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On meromorphic functions that share three values of finite weights ☆

Qi Han [a,b,*], Hong-Xun Yi [a]

[a] *Department of Mathematics, Shandong University, Jinan 250100, Shandong, PR China*
[b] *Department of Mathematics, Yamagata University, 1-4-12 Kojirakawa-machi, Yamagata-shi 990-8560, Japan*

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Abstract

A uniqueness theorem for two distinct non-constant meromorphic functions that share three values of finite weights is proved, which generalizes two previous results by H.X. Yi, and X.M. Li and H.X. Yi. As applications of it, many known results by H.X. Yi and P. Li, etc. could be improved. Furthermore, with the concept of finite-weight sharing, extensions on Osgood–Yang's conjecture and Mues' conjecture, and a generalization of some prevenient results by M. Ozawa and H. Ueda, ect. could be obtained.

Keywords: Meromorphic functions; Lacunary value; Finite-weight sharing; Bilinear transformation

1. Introduction and main result

In this paper, a meromorphic function always means meromorphic in the complex plane $\mathbb{C}$. For any non-constant meromorphic function f, we use the standard notations of *Nevanlinna's value distribution theory* of meromorphic functions such as the *characteristic function* $T(r, f)$, the *proximity function* $m(r, f)$, and the *counting function* $N(r, f)$ of poles (see [3,4,19]). We denote by $\mathbb{E}$ any set of finite linear measure in $\mathbb{R}^+$, not necessarily the same at each occurrence. For the function f, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ $(r \notin \mathbb{E})$.

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* Corresponding author.
E-mail addresses: k.l.han@mail.sdu.edu.cn (Q. Han), hxyi@sdu.edu.cn (H.-X. Yi).

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For a complex number $a \in \mathbb{C}$, we say that two non-constant meromorphic functions f and g share the value a *CM* (respectively *IM*), provided that they have the same a-points counting (respectively ignoring) multiplicities. As for the value ∞, we consider the functions $F = 1/f$ and $G = 1/g$ sharing the value 0 instead.

For a positive integer k, we denote by $N_{k)}(r, 1/(f-a))$ the counting function of the a-points of f with multiplicity $\leqslant k$, by $N_{(k}(r, 1/(f-a))$ the counting function of the a-points of f with multiplicity $\geqslant k$, while by $\bar{N}_{k)}(r, 1/(f-a))$ and $\bar{N}_{(k}(r, 1/(f-a))$ the reduced form of $N_{k)}(r, 1/(f-a))$ and $N_{(k}(r, 1/(f-a))$, respectively. Also, we denote by $N_0(r)$ the counting function of the zeros of $f-g$ but not the zeros of f, $f-1$ and $1/f$, and those of g, $g-1$ and $1/g$, respectively, with proper multiplicity, while by $\bar{N}_0(r)$ its reduced form.

In 1995, H.X. Yi proved the following

Theorem A. *(See [12] or [22].) Let f and g be two distinct non-constant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If, for some $a \in \mathbb{C} \setminus \{0, 1\}$, we have*

$$N\left(r, \frac{1}{f-a}\right) \neq T(r, f) + S(r, f),$$

then a is a lacunary value of f, and f is some bilinear transformation of g. Furthermore, f and g satisfy one of the following three relations:

(i) $f \equiv ag$;

(ii) $f + (a-1)g \equiv a$;

(iii) $(f-a)(g+a-1) \equiv a(1-a)$.

Nine years after that, X.M. Li and H.X. Yi extended Theorem A and obtained the following

Theorem B. *(See [12].) Let f and g be two distinct non-constant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If, for some $a \in \mathbb{C} \setminus \{0, 1\}$, not a lacunary value of f, we have*

$$N_{1)}\left(r, \frac{1}{f-a}\right) \neq T(r, f) + S(r, f),$$

then we could derive

$$N_{1)}\left(r, \frac{1}{f-a}\right) = \frac{k-2}{k} T(r, f) + S(r, f)$$

and

$$N_{(2}\left(r, \frac{1}{f-a}\right) = \frac{2}{k} T(r, f) + S(r, f),$$

and f and g assume one of the following six forms:

(i) $$f = \frac{e^{(k+1)\gamma} - 1}{e^{s\gamma} - 1}, \qquad g = \frac{e^{-(k+1)\gamma} - 1}{e^{-s\gamma} - 1},$$
$$\frac{(a-1)^{k+1-s}}{a^{k+1}} = \frac{s^s (k+1-s)^{k+1-s}}{(k+1)^{k+1}}, \quad a \neq \frac{k+1}{s};$$

(ii) $$f = \frac{e^{s\gamma} - 1}{e^{(k+1)\gamma} - 1}, \qquad g = \frac{e^{-s\gamma} - 1}{e^{-(k+1)\gamma} - 1},$$

$$a^s(1-a)^{k+1-s} = \frac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}, \quad a \neq \frac{s}{k+1};$$

(iii) $$f = \frac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1}, \qquad g = \frac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1},$$

$$\frac{(-a)^s}{(1-a)^{k+1}} = \frac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}, \quad a \neq -\frac{s}{k+1-s};$$

(iv) $$f = \frac{e^{k\gamma}-1}{\lambda e^{s\gamma}-1}, \qquad g = \frac{e^{-k\gamma}-1}{\lambda^{-1}e^{-s\gamma}-1},$$

$$\lambda^k \neq 0, 1, \quad \frac{(a-1)^{k-s}}{\lambda^k a^k} = \frac{s^s(k-s)^{k-s}}{k^k};$$

(v) $$f = \frac{e^{s\gamma}-1}{\lambda e^{k\gamma}-1}, \qquad g = \frac{e^{-s\gamma}-1}{\lambda^{-1}e^{-k\gamma}-1},$$

$$\lambda^s \neq 0, 1, \quad \lambda^s a^s(1-a)^{k-s} = \frac{s^s(k-s)^{k-s}}{k^k};$$

(vi) $$f = \frac{e^{s\gamma}-1}{\lambda e^{-(k-s)\gamma}-1}, \qquad g = \frac{e^{-s\gamma}-1}{\lambda^{-1}e^{(k-s)\gamma}-1},$$

$$\lambda^s \neq 0, 1, \quad \frac{(-\lambda a)^s}{(1-a)^k} = \frac{s^s(k-s)^{k-s}}{k^k},$$

where γ is a non-constant entire function, and s and $k \geqslant 2$ are two positive integers such that s and $k+1$ are mutually prime with $1 \leqslant s \leqslant k$ in cases (i)–(iii)*, and such that s and k are mutually prime with $1 \leqslant s \leqslant k-1$ in cases* (iv)–(vi).

Remark. In fact, Theorem B is an extension of Theorem A, since if two distinct non-constant meromorphic functions f and g share the values 0, 1 and ∞ *CM*, then for any $a \in \mathbb{C} \setminus \{0, 1\}$, we have $N_{(3}(r, 1/(f-a)) = S(r, f)$ (see [12, Lemma 3]). Therefore, the assumption on the value a in Theorem B is equivalent to

$$N\left(r, \frac{1}{f-a}\right) = T(r, f) + S(r, f) \quad \text{but} \quad N_{1)}\left(r, \frac{1}{f-a}\right) \neq T(r, f) + S(r, f).$$

It is natural to ask whether the value-sharing assumptions of Theorems A and B could be weakened anymore? The answer is affirmative. Now, let us introduce the definitions of *finite-weight sharing* due to I. Lahiri (see [5–8,26]).

Definition 1. Let k be a non-negative integer, let f be a non-constant meromorphic function, and let $a \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be a complex number. Then, we denote by $E_k(a, f)$ the set of all the a-points of f, where an a-point with multiplicity m is counted m times if $m \leqslant k$ while $k+1$ times if $m > k$.

Definition 2. Let k be a non-negative integer, let f and g be two non-constant meromorphic functions, and let $a \in \bar{\mathbb{C}}$ be a complex number. If $E_k(a, f) = E_k(a, g)$, then we say that f and g share the value a with weight k.

We also write f and g sharing (a, k) to mean that they share the value a with weight k. If f and g share (a, k), then they share (a, p) for all integers p $(0 \leqslant p < k)$. Clearly, f and g share a value a *CM* if and only if they share (a,k) for all positive integers k, while f and g share a value a *IM* if and only if they share $(a,0)$.

By using the concept of *finite-weight sharing*, our main result states

Theorem 1. *Let f and g be two distinct non-constant meromorphic functions sharing $(0,k_1)$, $(1,k_2)$ and (∞,k_3) with $k_1k_2k_3 > k_1+k_2+k_3+2$. If, for some $a \in \mathbb{C}\setminus\{0,1\}$, we have*

$$N_{1)}\left(r,\frac{1}{f-a}\right) \neq T(r,f)+S(r,f), \tag{1.1}$$

then f and g share the values 0, 1 and ∞ CM. Thus, the conclusions of Theorems A *and* B*, respectively, still hold.*

Since f and g share the values 0, 1 and ∞ *IM*, then by the *Second Main Theorem*, we could easily derive that $T(r,f)=O(T(r,g))$ and $T(r,g)=O(T(r,f))$, which implies $S(r,f)=S(r,g)$. In the following, we denote this term by $S(r)$.

2. Some lemmas

Lemma 1. *(See [26].) Let f and g be two distinct non-constant meromorphic functions sharing $(0,k_1)$, $(1,k_2)$ and (∞,k_3) with $k_1k_2k_3 > k_1+k_2+k_3+2$. Then, for $h \in \{f,g\}$, we have*

$$\bar{N}_{(2}\left(r,\frac{1}{h}\right)+\bar{N}_{(2}\left(r,\frac{1}{h-1}\right)+\bar{N}_{(2}(r,h)=S(r). \tag{2.1}$$

Lemma 2. *Let f and g be two distinct non-constant meromorphic functions sharing $(0,k_1)$, $(1,k_2)$ and (∞,k_3) with $k_1k_2k_3 > k_1+k_2+k_3+2$, and suppose that f is not any bilinear transformation of g. Then, for $h \in \{f,g\}$, we have*

$$\text{(i)}\quad T(r,f)+T(r,g)=\bar{N}\left(r,\frac{1}{h}\right)+\bar{N}\left(r,\frac{1}{h-1}\right)+\bar{N}(r,h)+N_0(r)+S(r); \tag{2.2}$$

$$\text{(ii)}\quad N\left(r,\frac{1}{f-g}\right)=\bar{N}\left(r,\frac{1}{f-g}\right)+S(r),\qquad N_0(r)=\bar{N}_0(r)+S(r). \tag{2.3}$$

Proof. The method we employed here for the proof of Lemma 2 is similar to that of the main result in [8] and [22], respectively. For the sake of convenience for the reader, we shall outline a proof of it.

Noting that f and g share $(0,k_1)$, $(1,k_2)$ and (∞,k_3) with $k_1k_2k_3 > k_1+k_2+k_3+2$, then we define

$$h_1 := \frac{f}{g},\qquad h_2 := \frac{f-1}{g-1},$$

and thus

$$f=\frac{h_2-1}{h_3-1},\qquad g=\frac{h_2^{-1}-1}{h_3^{-1}-1}\quad\left(h_3 := \frac{h_2}{h_1}\right). \tag{2.4}$$

It is obvious that for $h \in \{f,g\}$, we have $\sum_{j=1}^{3}T(r,h_j)=O(T(r,h))$ and $T(r,h)=O(\sum_{j=1}^{3}T(r,h_j))$. By Lemma 1, we see that

$$\sum_{j=1}^{3}\left(\bar{N}(r,h_j)+\bar{N}\left(r,\frac{1}{h_j}\right)\right)\leqslant O\left(\bar{N}_{(2}\left(r,\frac{1}{h}\right)+\bar{N}_{(2}\left(r,\frac{1}{h-1}\right)+\bar{N}_{(2}(r,h)\right)=S(r).$$

Therefore,

$$\sum_{j=1}^{3}T\left(r,\frac{h_j'}{h_j}\right)=S(r). \tag{2.5}$$

If one of h_1, h_2 and h_3 is a constant, then f would be some bilinear transformation of g, which contradicts the assumption. Thus, in the following, we suppose that none of h_1, h_2 and h_3 is a constant.

Now, we define

$$\varphi:=\frac{\frac{h_2'}{h_2}}{\frac{h_3'}{h_3}}=\frac{\frac{h_2'}{h_2}}{\frac{h_2'}{h_2}-\frac{h_1'}{h_1}}.$$

Then, $\varphi\not\equiv 0, 1$ and by (2.5), $T(r,\varphi)=S(r)$.

If

$$(\varphi-1)\frac{h_2'}{h_2}-\varphi'\equiv 0,$$

then we have $h_2\equiv c(\varphi-1)$ for some constant $c\neq 0$, and thus $T(r,h_2)=S(r)$. Also, we have

$$\frac{h_3'}{h_3}=\frac{h_2'}{\varphi h_2}=\frac{ch_2'}{h_2(h_2+c)}=\frac{h_2'}{h_2}-\frac{h_2'}{h_2+c},$$

and hence $h_3=c_1\frac{h_2}{h_2+c}$ for some constant $c_1\neq 0$, from which we have $T(r,h_3)=S(r)$, too. By (2.4), we derive that $T(r,f)=S(r)$ and $T(r,g)=S(r)$, a contradiction.

Hence, $(\varphi-1)\frac{h_2'}{h_2}-\varphi'\not\equiv 0$. Noting that

$$f-\varphi=\frac{h_2-\varphi h_3+\varphi-1}{h_3-1},$$

thus we define $\phi:=(f-\varphi)(h_3-1)=(h_2-\varphi h_3+\varphi-1)$, which combined with the expression of φ could yield

$$\frac{\phi'}{\phi}-\frac{h_2'}{h_2}=\frac{(h_2-\varphi h_3+\varphi-1)'-(h_2-\varphi h_3+\varphi-1)\frac{h_2'}{h_2}}{(f-\varphi)(h_3-1)}=\frac{(\varphi-1)\frac{h_2'}{h_2}-\varphi'}{f-\varphi}.$$

Hence, we obtain

$$\frac{1}{f-\varphi}=\frac{\frac{\phi'}{\phi}-\frac{h_2'}{h_2}}{(\varphi-1)\frac{h_2'}{h_2}-\varphi'},$$

which implies that

$$m\left(r,\frac{1}{f-\varphi}\right)=S(r)\quad\text{and}\quad N_{(2}\left(r,\frac{1}{f-\varphi}\right)=S(r). \tag{2.6}$$

Since $\frac{f-g}{g-1} = h_2 - 1$ and $g = \frac{h_2-1}{h_2-h_1}$, then we have

$$\frac{g'}{g}\frac{f-g}{g-1} = \frac{(\frac{h_1'}{h_1} - \frac{h_2'}{h_2})h_2 + h_3\frac{h_2'}{h_2} - \frac{h_1'}{h_1}}{h_3 - 1}.$$

Also, we have

$$(f-\varphi)\left(\frac{h_1'}{h_1} - \frac{h_2'}{h_2}\right) = \frac{(\frac{h_1'}{h_1} - \frac{h_2'}{h_2})h_2 + h_3\frac{h_2'}{h_2} - \frac{h_1'}{h_1}}{h_3 - 1},$$

too. Therefore,

$$(f-\varphi)\left(\frac{h_1'}{h_1} - \frac{h_2'}{h_2}\right) = \frac{g'}{g}\frac{f-g}{g-1}. \tag{2.7}$$

By (2.1), (2.5), the second equation of (2.6) and (2.7), and noting the fact that $T(r,\varphi) = S(r)$, we derive

$$\bar{N}\left(r, \frac{1}{f-\varphi}\right) = N_0(r) + N_0\left(r, \frac{1}{g'}\right) + S(r),$$

and hence $N_0(r) = \bar{N}_0(r) + S(r)$, which is the second equation of (2.3), where $N_0(r, 1/g')$ denotes the counting function of the zeros of g' but not the multiple zeros of $g(g-1)$.

Also, by (2.6), the above equation and the *First Main Theorem*, we have

$$T(r,f) = N_0(r) + N_0\left(r, \frac{1}{g'}\right) + S(r). \tag{2.8}$$

Applying the *Second Main Theorem* to the function g with the values 0, 1 and ∞, noting (2.8) and the fact that $\bar{N}(r,f) = \bar{N}(r,g) + O(1)$, to conclude

$$\begin{aligned}
T(r,f) + T(r,g) &\leqslant T(r,f) + \bar{N}(r,g) + \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{g-1}\right) - N_0\left(r, \frac{1}{g'}\right) + S(r)\\
&\leqslant N_0(r) + \bar{N}(r,g) + \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{g-1}\right) + S(r)\\
&\leqslant \bar{N}(r,g) + \bar{N}\left(r, \frac{1}{f-g}\right) + S(r)\\
&\leqslant \bar{N}(r,g) + N\left(r, \frac{1}{f-g}\right) + S(r)\\
&\leqslant T(r, f-g) + \bar{N}(r,g) + S(r)\\
&\leqslant m(r,f) + m(r,g) + N(r,f) + N(r,g) + S(r)\\
&\leqslant T(r,f) + T(r,g) + S(r),
\end{aligned}$$

which implies that (2.2) and the first equation of (2.3). □

Lemma 3. *Let f and g be two distinct non-constant meromorphic functions sharing $(0,k_1)$, $(1,k_2)$ and (∞,k_3) with $k_1k_2k_3 > k_1 + k_2 + k_3 + 2$. Then, for any $a \in \mathbb{C} \setminus \{0,1\}$ and $h \in \{f,g\}$, we have*

$$N_{(3}\left(r, \frac{1}{h-a}\right) = S(r). \tag{2.9}$$

Proof. Without loss of generality, we might assume $h = f$. If f is some bilinear transformation of g, then the conclusion is trivial since now, for any $a \in \mathbb{C} \setminus \{0, 1\}$, we have either $T(r, f) = N_{1)}(r, 1/(f - a)) + S(r)$ by the *Second Main Theorem*, or a is a lacunary value of f (see [12, Lemmas 1 and 2]). Thus, we suppose that f is not any bilinear transformation of g in the following.

With the same notations such as h_1, h_2, h_3 and φ in the proof of Lemma 2, we know that neither h_2 nor h_3 is a constant, and have

$$f - a = \frac{h_2 - ah_3 + (a-1)}{h_3 - 1}. \tag{2.10}$$

Take z_a to be an a-point of f with multiplicity $p \geqslant 3$ but not a zero or a pole of h_2 and h_3. Thus, by assumption, we have

$$h_2(z_a) - ah_3(z_a) + (a-1) = 0,$$
$$h_2'(z_a) - ah_3'(z_a) = 0,$$

and

$$h_2''(z_a) - ah_3''(z_a) = 0.$$

The above last two equations imply

$$\frac{h_2''(z_a)}{h_2'(z_a)} - \frac{h_3''(z_a)}{h_3'(z_a)} = 0.$$

If $\frac{h_2''}{h_2'} - \frac{h_3''}{h_3'} \equiv 0$, then by integrating it twice, we have $h_2 \equiv c_0 h_3 + c_1$ for two constants $c_0 \neq 0$, c_1. If $c_1 \neq 0$, then by the proof of Lemma 2, we have

$$\bar{N}\left(r, \frac{1}{h_3 + \frac{c_1}{c_0}}\right) = \bar{N}\left(r, \frac{1}{h_2}\right) + O(1) = S(r),$$

which means

$$T(r, h_3) \leqslant \bar{N}\left(r, \frac{1}{h_3 + \frac{c_1}{c_0}}\right) + \bar{N}\left(r, \frac{1}{h_3}\right) + \bar{N}(r, h_3) + S(r) = S(r).$$

Then, $T(r, h_2) = S(r)$, too. Hence, $T(r, f) = S(r)$ by (2.4), a contradiction. So, $c_1 = 0$. If $c_0 \neq a$, then by (2.10), the *First Main Theorem*, the *lemma of logarithmic derivative*, and the fact that

$$\sum_{j=1}^{3}\left(\bar{N}(r, h_j) + \bar{N}\left(r, \frac{1}{h_j}\right)\right) = S(r),$$

we obtain

$$\begin{aligned}
\bar{N}_{(3}\left(r, \frac{1}{f-a}\right) &\leqslant \bar{N}\left(r, \frac{1}{h_2'}\right) \leqslant \bar{N}\left(r, \frac{h_2}{h_2'}\right) + \bar{N}\left(r, \frac{1}{h_2}\right) + O(1) \\
&\leqslant T\left(r, \frac{h_2'}{h_2}\right) + \bar{N}\left(r, \frac{1}{h_2}\right) + O(1) \\
&\leqslant \bar{N}(r, h_2) + 2\bar{N}\left(r, \frac{1}{h_2}\right) + S(r) = S(r).
\end{aligned} \tag{2.11}$$

If $c_0 = a$, then by (2.10), we see that $\bar{N}_{(3}(r, 1/(f-a)) \leqslant \bar{N}(r, h_3) + O(1) = S(r)$. Hence, (2.11) holds, too.

If $\frac{h_2''}{h_2'} - \frac{h_3''}{h_3'} \not\equiv 0$, combining this with the *lemma of logarithmic derivative* and the facts that

$$\bar{N}\left(r, \frac{1}{h_j'}\right) \leqslant \bar{N}(r, h_j) + 2\bar{N}\left(r, \frac{1}{h_j}\right) + S(r) = S(r)$$

(shown in the proof in (2.11)) and $\bar{N}(r, h_j') = \bar{N}(r, h_j) + O(1) = S(r)$ for $j = 2, 3$, yields

$$\bar{N}_{(3}\left(r, \frac{1}{f-a}\right) \leqslant O\left(\sum_{j=2}^{3}\left(\bar{N}(r, h_j) + \bar{N}\left(r, \frac{1}{h_j}\right)\right)\right) + S(r) = S(r).$$

Also, (2.11) holds well.

Noting that z_a is an a-point of f with multiplicity $p \geqslant 3$, then it is a zero of $f'(f-g)$ with multiplicity at least $p - 1 \geqslant 2$. Combining the second equation of (2.6), (2.7) (interchanging positions of f and g, respectively) with (2.5) yields

$$\begin{aligned} N_{(3}\left(r, \frac{1}{f-a}\right) - \bar{N}_{(3}\left(r, \frac{1}{f-a}\right) &\leqslant N_{(2}\left(r, \frac{1}{f'}\right) + O(1) \\ &\leqslant N_{(2}\left(r, \frac{1}{f'(f-g)}\right) + O(1) \\ &\leqslant N_{(2}\left(r, \frac{1}{g-\varphi}\right) + S(r) = S(r), \end{aligned} \tag{2.12}$$

which together with (2.11) implies (2.9). □

Lemma 4. *Let f and g be two distinct non-constant meromorphic functions sharing* 0, 1 *and* ∞ *IM. Furthermore, if we assume that f is some bilinear transformation of g, then they satisfy one of the following six relations*:

(i) $fg \equiv 1$,
(ii) $f + g \equiv 1$,
(iii) $(f-1)(g-1) \equiv 1$,
(iv) $f \equiv \alpha g$,
(v) $f - 1 \equiv \alpha(g-1)$,
(vi) $(f-\alpha)(g+\alpha-1) \equiv \alpha(1-\alpha)$,

where $\alpha \neq 0, 1$ is a constant.

Proof. Without loss of generality, we may suppose that

$$f = \frac{ag+b}{cg+d}, \quad (ad - bc \neq 0).$$

Noting that f and g are distinct, we shall discuss the following six cases.

Case (i). If 0 and ∞ are lacunary values of f and g, then $a + b = c + d$ and $a = d = 0$, since f and g have infinitely many 1-points, which means $fg \equiv 1$.

Case (ii). If 0 and 1 are lacunary values of f and g, then $c = 0$ and $f = \frac{a}{d}g + \frac{b}{d}$. Further, we have $\frac{a}{d} + \frac{b}{d} = 0$ and $\frac{b}{d} = 1$, which is just $f + g \equiv 1$.

Case (iii). If 1 and ∞ are lacunary values of f and g, then $b=0$ and $c=-d$. Thus, $f=\frac{ag}{c(g-1)}$, which could be rewritten as $(f-1)(g-1)\equiv 1$.

Case (iv). If only 1 is a lacunary value of f and g, then $b=c=0$ and thus we have $f\equiv\alpha g$, where $\alpha=\frac{a}{d}\neq 0,1$.

Case (v). If only 0 is a lacunary value of f and g, then $c=0$ and $d=a+b$. Then, we have $f-1\equiv\alpha(g-1)$ with $\alpha=\frac{a}{d}\neq 0,1$.

Case (vi). If only ∞ is a lacunary value of f and g, then $b=0$ and $a=c+d$. Therefore, $(f-\alpha)(g+\alpha-1)\equiv\alpha(1-\alpha)$ with $\alpha=1+\frac{d}{c}\neq 0,1$. □

Lemma 5. *(See [2] or [27].) Let ω_1 and ω_2 be two non-constant meromorphic functions satisfying $\bar{N}(r,\omega_j)+\bar{N}(r,1/\omega_j)=S^*(r)$ for $j=1,2$. If $\omega_1^s\omega_2^t-1$ is not identically equal to zero for all integers s and t satisfying $|s|+|t|>0$, then we have $N_0(r,1;\omega_1,\omega_2)=S^*(r)$. Where $N_0(r,1;\omega_1,\omega_2)$ denotes the reduced counting function of the common 1-points of ω_1 and ω_2, and $S^*(r)=o(T(r):=T(r,\omega_1)+T(r,\omega_2))$ $(r\notin\mathbb{E})$ only depends on ω_1 and ω_2.*

3. Proof of Theorem 1

Let us proceed the proof of Theorem 1 with two cases.

Case 1.

$$N\left(r,\frac{1}{f-a}\right)\neq T(r,f)+S(r). \tag{3.1}$$

If f is a bilinear transformation of g, then from the conclusions of Lemma 4, we could easily see that f and g share the values 0, 1 and ∞ *CM*. Cases (i)–(iii) in Lemma 4 contradict (3.1), and thus might be ruled out. Case (iv) in Lemma 4 means $a=\alpha$, a lacunary value of f, and then case (i) in Theorem A occurs. Also, case (v) in Lemma 4 means $a=1-\alpha$, a lacunary value of f, and then case (ii) in Theorem A occurs. At last, case (vi) in Lemma 4 means $a=\alpha$, a lacunary value of f, and hence case (iii) in Theorem A occurs.

If f is not any bilinear transformation of g, then by (2.2) and (2.8) (with interchanged positions of f and g), plus the *Second Main Theorem*, we see

$$\begin{aligned}2T(r,f)&\leqslant N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-1}\right)+\bar{N}(r,f)+N\left(r,\frac{1}{f-a}\right)-N\left(r,\frac{1}{f'}\right)+S(r)\\&\leqslant \bar{N}\left(r,\frac{1}{f}\right)+\bar{N}\left(r,\frac{1}{f-1}\right)+\bar{N}(r,f)+N\left(r,\frac{1}{f-a}\right)-\bar{N}_0\left(r,\frac{1}{f'}\right)+S(r)\\&\leqslant T(r,f)+N\left(r,\frac{1}{f-a}\right)+S(r)\leqslant 2T(r,f)+S(r),\end{aligned}$$

which implies that $T(r,f)=N(r,1/(f-a))+S(r)$, a contradiction against (3.1).

Case 2.

$$N\left(r,\frac{1}{f-a}\right)=T(r,f)+S(r). \tag{3.2}$$

Noting (1.1) and (2.9), we know that

$$\bar{N}_{(2}\left(r, \frac{1}{f-a}\right) \neq S(r). \tag{3.3}$$

We continue to use those notations such as h_1, h_2 and h_3 in the proof of Lemma 2. Then, from (2.10), (3.3) and the fact that $\sum_{j=1}^{3}(\bar{N}(r, h_j) + \bar{N}(r, 1/h_j)) = S(r)$, it is not difficult to claim that $T(r, h_j) \neq S(r)$ for $j = 1, 2, 3$.

In fact, if $T(r, h_1) = S(r)$, then we rewrite (2.10) as

$$f - a = \frac{(h_1 - a)h_3 + a - 1}{h_3 - 1},$$

and obtain $T(r, f) = T(r, h_3) + S(r)$. Obviously, $h_1 - a \not\equiv 0$. Otherwise, it might derive that $f \equiv ag$, which implies that 1 and a are lacunary values of f, a contradiction against (3.2). Now, applying the *Second Main Theorem* concerning three small functions (see [19, Theorem 1.36]) to the function h_3 with its small functions 0, ∞ and $\beta := -(a-1)/(h_1 - a)$ to conclude that

$$\begin{aligned} T(r, h_3) &\leqslant \bar{N}(r, h_3) + \bar{N}\left(r, \frac{1}{h_3}\right) + \bar{N}\left(r, \frac{1}{h_3 - \beta}\right) + S(r) \\ &\leqslant \bar{N}\left(r, \frac{1}{h_3 - \beta}\right) + S(r) \leqslant \bar{N}\left(r, \frac{1}{h_3 - \beta}\right) + \frac{1}{2}N_{(2}\left(r, \frac{1}{h_3 - \beta}\right) + S(r) \\ &\leqslant N\left(r, \frac{1}{h_3 - \beta}\right) + S(r) \leqslant T(r, h_3) + S(r), \end{aligned}$$

which implies that $N_{(2}(r, 1/(h_3 - \beta)) = S(r)$. Hence, we could immediately derive that $\bar{N}_{(2}(r, 1/(f-a)) \leqslant N_{(2}(r, 1/(h_3 - \beta)) + S(r) = S(r)$, a contradiction against (3.3). Analogous discussions could yield $T(r, h_2) \neq S(r)$ and $T(r, h_3) \neq S(r)$.

Let z_a be a multiple a-point of f but not a zero or a pole of $\frac{h_2'}{h_2}$, $\frac{h_3'}{h_3}$ and $\frac{h_2'}{h_2} - \frac{h_3'}{h_3}$. Since $h_2(z_a) - ah_3(z_a) + (a-1) = 0$ and $h_2'(z_a) - ah_3'(z_a) = 0$, we have

$$h_2(z_a) = \frac{(a-1)\frac{h_3'(z_a)}{h_3(z_a)}}{\frac{h_2'(z_a)}{h_2(z_a)} - \frac{h_3'(z_a)}{h_3(z_a)}} \quad \text{and} \quad h_3(z_a) = \frac{(a-1)\frac{h_2'(z_a)}{h_2(z_a)}}{a(\frac{h_2'(z_a)}{h_2(z_a)} - \frac{h_3'(z_a)}{h_3(z_a)})}.$$

Now, let us define

$$\omega_1 := \frac{h_2(\frac{h_2'}{h_2} - \frac{h_3'}{h_3})}{(a-1)\frac{h_3'}{h_3}}, \qquad \omega_2 := \frac{ah_3(\frac{h_2'}{h_2} - \frac{h_3'}{h_3})}{(a-1)\frac{h_2'}{h_2}}, \tag{3.4}$$

and

$$T(r) := T(r, \omega_1) + T(r, \omega_2), \qquad S^*(r) := o\big(T(r)\big) \quad (r \notin \mathbb{E}).$$

It is easily seen that for $h \in \{f, g\}$, we have

$$\sum_{j=1}^{2} T(r, \omega_j) = O\big(T(r, h)\big), \qquad T(r, h) = O\left(\sum_{j=1}^{2} T(r, \omega_j)\right),$$

and

$$S^*(r) = S(r), \qquad \sum_{j=1}^{2}\left(\bar{N}(r, \omega_j) + \bar{N}\left(r, \frac{1}{\omega_j}\right)\right) = S(r).$$

Since now

$$\bar{N}_{(2}\left(r, \frac{1}{f-a}\right) \leqslant N_0(r, 1; \omega_1, \omega_2) + S(r),$$

thus by (3.3) and the conclusion of Lemma 5, we know that there exist two integers s and t such that $|s| + |t| > 0$, and such that $\omega_1^s \omega_2^t \equiv 1$. It could be rewritten as

$$h_2^s h_3^t \equiv \left(\frac{(a-1)\frac{h_3'}{h_3}}{\frac{h_2'}{h_2} - \frac{h_3'}{h_3}}\right)^s \left(\frac{(a-1)\frac{h_2'}{h_2}}{a(\frac{h_2'}{h_2} - \frac{h_3'}{h_3})}\right)^t. \tag{3.5}$$

Applying logarithmic differentiation to (3.5) to obtain

$$s\frac{h_2'}{h_2} + t\frac{h_3'}{h_3} \equiv \frac{s\frac{h_2'}{h_2} + t\frac{h_3'}{h_3}}{\frac{h_2'}{h_2}(1 - \frac{h_2'}{h_2}\frac{h_3}{h_3'})}\left(\frac{h_2'}{h_2}\frac{h_3}{h_3'}\right)'. \tag{3.6}$$

If $s\frac{h_2'}{h_2} + t\frac{h_3'}{h_3} \not\equiv 0$, then we have

$$\frac{h_2'}{h_2} \equiv \frac{(\frac{h_2'}{h_2}\frac{h_3}{h_3'})'}{1 - \frac{h_2'}{h_2}\frac{h_3}{h_3'}}.$$

Applying integration to it twice, we obtain $h_3 \equiv c_2(h_2 - c_1)$, where c_1, c_2 are two non-zero constants. So,

$$T(r, h_2) \leqslant \bar{N}\left(r, \frac{1}{h_2 - c_1}\right) + S(r) = \bar{N}\left(r, \frac{1}{h_3}\right) + S(r) = S(r),$$

a contradiction against the fact that $T(r, h_j) \neq S(r)$ for $j = 1, 2, 3$.

Therefore, $s\frac{h_2'}{h_2} + t\frac{h_3'}{h_3} \equiv 0$. Hence, by integration, we get $h_2^s h_3^t \equiv c$ for a non-zero constant c. Since none of h_1, h_2 and h_3 is a constant, then we have $st(s+t) \neq 0$. Rewrite it in terms of f and g as

$$\left(\frac{g}{f}\right)^t \left(\frac{f-1}{g-1}\right)^{s+t} \equiv c,$$

which implies that f and g share the values 0, 1 and ∞ *CM*.

4. Applications of Theorem 1

In the same paper, X.M. Li and H.X. Yi obtained the following two theorems, the former of which was an extension of a previous result by P. Li (see [10]).

Theorem C. *(See [12].) Let f and g be two distinct non-constant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If, for some $a \in \mathbb{C} \setminus \{0, 1\}$, not a lacunary value of f, we have*

$$N_{1)}\left(r, \frac{1}{f-a}\right) \leqslant uT(r, f) + S(r),$$

where $u < \frac{1}{3}$, then

$$N_{1)}\left(r, \frac{1}{f-a}\right) = O(1),$$

and f and g assume one of the following nine forms:

$$\text{(i)} \quad f = \frac{e^{3\gamma}-1}{e^{\gamma}-1}, \qquad g = \frac{e^{-3\gamma}-1}{e^{-\gamma}-1}, \qquad a = \frac{3}{4};$$

$$\text{(ii)} \quad f = \frac{e^{3\gamma}-1}{e^{2\gamma}-1}, \qquad g = \frac{e^{-3\gamma}-1}{e^{-2\gamma}-1}, \qquad a = -3;$$

$$\text{(iii)} \quad f = \frac{e^{\gamma}-1}{e^{3\gamma}-1}, \qquad g = \frac{e^{-\gamma}-1}{e^{-3\gamma}-1}, \qquad a = \frac{4}{3};$$

$$\text{(iv)} \quad f = \frac{e^{2\gamma}-1}{e^{3\gamma}-1}, \qquad g = \frac{e^{-2\gamma}-1}{e^{-3\gamma}-1}, \qquad a = -\frac{1}{3};$$

$$\text{(v)} \quad f = \frac{e^{2\gamma}-1}{e^{-\gamma}-1}, \qquad g = \frac{e^{-2\gamma}-1}{e^{\gamma}-1}, \qquad a = \frac{1}{4};$$

$$\text{(vi)} \quad f = \frac{e^{\gamma}-1}{e^{-2\gamma}-1}, \qquad g = \frac{e^{-\gamma}-1}{e^{2\gamma}-1}, \qquad a = 4;$$

$$\text{(vii)} \quad f = \frac{e^{2\gamma}-1}{\lambda e^{\gamma}-1}, \qquad g = \frac{e^{-2\gamma}-1}{\lambda^{-1}e^{-\gamma}-1}, \qquad \lambda^2 \neq 1, \quad a^2\lambda^2 = 4(a-1);$$

$$\text{(viii)} \quad f = \frac{e^{\gamma}-1}{\lambda e^{2\gamma}-1}, \qquad g = \frac{e^{-\gamma}-1}{\lambda^{-1}e^{-2\gamma}-1}, \qquad \lambda \neq 1, \quad 4a(1-a)\lambda = 1;$$

$$\text{(ix)} \quad f = \frac{e^{\gamma}-1}{\lambda e^{-\gamma}-1}, \qquad g = \frac{e^{-\gamma}-1}{\lambda^{-1}e^{\gamma}-1}, \qquad \lambda \neq 1, \quad (1-a)^2 + 4a\lambda = 0,$$

where γ is a non-constant entire function.

Theorem D. *(See [12].) Let f and g be two distinct non-constant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If, for some $a \in \mathbb{C} \setminus \{0, 1\}$, we have*

$$N_{1)}\left(r, \frac{1}{f-a}\right) \leqslant uT(r, f) + S(r),$$

$$\bar{N}(r, f) \leqslant vT(r, f) + S(r),$$

and

$$N_{1)}\left(r, \frac{1}{g-a}\right) \neq T(r, g) + S(r),$$

where $u < \frac{1}{3}$ and $v < \frac{1}{2}$, then

$$N_{1)}\left(r, \frac{1}{f-a}\right) = O(1),$$

and one of the following three cases holds:

(i) $\left(f-\frac{1}{2}\right)\left(g-\frac{1}{2}\right)\equiv\frac{1}{4}, \qquad a=\frac{1}{2};$

(ii) $f=-e^{2\gamma}-e^{\gamma}, \qquad g=-e^{-2\gamma}-e^{-\gamma}, \qquad a=\frac{1}{4};$

(iii) $f=e^{2\gamma}+e^{\gamma}+1, \qquad g=e^{-2\gamma}+e^{-\gamma}+1, \qquad a=\frac{3}{4},$

where γ is a non-constant entire function.

From the conclusions of Theorem 1, we could show the following

Theorem 2. *Let f and g be two distinct non-constant meromorphic functions sharing $(0,k_1)$, $(1,k_2)$ and (∞,k_3) with $k_1k_2k_3>k_1+k_2+k_3+2$. If we retain the other assumptions in Theorems* C *and* D*, respectively, then their conclusions hold.*

Proof. Since we have the inequality $N_{1)}(r,1/(f-a))\leqslant uT(r,f)+S(r)$ anyway, thus (1.1) holds. Combining the methods in the original proof of Theorems C and D, respectively, with the conclusions of Theorem 1 yields the desired results. □

Also, in that same paper, X.M. Li and H.X. Yi gave a concise proof of the following Theorem E, which was a beautiful result in the earlier studies of unique range sets and also an extension of a previous result of H.X. Yi in [20].

Now, let us introduce the idea of *finite-weight sharing concerning sets* firstly.

Definition 3. Let k be a non-negative integer, let f and g be two non-constant meromorphic functions, and let $\mathbb{S}\subseteq\mathbb{C}\cup\{\infty\}$ be a set with distinct elements. Then, we define $E_k(\mathbb{S},f):=\bigcup_{a\in\mathbb{S}}E_k(a,f)$. If $E_k(\mathbb{S},f)=E_k(\mathbb{S},g)$, then we say that f and g share the set $\mathbb{S}$ with weight k.

Theorem E. *(See [10] or [12] or [23].) Let $\mathbb{S}_1=\{a_1,a_2\}$ and $\mathbb{S}_2=\{b_1,b_2\}$ be two sets of distinct elements with $a_1+a_2=b_1+b_2$ but $a_1a_2\neq b_1b_2$, and set $\mathbb{S}_3=\{\infty\}$. Suppose that two distinct non-constant meromorphic functions f and g share $\mathbb{S}_1$, $\mathbb{S}_2$ and $\mathbb{S}_3$ CM, then f and g have one of the following four relations*:

(i) $f+g\equiv a_1+a_2;$

(ii) $\left(f-\frac{c}{2}\right)\left(g-\frac{c}{2}\right)\equiv\left(\frac{a_1-a_2}{2}\right)^2 \quad (c=a_1+a_2);$

(iii) $(f-a_j)(g-a_k)\equiv(-1)^{j+k}(a_1-a_2)^2 \quad (j,k=1,2);$

(iv) $(f-b_j)(g-b_k)\equiv(-1)^{j+k}(b_1-b_2)^2 \quad (j,k=1,2),$

where case (ii) *occurs only for $(a_1-a_2)^2+(b_1-b_2)^2=0$, case* (iii) *occurs only for $3(a_1-a_2)^2+(b_1-b_2)^2=0$, while case* (iv) *occurs only for $(a_1-a_2)^2+3(b_1-b_2)^2=0$.*

Combining analogous method as that in the proof of Theorem E in [12] with the conclusions of Theorem 1 could yield the same conclusions if we weaken the assumption that f and g share the sets $\mathbb{S}_1$, $\mathbb{S}_2$ and $\mathbb{S}_3$ *CM* to $E_{k_1}(\mathbb{S}_1,f)=E_{k_1}(\mathbb{S}_1,g)$, $E_{k_2}(\mathbb{S}_2,f)=E_{k_2}(\mathbb{S}_2,g)$ and $E_{k_3}(\mathbb{S}_3,f)=E_{k_3}(\mathbb{S}_3,g)$ for three positive integers k_1, k_2 and k_3 such that $k_1k_2k_3>k_1+k_2+k_3+2$.

5. On conjectures of Osgood–Yang and Mues

It is well known that C.F. Osgood and C.C. Yang conjectured that if two distinct non-constant entire functions f and g share the values 0 and 1 *CM*, then

$$T(r,f) \sim T(r,g) \quad (r \to \infty,\ r \notin \mathbb{E}).$$

Nineteen years after they proposed the above conjecture, in 1995, E. Mues extended it to meromorphic functions and conjectured that if two distinct non-constant meromorphic functions f and g share the values 0, 1 and ∞ *CM*, then

$$\left(\frac{1}{2}+o(1)\right) \leqslant \frac{T(r,f)}{T(r,g)} \leqslant \bigl(2+o(1)\bigr) \quad (r \to \infty,\ r \notin \mathbb{E}).$$

Also, the bounds $1/2$ and 2 could not be sharpened any more as shown in [1].

The first promising result that shows the above two conjectures could be solved was obtained by P. Li and C.C. Yang in 1998 (see [9]). Then, in 1999, by employing a result of Y.H. Li and Q.C. Zhang (see [11]), which plays quite an important role in sharpening the *Second Main Theorem* concerning small functions (see [13,15,18]), the second author of that paper proved the following result, whose embryonic form could be found in [9].

Theorem F. *(See [2] or [27].) Let f and g be two distinct non-constant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If $N_0(r) \neq S(r)$, then f is not any bilinear transformation of g if and only if*

$$0 < \limsup_{r\to\infty,\, r\notin\mathbb{E}} \frac{N_0(r)}{T(r,f)} \leqslant \frac{1}{2},$$

and then

$$N_0(r) = \frac{1}{k}T(r,f) + S(r).$$

Furthermore, f and g assume one of the following three forms:

$$\text{(i)} \quad f = \frac{e^{s\gamma}-1}{e^{(k+1)\gamma}-1}, \qquad g = \frac{e^{-s\gamma}-1}{e^{-(k+1)\gamma}-1}, \quad 1 \leqslant s \leqslant k;$$

$$\text{(ii)} \quad f = \frac{e^{(k+1)\gamma}-1}{e^{(k+1-s)\gamma}-1}, \qquad g = \frac{e^{-(k+1)\gamma}-1}{e^{-(k+1-s)\gamma}-1}, \quad 1 \leqslant s \leqslant k;$$

$$\text{(iii)} \quad f = \frac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1}, \qquad g = \frac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1}, \quad 1 \leqslant s \leqslant k,$$

where s and $k \geqslant 2$ are two positive integers such that s and $k+1$ are mutually prime, and γ is a non-constant entire function.

In 2003, by using the conclusions of Theorem F and an equality in [22] like (2.2), H.X. Yi and Y.H. Li completely solved the above two conjectures (see [24]). Some extensions on their results could be found in [2,8]. Here, we give a concise proof of the above two conjectures with *finite-weight sharing* assumptions.

Theorem 3. *Let f and g be two distinct non-constant meromorphic functions sharing $(0,k_1)$, $(1,k_2)$ and (∞,k_3) with $k_1k_2k_3 > k_1+k_2+k_3+2$. Then,*

$$\left(\frac{1}{2}+o(1)\right) \leqslant \frac{T(r,f)}{T(r,g)} \leqslant \big(2+o(1)\big) \quad (r \to \infty,\ r \notin \mathbb{E}). \tag{5.1}$$

In particular, if f and g are entire, then we just consider f and g sharing $(0,k_1)$ and $(1,k_2)$ with $k_1k_2>1$ and have

$$T(r,f) \sim T(r,g) \quad (r \to \infty,\ r \notin \mathbb{E}). \tag{5.2}$$

Proof. If f is some bilinear transformation of g, then (5.2) holds well. If f is not any bilinear transformation of g, then, with the same notations such as h_1, h_2 and h_3 in the proof of Lemma 2, we know that none of h_1, h_2 or h_3 is a constant. Furthermore, if $N_0(r) \neq S(r)$, then by the second equality in (2.3), and noting the fact that $\bar{N}_0(r) \leqslant N_0(r,1;h_1,h_2)+S(r)$ anyway, it implies that f and g share the values 0, 1 and ∞ *CM* from the conclusion of Lemma 5. In fact, there exist two integers u and v such that $h_1^u h_2^v \equiv 1$ with $uv(u+v) \neq 0$. Hence,

$$\left(\frac{f}{g}\right)^u \left(\frac{f-1}{g-1}\right)^v \equiv 1,$$

from which we could immediately derive that f and g share the values 0, 1 and ∞ *CM*. So, the conclusions of Theorem F hold well. Valiron's theorem (see [17, pp. 34–37] and [19, Theorem 1.13]) applied to the three cases in Theorem F yields (5.2). If $N_0(r)=S(r)$, then we get (5.1) by (2.2). When f and g are entire, we just assume that f and g share $(0,k_1)$ and $(1,k_2)$. Hence, the inequality $k_1k_2k_3 > k_1+k_2+k_3+2$ with $k_3=\infty$ turns out to be $k_1k_2>1$. Noting the fact that $N(r,f)=O(1)$ and $N(r,g)=O(1)$, we have (5.2) similarly. □

Remark. If $\{k_1,k_2,k_3\}=\{1,2,6\}$ or $\{1,3,4\}$ or $\{2,2,3\}$ for two distinct non-constant meromorphic functions f and g, and $\{k_1,k_2\}=\{1,2\}$ for two distinct non-constant entire functions f and g, then the conclusions of Theorem 3 hold.

6. On results of Ozawa and Ueda

In 1976, M. Ozawa proved the following

Theorem G. *(See [14].) Let f and g be two non-constant entire functions of finite order sharing* 0 *and* 1 *CM. If $\delta(0,f)>\frac{1}{2}$, then either $fg\equiv 1$ or $f\equiv g$.*

Seven years later, H. Ueda removed the restriction on order and extended the above result to meromorphic functions as the following

Theorem H. *(See [16].) Let f and g be two non-constant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If*

$$\limsup_{r\to\infty,\, r\notin\mathbb{E}} \frac{N(r,f)+N(r,\frac{1}{f})}{T(r,f)} < \frac{1}{2},$$

then either $fg\equiv 1$ or $f\equiv g$.

In 1990, H.X. Yi generalized the above two theorems and obtained

Theorem I. *(See [21].) Let f and g be two non-constant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If*

$$\limsup_{r\to\infty,\, r\notin\mathbb{E}} \frac{N_{1)}(r,f)+N_{1)}(r,\frac{1}{f})}{T(r,f)} < \frac{1}{2},$$

then either $fg\equiv 1$ or $f\equiv g$.

Some results concerning weighted sharing on this topic and its related problems could be found in [5–7,25,26]. Here, we derive a theorem which generalizes Theorems G–I and some other results through the conclusions of Theorem F.

Theorem 4. *Let f and g be two distinct non-constant meromorphic functions sharing $(0,k_1)$, $(1,k_2)$ and (∞,k_3) with $k_1k_2k_3>k_1+k_2+k_3+2$. If, for $h\in\{f,g\}$, we have*

$$\limsup_{r\to\infty,\, r\notin\mathbb{E}} \frac{N_{1)}(r,h)+N_{1)}(r,\frac{1}{h})+N_{1)}(r,\frac{1}{h-1})}{T(r,f)+T(r,g)} < 1, \tag{6.1}$$

then f and g assume the following three forms:

$$\text{(i)}\quad f=\frac{e^{s\gamma}-1}{e^{(k+1)\gamma}-1},\qquad g=\frac{e^{-s\gamma}-1}{e^{-(k+1)\gamma}-1},\quad 1\leqslant s\leqslant k;$$

$$\text{(ii)}\quad f=\frac{e^{(k+1)\gamma}-1}{e^{(k+1-s)\gamma}-1},\qquad g=\frac{e^{-(k+1)\gamma}-1}{e^{-(k+1-s)\gamma}-1},\quad 1\leqslant s\leqslant k;$$

$$\text{(iii)}\quad f=\frac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1},\qquad g=\frac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1},\quad 1\leqslant s\leqslant k,$$

where s and k are two positive integers such that s and $k+1$ are mutually prime, and γ is a non-constant entire function. Furthermore, we have

$$N_{1)}(r,h)+N_{1)}\left(r,\frac{1}{h}\right)+N_{1)}\left(r,\frac{1}{h-1}\right)=\left(2-\frac{1}{k}\right)T(r,h)+S(r). \tag{6.2}$$

Proof. Let us also proceed the proof with two cases.

Case 1. If f is some bilinear transformation of g, then from the conclusions of Lemma 4, we derive that

If $fg\equiv 1$, then 0 and ∞ are lacunary values of f and g, thus we may write $f=e^{\beta}$ and $g=e^{-\beta}$, which means that f and g satisfy case (iii) in the statement of Theorem 4 with $k=s=1$ and $\gamma=\beta+(2\mu+1)\cdot\pi i$ ($\mu\in\mathbb{Z}$) and (6.2) holds.

If $f+g\equiv 1$, then 0 and 1 are lacunary values of f and g, thus we may write $f=\frac{1}{1-e^{\beta}}$ and $g=\frac{e^{\beta}}{e^{\beta}-1}$, which implies that f and g satisfy case (i) in the statement of Theorem 4 with $k=s=1$ and $\gamma=\beta+(2\mu+1)\cdot\pi i$ ($\mu\in\mathbb{Z}$) and (6.2) holds.

If $(f-1)(g-1)\equiv 1$, then 1 and ∞ are lacunary values of f and g. Set $f=e^{\beta}+1$ and $g=1+e^{-\beta}$. Then, f and g satisfy case (ii) in the statement of Theorem 4 with $k=s=1$ and $\gamma=\beta+2\mu\cdot\pi i$ ($\mu\in\mathbb{Z}$) and (6.2) holds.

If $f\equiv\alpha g$ ($\alpha\neq 0,1$), then 1 and α are lacunary values of f, and 1 and $1/\alpha$ are lacunary values of g. By the *Second Main Theorem* and (2.1), $T(r,h)=\bar{N}(r,h)+S(r)=N_{1)}(r,h)+S(r)$ and

$T(r,h)=\bar{N}(r,1/h)+S(r)=N_{1)}(r,1/h)+S(r)$. Hence, we have a contradiction against (6.1) since now $T(r,f)=T(r,g)+S(r)$.

If $f-1\equiv\alpha(g-1)$ $(\alpha\neq 0,1)$, then 0 and $1-\alpha$ are lacunary values of f, and 0 and $1-1/\alpha$ are lacunary values of g. A contradiction follows analogously.

If $(f-\alpha)(g+\alpha-1)\equiv\alpha(1-\alpha)$ $(\alpha\neq 0,1)$, then ∞ and α are lacunary values of f, and ∞ and $1-\alpha$ are lacunary values of g. Similarly, we get a contradiction.

Case 2. If f is not any bilinear transformation of g, then by (2.1), (2.2), the second equality of (2.3) and (6.1), we see $\bar{N}_0(r)\neq S(r)$. Therefore, it implies that f and g share the values 0, 1 and ∞ *CM* by the proof of Theorem 3. Employing the conclusions of Theorem F, we know that f and g assume the three forms in the statement of Theorem 4 with $k\geqslant 2$.

If case (i) holds, then we have

$$\begin{aligned}
&T(r,f)=kT\bigl(r,e^{\gamma}\bigr)+S(r),\\
&N_{1)}(r,f)=kT\bigl(r,e^{\gamma}\bigr)+S(r),\\
&N_{1)}\left(r,\frac{1}{f}\right)=(s-1)T\bigl(r,e^{\gamma}\bigr)+S(r),\\
&N_{1)}\left(r,\frac{1}{f-1}\right)=(k-s)T\bigl(r,e^{\gamma}\bigr)+S(r).
\end{aligned}$$

Thus, we have (6.2).

If case (ii) holds, then we have

$$\begin{aligned}
&T(r,f)=kT\bigl(r,e^{\gamma}\bigr)+S(r),\\
&N_{1)}(r,f)=(k-s)T\bigl(r,e^{\gamma}\bigr)+S(r),\\
&N_{1)}\left(r,\frac{1}{f}\right)=kT\bigl(r,e^{\gamma}\bigr)+S(r),\\
&N_{1)}\left(r,\frac{1}{f-1}\right)=(s-1)T\bigl(r,e^{\gamma}\bigr)+S(r).
\end{aligned}$$

Thus, we have (6.2), too.

If case (iii) holds, then we have

$$\begin{aligned}
&T(r,f)=kT\bigl(r,e^{\gamma}\bigr)+S(r),\\
&N_{1)}(r,f)=(k-s)T\bigl(r,e^{\gamma}\bigr)+S(r),\\
&N_{1)}\left(r,\frac{1}{f}\right)=(s-1)T\bigl(r,e^{\gamma}\bigr)+S(r),\\
&N_{1)}\left(r,\frac{1}{f-1}\right)=kT\bigl(r,e^{\gamma}\bigr)+S(r).
\end{aligned}$$

Thus, we have (6.2), too. $\square$

Corollary. *Let f and g be two distinct non-constant meromorphic functions sharing $(0,k_1)$, $(1,k_2)$ and (∞,k_3) with $k_1k_2k_3>k_1+k_2+k_3+2$. If*

$$\limsup_{r\to\infty,\, r\notin\mathbb{E}}\frac{N_{1)}(r,f)+N_{1)}(r,\frac{1}{f})}{T(r,f)}<\frac{1}{2},$$

then $fg \equiv 1$.

Remark. Under the same value-sharing assumptions as those in Theorem 4 and its Corollary, and furthermore, if we assume that

$$\limsup_{r\to\infty,\, r\notin\mathbb{E}} \frac{N_{1)}(r,f)+N_{1)}(r,\frac{1}{f-1})}{T(r,f)} < \frac{1}{2},$$

or

$$\limsup_{r\to\infty,\, r\notin\mathbb{E}} \frac{N_{1)}(r,\frac{1}{f})+N_{1)}(r,\frac{1}{f-1})}{T(r,f)} < \frac{1}{2},$$

then either $(f-1)(g-1)\equiv 1$ or $f+g\equiv 1$.

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