therefore, all systems of the form (1) are split into equivalence classes of the form (2) so that each class is specified by a certain reflecting function referred to as the RF of the class. For all systems of one class, the shift operator [3, pp. 11–13] on the interval $[-\omega, \omega]$ is the same. Therefore, all equivalent $2\omega$-periodic systems have a common mapping over the period $[-\omega, \omega]$.

Let system (1) be linear, i.e.,

$$\dot{x} = P(t)x, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n,$$

and $\Phi(t)$ is its fundamental matrix of solutions. Then general solution of system (3) is $\varphi(t; t_0, x_0) \equiv \Phi(t)\Phi^{-1}(t_0)x_0$. Therefore, RF of system (3) is linear and $F(t, x) \equiv F(t)x$, where $F(t) := \Phi(-t)\Phi^{-1}(t)$. This matrix $F(t)$ is referred to as a reflecting matrix (RM) of system (3).

RM of any system satisfies the relations $F(-t)F(t) \equiv F(0) = E$, where $E$ is the $n \times n$ unit matrix. Differentiable matrix $F(t)$ is a RM of system (3) if and only if it is a solution of the system (basic relation)

$$\dot{F}(t) + F(t)P(t) + P(-t)F(t) = 0$$

with the initial condition $F(0) = E$. Any linear system with reflecting matrix $F(t)$ can be reduced in the form

$$\dot{x} = \left[-\frac{1}{2} F(-t)\dot{F}(t) + F(-t)R(t) - R(-t)F(t)\right]x,$$

where $R(t)$ is an arbitrary continuous real matrix.

If matrix $P(t)$ is $2\omega$-periodic and $F(t)$ is RM of system (3), then for this system the matrix $F(-\omega)$ is the monodromy matrix on the interval $[-\omega, \omega]$. Thus solutions $\mu_i, i = 1, n$ of the equation $\det(F(-\omega) - \mu E) = 0$ are multiplicators of system (3).

See articles [4–13] which are also devoted to investigations of qualitative behaviour of solutions of differential systems with the help of reflecting function.

Consider the autonomous differential system

$$\dot{x} = f(x), \quad x \in D \subset \mathbb{R}^n,$$

where $f$ is a continuously differentiable vector function. The solution $\eta = \eta(t)$ ($t_0 \leq t < \infty$) of system (4) is called an orbitally stable solution [14, pp. 299–303] if for any solutions $x = x(t)$ ($t_0 \leq t < \infty$) and any $\varepsilon > 0$ exist $\delta = \delta(\varepsilon, t_0) > 0$ such that if $\|x(t_0) - \eta(t_0)\| < \delta$
then $\rho(x(t), L^+) < \epsilon$ for $t \geq t_0$, where $L^+ = \{ \eta(t) \mid t_0 \leq t < \infty \}$ is a positive half-orbit of the solution $\eta(t)$; $\rho(z, L)$ is the distance from point $z \in \mathbb{R}^n$ up to set $L \subset \mathbb{R}^n$, i.e., $\rho(z, L) = \inf_{t\in L} \|z - x\|$.

The orbitally stable solution $\eta(t)$ called an asymptotically orbitally stable solution [14, pp. 299–303] if exist $\Delta > 0$ such that for any solutions $x(t)$ satisfying the inequality $\|x(t_0) - \eta(t_0)\| < \Delta$ follows $\lim_{t\to\infty} \rho(x(t), L^+) = 0$.

Conditions of the existence and stability of a periodic solutions for a periodic ordinary differential systems are established in this article. Present research is a prolongation of article [5], where the set of systems with RF of the form

$$F(t, x) \equiv e^{A_0\alpha_0(t)} e^{A_1\alpha_1(t)} \ldots e^{A_m\alpha_m(t)} x$$

(where $A_i$, $i = \overline{0, m}$ are constant $n \times n$ matrixes; $\alpha_i(t)$, $i = \overline{0, m}$ are odd and a necessary number of times continuously differentiable scalar functions) for which

$$\frac{\partial F}{\partial t}(t, x) \equiv e^{A_0\alpha_0(t)} e^{A_1\alpha_1(t)} \ldots e^{A_m\alpha_m(t)} \sum_{i=0}^m A_i \dot{\alpha}_i(t) x$$

are investigated.

2. Linear systems

Consider the linear differential system

$$\dot{x} = P(t)x, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n, \quad (5)$$

where $P(t)$ is a twice continuously differentiable $n \times n$ matrix. In some cases (as it takes place for a periodic systems) the fundamental matrix $X(t)$ of system (5) can be represented in the form $X(t) \equiv \Phi(t)e^{-\frac{t}{\Delta}B}$, where $\Phi(t)$ is a continuous periodic $n \times n$ matrix; $B$ is a constant $n \times n$ matrix. RM of such systems is $F(t) \equiv X(-t)X^{-1}(t) \equiv \Phi(-t)e^{Bt}\Phi^{-1}(t)$. With this in mind we suppose what RM of system (5) is given by $F(t) \equiv e^{\alpha(t)A} e^{\beta(t)B} e^{-\alpha(-t)A}$, where $\alpha(t)$ and $\beta(t)$ are arbitrary scalar functions; $A$ and $B$ are constant $n \times n$ matrices.

Lemma 1. Let RM of system (5) be $F(t) \equiv e^{\alpha(t)A} e^{\beta(t)B} e^{-\alpha(-t)A}$, where $\alpha(t)$ and $\beta(t)$ are thrice differentiable scalar functions, moreover, $\beta(t)$ is odd and $\dot{\beta}(0) = 1$; $A$ and $B$ are constant $n \times n$ matrices. Then

$$B = -2e^{-\alpha(0)A}(\dot{\alpha}(0)A + P(0))e^{\alpha(0)A}$$

and

$$-\dot{\alpha}^2(0)(\dot{\alpha}^2P(0) + P(0)A^2) + 3\ddot{\alpha}(0)(P(0)A - AP(0)) + \dddot{\alpha}(0)A$$

$$+ 2\dot{\alpha}^2(0)AP(0)A + 2\dot{\alpha}(0)(P^2(0)A + AP^2(0)) - 4\dot{\alpha}(0)P(0)AP(0)$$

$$- (\dot{\alpha}(0)A + P(0))\ddot{\beta}(0) + \dddot{P}(0) - 2P(0)\dot{P}(0) + 2\dot{P}(0)P(0) = 0. \quad (6)$$

Proof. Writing out BR for the considered RM, we obtain the identity

$$F(t)(A\dot{\alpha}(-t) + P(t)) + (\dot{\alpha}(t)A + P(-t))F(t) + e^{\alpha(t)A}B e^{\beta(t)B} e^{-\alpha(-t)A} \dot{\beta}(t) \equiv 0.$$  

By setting $t = 0$, we obtain matrix $B$. Twice differentiating the above obtained identity and putting $t = 0$, we get (6). \qed
Lemma 2. Let RM of system (5) be \( F(t) \equiv e^{At} e^{Bt} e^{At} \), where \( A \) and \( B \) are constant \( n \times n \) matrices. Suppose also that \( S(t) \equiv e^{At} P(t) e^{-At} - P(0) \) is odd matrix. Then \( B = -2(A + P(0)) \) and
\[
(P^2(0) - \dot{P}(0)) A + A(P^2(0) + \dot{P}(0)) - 2P(0)AP(0) = P(0) \dot{P}(0) - \ddot{P}(0)P(0) - \dddot{P}(0).
\]

Proof. We use Lemma 1 and put \( \alpha(t) \equiv \beta(t) \equiv t \), then we obtain \( B = -2(A + P(0)) \) and
\[
2AP(0) - A^2P(0) - P(0)A^2 + 2(P^2(0)A + AP^2(0))
- 4P(0)AP(0) + \dot{P}(0) - 2P(0)\dot{P}(0) + 2\ddot{P}(0)P(0) = 0.
\]
Since \( S(t) \) is odd, then by summing \( S(t) \) and \( S(-t) \), we get
\[
e^{-At} P(-t)e^{At} + e^{At} P(t)e^{-At} \equiv 2P(0).
\]
Twice differentiating the obtained identity and putting \( t = 0 \), we get
\[
A^2P(0) - 2AP(0)A + P(0)A^2 + 2A\dot{P}(0) - 2\dot{P}(0)A + \dddot{P}(0) = 0.
\]
By summing last equation and (8), we get (7). \( \square \)

Theorem 3. Let \( A \) be an \( n \times n \) matrix satisfying condition (7). Suppose that \( B = -2(A + P(0)) \). If \( e^{At} P(t)e^{-At} - P(0) \) is odd matrix that commutes with \( B \), then

1. the mapping of the 2\( \omega \)-periodic system (5) over the period \( [-\omega, \omega] \) is \( F(-\omega, x) = e^{-A\omega} e^{-B\omega} e^{-A\omega} x \);
2. solution \( x(t) \) of system (5) with initial condition \( x(-\omega) = x_0 \) is 2\( \omega \)-periodic solution if and only if \( F(-\omega, x_0) = x_0 \);
3. for any solution \( x(t) \) of system (5) the identity \( x(-t) \equiv e^{At} e^{Bt} e^{At} x(t) \) is true.

Proof. By checkout of the BR it is proved, that the function \( F(t, x) \equiv e^{At} e^{Bt} e^{At} x \) is RF of system (5). Therefore, all assertions of the theorem follow from [1, p. 11]. \( \square \)

Example 4. Consider the system
\[
\begin{align*}
\dot{x} &= \frac{1}{2} (f(t) - k + (g(t) - k) \cos(l - k) t) x + \frac{1}{2} (l - k - (g(t) - k) \sin(l - k) t) y, \\
\dot{y} &= \frac{1}{2} (k - l - (g(t) - k) \sin(l - k) t) x + \frac{1}{2} (f(t) - k - (g(t) - k) \cos(l - k) t) y,
\end{align*}
\]
where \( f(t) \) and \( g(t) \) are odd continuous scalar functions; \( k, l \in \mathbb{R} \).

From Eq. (7) we find the matrix
\[
A = \begin{pmatrix} m & \frac{k-l}{2} \\ \frac{l-k}{2} & m \end{pmatrix},
\]
where \( m \) is an arbitrary real number. It is easy to test, that \( e^{At} P(t)e^{-At} - P(0) \) is odd matrix that commutes with \( B = -2(A + P(0)) \), where \( P(t) \) is the matrix of the considered system.

If \( f(t) \) and \( g(t) \) are \( \frac{2\pi}{l-k} \)-periodic functions \( (l > k) \), then the considered system is also \( \frac{2\pi}{l-k} \)-periodic with respect to \( t \). Thus, by Theorem 3,
\[
F\left(-\frac{\pi}{l-k}\right) = \begin{pmatrix} -1 & 0 \\ 0 & -\exp\left(\frac{2\pi k}{l-k}\right) \end{pmatrix}
\]
is the monodromy matrix on the interval $[-\pi / l, \pi / l]$ of the considered system. Consequently, this system has, at least one-parameter family of $\frac{4\pi}{l-k}$-periodic solutions. Since $l > k$, then for $k > 0$ the considered system is stable. For $k < 0$ the system is unstable. If $k = 0$, then any solution of the considered system is $\frac{4\pi}{l-k}$-periodic.

**Lemma 5.** Let matrix $F(t) \equiv e^{A \sin^3 t} e^{B \sin t} e^{A \sin^3 t}$ be RM of system (5), where $A$ and $B$ are constant $n \times n$ matrices. Then $A = \frac{1}{6} (2P(0)\dot{P}(0) - 2\dot{P}(0)P(0) - P(0) - \ddot{P}(0))$, $B = -2P(0)$.

**Proof.** Since $F(t)$ is RM of system (5), then $BR \dot{F}(t) + F(t)P(t) + P(-t)F(t) = 0$ is valid. We find matrix $B$ from $BR$ by setting $t = 0$. Twice differentiating $BR$ and putting $t = 0$ we obtain equation for matrix $A$. □

By combine Lemma 5 and properties of reflecting function we obtain next theorem.

**Theorem 6.** Suppose that $A = \frac{1}{6} (2P(0)\dot{P}(0) - 2\dot{P}(0)P(0) - P(0) - \ddot{P}(0))$ and $B = -2P(0)$. Let matrices $P(t)$ and $F(t) \equiv e^{A \sin^3 t} e^{B \sin t} e^{A \sin^3 t}$ satisfy the $BR \dot{F}(t) + F(t)P(t) + P(-t) \times F(t) = 0$. Then

1. the mapping of $2\omega$-periodic system (5) over the period $[-\omega, \omega]$ is $F(-\omega, x) \equiv e^{-A \sin^3 \omega} e^{-B \sin \omega} e^{-A \sin^3 \omega} x$;
2. solution $x(t)$ of system (5) with initial condition $x(-\omega) = x_0$ is $2\omega$-periodic solution if and only if $F(-\omega, x_0) = x_0$;
3. for any solution $x(t)$ of system (5) the identity $x(-t) \equiv e^{A \sin^3 t} e^{B \sin t} e^{A \sin^3 t} x(t)$ is true.

**Proof.** Since $BR$ is valid, therefore all assertions of the theorem follow from properties of RF [1, p. 11]. □

**Example 7.** Consider the system

$$\dot{x} = \frac{1}{2} \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} x,$$

where

$$a(t) \equiv (2\cos(2l \sin^3 t) - 3\sin^2 t(2k - l \sin(2l \sin^3 t) \sinh(4\sin t))) \cos t,$$
$$b(t) \equiv (3l \sin^2 t(1 + e^{4\sin t}) \cos^2(l \sin^3 t) + e^{-4\sin t} \sin^2(l \sin^3 t)) - 2\sin(2l \sin^3 t)) \cos t,$$
$$c(t) \equiv -(3l \sin^2 t(1 + e^{-4\sin t}) \cos^2(l \sin^3 t) + e^{4\sin t} \sin^2(l \sin^3 t)) + 2\sin(2l \sin^3 t)) \cos t,$$
$$d(t) \equiv -(2\cos(2l \sin^3 t) + 3\sin^2 t(2k + l \sin(2l \sin^3 t) \sinh(4\sin t))) \cos t;$$

$k, l \in \mathbb{R}$.

Suppose that $F(t) \equiv e^{A \sin^3 t} e^{B \sin t} e^{A \sin^3 t}$ is RM of the considered system, then, by Lemma 5, we have

$$A = \begin{pmatrix} k & -l \\ l & k \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$$
Therefore,
\[ F(t) \equiv e^{2k \sin^3 t} \begin{pmatrix} F_1(t) & F_2(t) \\ F_3(t) & F_4(t) \end{pmatrix}, \]
where
\[ F_1(t) \equiv e^{-2 \sin t} \cos^2(l \sin^3 t) - e^{2 \sin t} \sin^2(l \sin^3 t), \]
\[ F_3(t) \equiv \cosh(2 \sin t) \sin(2l \sin^3 t), \]
\[ F_2(t) \equiv -F_3(t), \]
\[ F_4(t) \equiv F_1(-t). \]

By checkout of the BR, one can show that this matrix \( F(t) \) is actually RM of the considered 2\( \pi \)-periodic system.

Thus, by Theorem 6, \( F(-\pi) = E \) is the monodromy matrix of this system. Hence any solution of the considered system is 2\( \pi \)-periodic.

**Example 8.** Consider the system
\[ \dot{x} = \frac{1}{8} \cos t \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} x, \]
where
\[ a(t) \equiv 8 - 12k(1 - \cos 2t) - 24l \cos(2l \sin^3 t) \sin^3 t + (4 + 3l(3 + \cos 4t - 4 \cos 2t)) \sin(2l \sin^3 t), \]
\[ b(t) \equiv 4 + 3l(7 - 8 \cos 2t + \cos 4t) + 4(1 + 6l \sin^4 t) \cos(2l \sin^3 t) + 24l \sin^3 t \sin(2l \sin^3 t), \]
\[ c(t) \equiv -4 - 3l(7 - 8 \cos 2t + \cos 4t) + 4(1 + 6l \sin^4 t) \cos(2l \sin^3 t) + 24l \sin^3 t \sin(2l \sin^3 t), \]
\[ d(t) \equiv 8 - 12k(1 - \cos 2t) + 24l \cos(2l \sin^3 t) \sin^3 t - (4 + 3l(3 + \cos 4t - 4 \cos 2t)) \sin(2l \sin^3 t); \]
\( k, l \in \mathbb{R}. \)

Suppose that \( F(t) \equiv e^{A \sin^3 t} e^{B \sin t} e^{A \sin^3 t} \) is RM of the considered system, then, by Lemma 5, we have
\[ A = \begin{pmatrix} k & -l \\ l & k \end{pmatrix}, \quad B = \begin{pmatrix} -2 & -2 \\ 0 & -2 \end{pmatrix}. \]

Therefore,
\[ F(t) \equiv e^{-2(1-k \sin^2 t) \sin t} \begin{pmatrix} F_1(t) & F_2(t) \\ F_3(t) & F_4(t) \end{pmatrix}, \]
where
\[ F_1(t) \equiv F_4(t) \equiv \cos(2l \sin^3 t) - \sin(2l \sin^3 t) \sin t, \]
\[ F_2(t) \equiv -2 \cos^2(l \sin^3 t) \sin t - \sin(2l \sin^3 t), \]
\[ F_3(t) \equiv \sin(2l \sin^3 t) - 2 \sin^2(l \sin^3 t) \sin t. \]
By checkout of the BR, one can show that this matrix $F(t)$ is actually RM of the considered $2\pi$-periodic system.

Thus, by Theorem 6, $F(-\pi) = E$ is the monodromy matrix of this system. Hence any solution of the considered system is $2\pi$-periodic.

### 3. Nonlinear systems

The obtained results for linear differential system can be extended for nonlinear systems. We mark the case when a nonlinear system has the linear RF.

Consider the $2\pi \frac{p}{q}$-periodic system with respect to $t$ ($p, q \in \mathbb{N}$, $\frac{p}{q}$ is an irreducible fraction)

$$
\dot{x} = X(t, x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n,
$$

where $X(t, x)$ is a continuously differentiable function with respect to all of variables, and $X(t, 0) = 0$. By [7], if RF of system (9) is linear, then this RF is RF of system (5) with $P(t) \equiv X'(t, 0)$ too.

**Theorem 9.** Let the conditions of Theorem 6 be satisfied for system (5), where matrix $P(t) \equiv X'(t, 0)$. Moreover, suppose that

$$
\dot{F}(t)x + F(t)X(t, x) + X(-t, F(t)x) \equiv 0.
$$

Then any continuable on $[-\pi p; \pi p]$ solution of system (9) is $2\pi p$-periodic.

**Proof.** By checkout of the BR it is proved, that function $F(t, x) \equiv F(t)x$ is RF of system (9). Therefore, the assertion of the theorem follows from properties of RF [1, p. 14].

**Theorem 10.** Let the conditions of Theorem 3 be satisfied for system (5), where matrix $P(t) \equiv X'(t, 0)$. Then

1. if for any solution $\mu_i$, $i = 1, n$ of the equation
   
   $$
   \det(e^{-A\pi p}e^{-B\pi p}e^{-A\pi p} - \mu E) = 0
   $$

   the inequality $|\mu_i| < 1$ is true, then solution $x \equiv 0$ of system (9) is asymptotically stable;

2. if exist $\mu_i$ for which $|\mu_i| > 1$, then solution $x \equiv 0$ of system (9) is instable;

3. if identity (10) is true, where $F(t) \equiv e^{At}e^{Bt}e^{At}$, and exist $\mu_j = 1$, then exist at least one-parameter family of $2\pi p$-periodic solutions of system (9).

**Proof.** By Theorem 3, the multipicators $\mu_i$, $i = 1, n$ of system (5) are solutions of the equation

$$
\det(e^{-A\pi p}e^{-B\pi p}e^{-A\pi p} - \mu E) = 0.
$$

Assertions (1) and (2) of the theorem follow from properties of RF [15, p. 229]. If identity (10) is true, then mappings over the period of system (9) and system $\dot{x} = X'(t, 0)x$ are coinciding. Whence follows assertion (3) of the theorem.

### 4. Autonomous nonlinear systems

The obtained results for linear differential systems can also use for the investigation of the asymptotic orbital stability of a cycle of multidimensional autonomous differential systems.

Consider the system

$$
\dot{x} = f(x), \quad x \in D \subset \mathbb{R}^n,
$$

(11)
where \( f(x) \) is a continuously differentiable vector function. Let \( \eta(t) \) (\( \dot{\eta}(t) \neq 0 \)) be a \( 2\pi \frac{p}{q} \)-periodic solution of system (11), where \( p, q \in \mathbb{N} \), and \( \frac{p}{q} \) is an irreducible fraction.

**Theorem 11.** Let the conditions of Theorem 3 be satisfied for system (5), where matrix \( P(t) \equiv f_x'(\eta(t)) \). If among solutions \( \mu_i \) of the equation
\[
\det(e^{-A\pi p} e^{-B\pi p} e^{-A\pi p} - \mu E) = 0
\]
there is unique simple unit, and any others satisfy the inequality \( |\mu_i| < 1 \), then solution \( \eta(t) \) of system (11) is asymptotically orbitally stable.

**Proof.** By Theorem 3, the multiplicators \( \mu_i, i = 1, n \) of system (5) are solutions of the equation
\[
\det(e^{-A\pi p} e^{-B\pi p} e^{-A\pi p} - \mu E) = 0.
\]
The validity of this theorem readily follows from theorem in [14, p. 309]. \( \square \)

References